Analog Integrated Circuit Design
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Lecture – 59
Discrete -Time Active Filters

Hello and welcome to lecture 59 of analog integrated circuit design, we have looked at how to design continues time filters. In this lecture we will briefly look at how to design discrete time filters these filters are useful when the signal is available in a sample data form, that is the input signal is sampled at some uniform sampling interval. There are also useful in some other systems which operate on the sample data which are not necessarily filters, but some other signal processing system such as delta sigma modulators.

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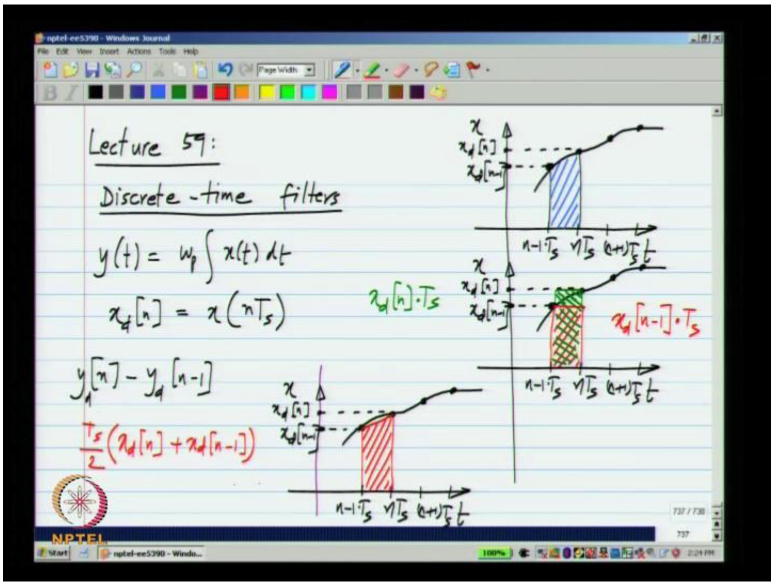

Now what are discrete time filters we have a continuous time signal x of t. Where signal x is defined for all values of time and discrete time signal is where you take samples of these and it is defined only at these discrete instance of time or sometimes it is defined as something that changes only at discrete instance of time and this interval between successive samples is the sampling interval T s and the reciprocal of that is the sampling frequency f s.

Now when we say it is a discrete time filter it will operate on these samples and it will perform some functions low pass band pass or high pass whatever filtering function, we require and provide output samples the relationship between output samples. And input samples in the frequency domain will be similar to the relationship between the outputs and the input for a continuous time filter in the frequency time domain that is for a filter of the corresponding type low pass filter will filter out signals of low frequency and so on.

Now you know from a basics of sampling that the signal has to be sampled at at least twice the bandwidth for it to be properly represented. Now we will not going to those details it is assumed that the signals would be sampled at the appropriate frequency and things like that now what we tried to do in this lecture is to see how we can make these discrete time filters. We already know how to make continuous time filters.

So, we will take that as the starting point and make our discrete time filters we could start from the description of a discrete time filter in the sample domain that is in the z transfer domain, but usually what is done is because there has been a large body of work trying to find out optimum transform functions for continuous time filters the same is used, but adapted to discrete time filters. Now we know that for continuous time filters the basic building block is the integrator, we saw that we could synthesize any filter by representing each state variable output as an integration of a linear combination of number of other quantities the, which could be the input and other state variables.

Now what is done is to approximate this integration in the discrete time domain that is for the discrete time signals and from there we define the discrete time equivalent of the continuous time filter, but this is not strictly necessary, you could start with any arbitrary transform function in the discrete domain that is defined in the z domain and then you can use the circuits that I am going to describe to realize those transform functions as well.

Now let us say we had an integral relationship between y and x that is y is the integral of x with over time now how do we implement this in the discrete time domain. Let us say x varies with time like this and I will show the sampling instance and the sample values may be I will show it like that and let me call this the nth sample n T s n minus 1 T s n plus 1 T s and so on. And these sample values are what constitute the discrete time signal

I will define the discrete time signal x d such that the nth sample of x d is nothing, but the value of x at n times T s. So, this particular value is x d of n minus 1 and this x d of n and so on. Now what is the integral of this we know that the integral of a curve is the area under the curve.

So, the integral of this curve between n minus 1 T s and n T s is nothing, but this area that is this is the difference between value of y at n T s and value of y at n minus 1 T s this is written as a indefinite integral, if you look at the definite integral from n minus 1 T s to n T s that will be nothing, but y d of n minus y d of n minus 1 the value of integral up to this time minus the value of integral up to that time that is the area here that is given by this blue area.

Now what we have to do is we can of course, compute this area, but to compute this area under this curve we need the continuous time values of x what we are trying to do here is work only with the samples x d that is x d of n minus 1 and x d of n. So, how do we compute the area knowing only x d the discrete samples x d, we have to resort approximations and there are few things that we can do these are very familiar to you from your basic mathematics. So, what are the possibilities I do not know the continuous curve.

So, there are a few things that I can do first of all I will take this area that is I will take the value of x d of n minus 1 and find that area that is one possibility and a second possibility is I will take x d of n and take that area this entire area what I am doing is either approximating this curve by this constant which is the previous sample or this constant which is the next sample and clearly you see that the approximation will be more refined if I do this which is instead of this I still have to work with samples. So, this is the actual curve.

So, for working with samples I will use this straight line and I will approximate the curve by a straight line instead of a constant this line which connects x d of n minus 1 and x d of n and take that area now because this is a straight line I do not need to know the intermediate values I only need to know the values at the nth sample and n minus 1 sample. So, these three approximations for integration form the basis of transforming continuous time filters to discrete time filters.

So, if I look at this red area here what is that area that is nothing, but x d of n minus 1 times T s which is the width the height is x d of n minus 1 and width is T s if I look at the green area that is x d of n times T s again the height is x d of n and the width is T s and what is the area in this more refined case you can see that it is basically the average value of x t of n minus 1 and x t of n times the width. So, it is x d of n plus x do f n minus 1 times T s by 2 that is the area of this quadrilateral.

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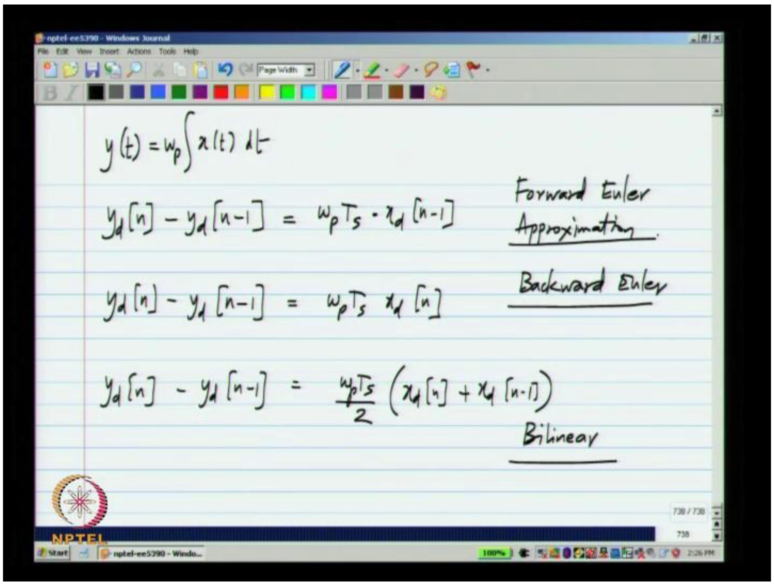

Based on these we can make our approximations we have the relationship between the continuous time input and output y and x and this omega p is a constant of proportionality which makes y and x as a same dimension let us say x and y are voltages this omega p is frequency. So, three dimensions are consistent now what are the three approximations for this we have in terms of the discrete time values y d of n minus y d of n minus 1 is basically omega p times the approximated integral which is either this or that or finally, now it turns out all these are valid approximations clearly.

This is the most refined and preferred for designing filters, but we can also use these things and it turns out these are all also used as numerical approximations for integration and in this case and this particular approximation is known as forward Euler approximation and this is backward Euler and this is known as bilinear. Now I will describe the following steps using the bilinear approximation to integration, but you can equally well do that with the other two.

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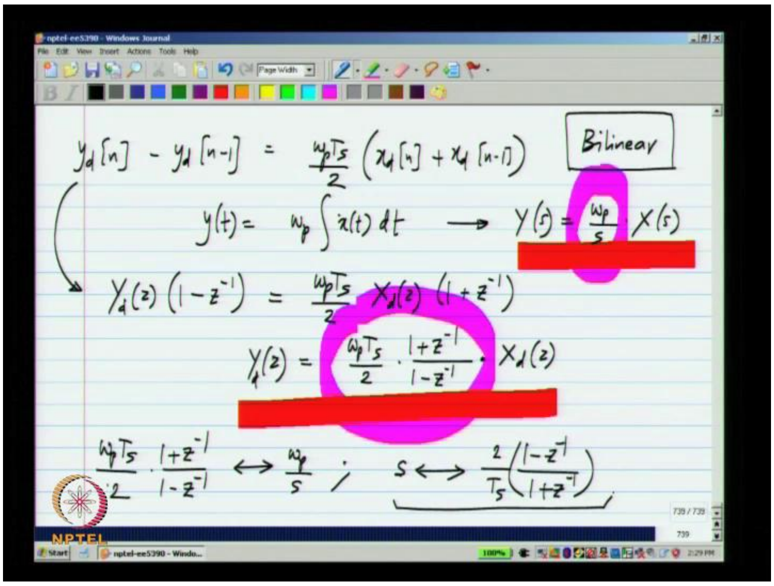

Now the original stuff that we started off with is y of t is omega p integral x of t d t now in the frequency domain this turns out to be y of s equals omega p by s times x of s if I take Laplace transforms if I take the z transform of this, I will get y d of z 1 minus z inverse equals omega p T s by 2 x d of z 1 plus z inverse if I rewrite this as that one y d of z will be omega p T s by 2 1 plus z inverse by 1 minus z inverse times x d of z.

So, this is an approximate realisation of that where y d represents a samples of y and x d represents the samples of x. So, you to transform a filter from continuous time to discrete time the continuous time filter will be described in terms of s as polynomials in s. So, what we have to do is substitute this part with that then basically what we are doing is each integrator in the filter is approximated by its discrete time counterpart. So, the overall filter will be a reasonable approximation to the original continuous time filter you start off with that is the idea.

So, what does this mean in this means that if I substitute s by omega p T s by 2 1 plus z inverse by 1 minus z inverse corresponds to omega p by s and this means that s corresponds to 2 over T s 1 minus z inverse by 1 plus z inverse this is the transformation from continuous time to discrete time. So, any transform function described in terms of s you can now describe in terms of z and then implement it.

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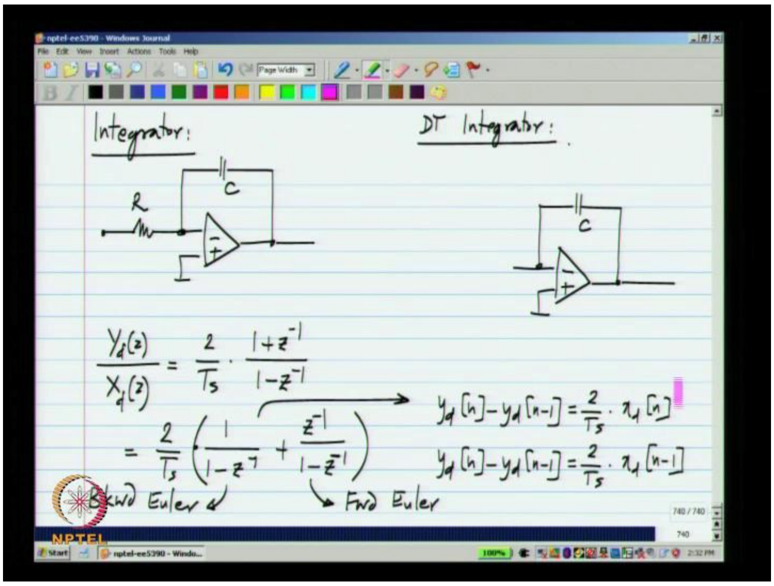

So, we will quickly look at this for an integrated itself and for a second order filter I will show an opamp integrator here. Now the discrete time integrator is built in a similar way to this this capacitor you see holds the state variable corresponding to the state of the integrator in the discrete time integrator as well there is a capacitor C which holds the state of the accumulator. Now what else do we need to have y d of z by x d of z equals 2 over T s 1 plus z inverse by 1 minus z inverse which I will write as two parts which are 1 by 1 minus z inverse plus z inverse by 1 minus z inverse and in fact, if you work out the forward and backward Euler they will correspond to each of these terms. So, this is the backward Euler and this is the forward Euler.

Now how do we implement these things that is basically the voltage on this capacitor should be such that it will follow this relationship and the input to the system are samples of x d we will assume that the inputs are changing every time period T s. So, if I write the differential equation corresponding to each of these parts this will be and the other one will be instead of x d of 1 we will have x d of n minus 1.

So, what it means is the difference of the voltage on the capacitor between one cycle and the next should be related to the input sample like this. Now how do you change the voltage on a capacitor it is by dumping charge into the capacitor and the amount of voltage change equals the amount of charge that is added to the capacitor divided by the capacitance value. So, what these two differential equations say the amount of voltage change is proportional to input sample either the present input sample or the previous input sample. So, what we need to do is to dump an amount of charge on this capacitor which is proportional to either the present sample or the previous sample. So, what is done is the input sample value either the present input or the previous input is stored on a capacitor and that is transferred to the capacitor C.

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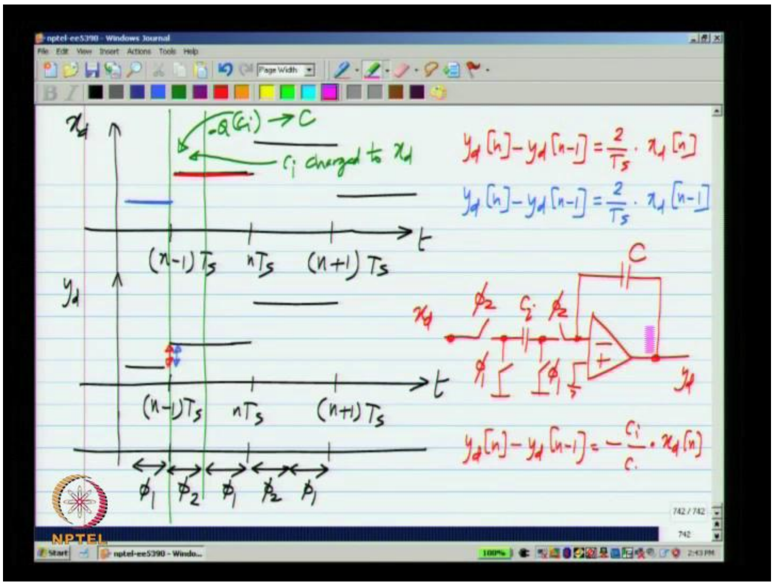

Now, we need to define a certain convention for timing. So, this is t and the input samples x d are changing like this. Let us say they are held constant for one whole period and they are changing at these instance. Now x d and represented as a sample signal and y d represented as the sample signal also will do something of that sort and so on.

So, what this expression is saying is that this y d of n minus y d of n minus 1 should be related to x d of n that is this part and what the next one is saying the other term is that in that case this difference should be related to that one. So, how are these things implemented if let us first consider the stuff shown in blue this corresponds to the stuff shown in blue and let us say that each clock period this period is divided into two phases because we need to look at the value of the input capacitor and we need to change the voltage on the feedback capacitor let me call this phi 2 and this one phi 1 this is just some convention that I have used.

So, what this means is when during the phase phi 2 the output voltage changes that is what this is showing and phi 1 it is constant. So, if this switched is operated with phi 2 what happens is you know that if this switched is open clearly the voltage across the capacitor C cannot change at all it is when this switched is closed that some charge can flow into it and the voltage can be changed. Now if I look at what I want to have here in blue what it means is that when I close the switched phi 2 a certain amount of charge should flow and that charge should be related to whatever happened in the previous period.

So, the capacitor should be charged to the previous sample voltage and not the present sample voltage and that can be arranged by having switches like this. So, what; that means, is that let me call this C i. So, during this period during this interval C i gets charged to x d of n minus 1 that is when Phi 1 is closed here and phi is closed over there simply that capacitor C i is placed across x d and during the next phase phi 2 what happens is you see that this C i is connected between this negative terminal of the opamp and ground it is assumed that the opamp is operating in negative feedback if you assume that it is an ideal opamp or something that has a very large gain the voltage here will be 0 so; that means, that all the charge from C i will be transferred to C.

So, in this other phase the following phase the charge on C i goes to C and it turns out that if you write out the difference equation you will see y d of n minus y d of n minus 1 equals C i by C times x d of n minus 1. I hope this is clear the output is changing in phi 2. So, the input capacitor is charged to the previous sample and then that charge is dumped on the capacitor C during phi 2. Now, we would like to instead implement the other one where the change in the output voltage is proportional to the present sample.

So, in this case what should happen is when phi 2 is connected to this that is during this period that is when the voltage on the capacitor C changes and amount of charge should be related to the present sample so; that means, that this has to be phi 2 and these two have to be phi 1. I am showing you the topology which implements this function now a later I will show how to analyse the circuits like. This is a general class of circuits known as switched capacitor circuits and they are very useful and they can implement any discrete time transform function that you would like to realize.

So, what happens in this case the phase is being used are slightly different during this period during this phi 1 just before the transition the capacitor C i is discharged. because of these two switches which connected to ground. Now during the phase phi 2 what happens is that the left side this is connected to x d and this point will be at virtual ground due to negative feedback around a high gain opamp and there will be a certain change of a charge in this capacitor and that will be related to how much charge is there on C i because if you look at this node no charge can escape this node between phi 1 and phi 2. So, the amount of charge change inc will be related to the amount of charge change in C i which in turn is proportional to the present sample.

So, in this both of these happen in the same phase C i charged to x d and. In fact, the negative of q of C i goes to c and it turns out that if you write out the difference equation for this you will get Y d of n minus y d of n minus 1 equals minus C i by C times x d of n you see that it very much looks like the first one and except for this minus sign now if youre implementing the circuit in fully next differential manner which is the most common way of implementing these filters there's no problem you simply swap the wires and then you get the negative voltage. So, this is a circuit that implements this backward Euler integration.

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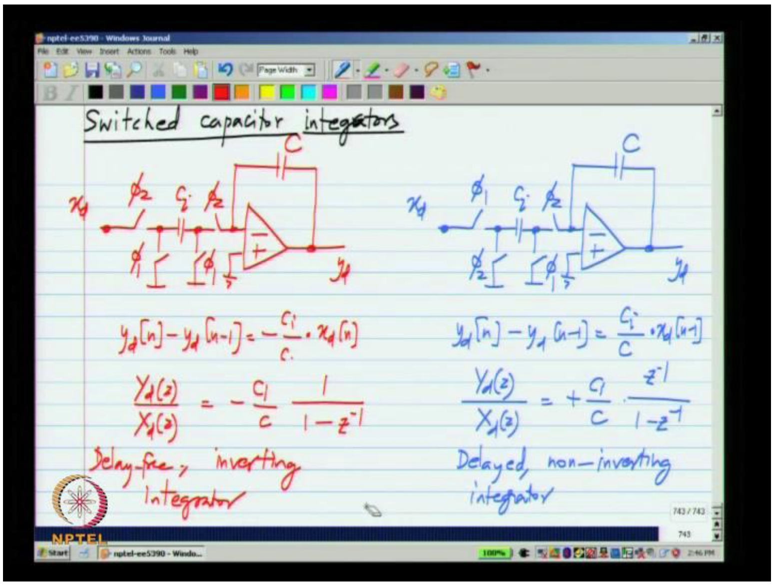

As I mentioned before these are known as switched capacitor circuits, because we have a switches in capacitors these are features in the circuit, where we need changes in voltages at discrete instance of time, and the way to accomplish that is to dump appropriate amount of charges on a capacitor. In this case, if you connect a capacitor in feedback across an op amp and connect another capacitor to the input what happens is, because of the virtual ground the amount of charge change in the input capacitor will be equal to the amount of charge change in the capacitor connected across the op amp that is the basic principle behind these circuits.

And these circuits because they implement integrators there are known as switched capacitors, integrators. If you write down the transform function of this you will get Y d by X d equals minus CI by C 1 by 1 minus z inverse, and similarly for the other one you will get Y d by X d equals plus C I by C z inverse by 1 minus z inverse. Now, you see what differentiates between these two is first of all the sign. Now, the sign is of not much consequence in a fully differential circuit, but the other thing is you have an extra factor of z inverse here, whereas you do not have it there. So this is called, delayed non inverting integrator, and the other one is called delay free inverting integrator.

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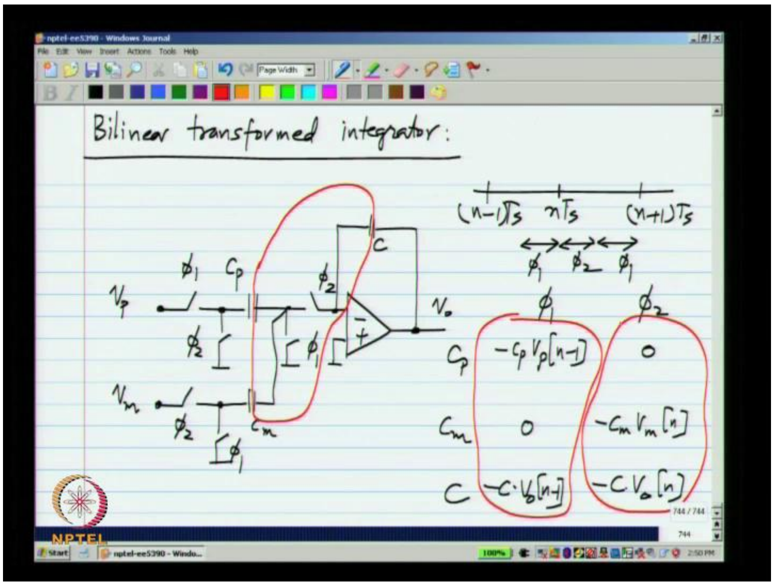

Now by combining these two you can get a bilinear transformed integrator that you get by this switched is operated in phi 2, and let me call this C p, and this one C m, the input voltage can be the same or different for these two arms let me for now call it V p and V m, and this will be phi 2 and phi 1 they will be in the opposite phases to that one c, if I call this V o the way to analyse this is to look at the charge on these plates; On these plates just before this phi two is closed and just after the phi two closed now between those two instances no charge can escape these and the total charge will be conserved that is the general way of analysing any switched capacitor filter that you are given and in this case we already know the answer, but we will still do that just as an exercise, these are the sampling instance something is changing over there, and this is phi 1, this is phi 2, and that is phi 1 again.

If you look at this phi1 what happens is this capacitor C p is charged to V p and this C m is completely discharged and this C has whatever value it has during this cycle C p will have minus C p times V p that is the charge on this particular plate, and C m will have zero and C will have minus C times V o of n minus 1, this is also by the way minus C p V p of n minus 1. So that is the charge in phi 1 and phi 2, this will have 0, this will have minus C m V m of n, and this will have minus C V naught of n, and the sum of these must be equal to the sum of those because the charge cannot go anywhere.

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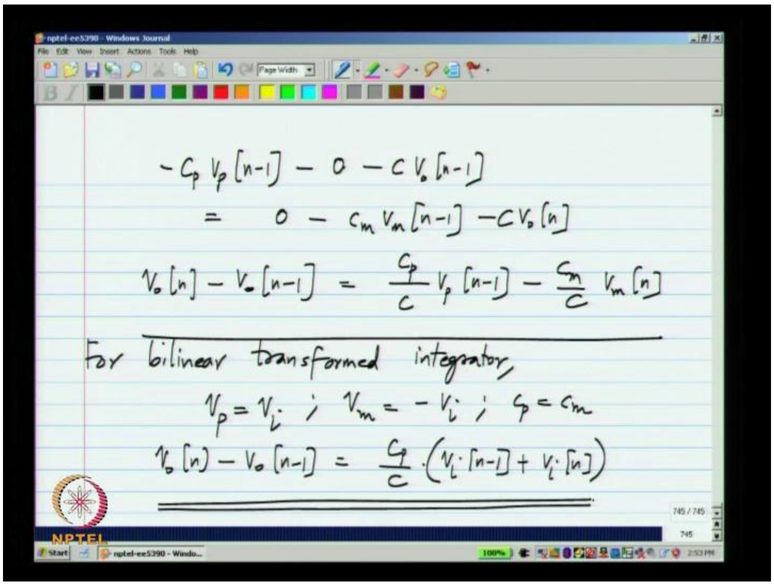

And that will give you the difference equation that C p V p of n minus 1, minus 0, minus C V o of n minus1equals minus C m V m, equals 0, minus C m V m of n minus 1, minus C V o of n, and this gives you the result that V o of n minus V o of n minus 1 equals C p by C V p of n minus 1 minus C m by C V m of n. So you get simply the combination of the delayed non inverting integration and the delay free inverting integration, now to realize the bilinear transformed integrator from this all you have to do is let us say you make V p equals V i and V m equals minus V i and C p equals C m and C p by C m should be this quantity.

So, if you do that what we will see is V naught of n minus V naught of n minus one equals C p by C V i of n minus one plus V i of n which is the differential equation that corresponds to bilinear transformation. So, by using this integrator as a building block you'll be able to realize any by linearly transformed filter. Any filter that has been that uses bilinear transform to go from continues time to discrete time. So, that is a general way of doing things and I will also show one particular addition that is sometimes useful.

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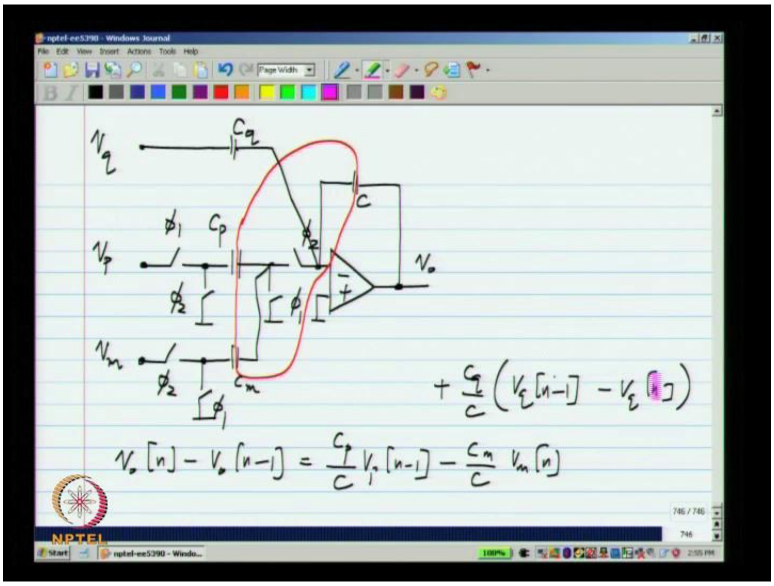

Noise it is also possible to add a capacitor directly without switching. I will call that C 0 and V 0 and this just gives you more possibilities actually it turns out with switched capacitor filters you can generate a greater variety of transform functions in integrator like this then in a continuous time version. And in this particular case let me call this V q and maybe this one as C q in that case V o of n minus V o of n minus one turns out to have C p by C V p of n minus one minus C m of C V m of n and plus C q by C V q of n minus 1 minus V q of n.

So, you can implement more signal processing functions by making V p V m and V q of appropriate values and choosing the three capacitors C p C m and C q. So, everything that we implemented with continuous time integrators each integrator can be substituted by this bilinear transform integrator. So, that is one way to do it now on this course we will not deal with these switched capacitor filters in such great detail. So, what I will do is I will show you a switched capacitor topology for a second order filter which, In fact, it turns out to be slightly more efficient than transforming each integrator using bilinear transform. Transforming each integrator using bilinear transform is certainly possible, but the topology I am going to show you happen to be more efficient and it is a well minus known topology for generating all kinds of second minus order transform functions.

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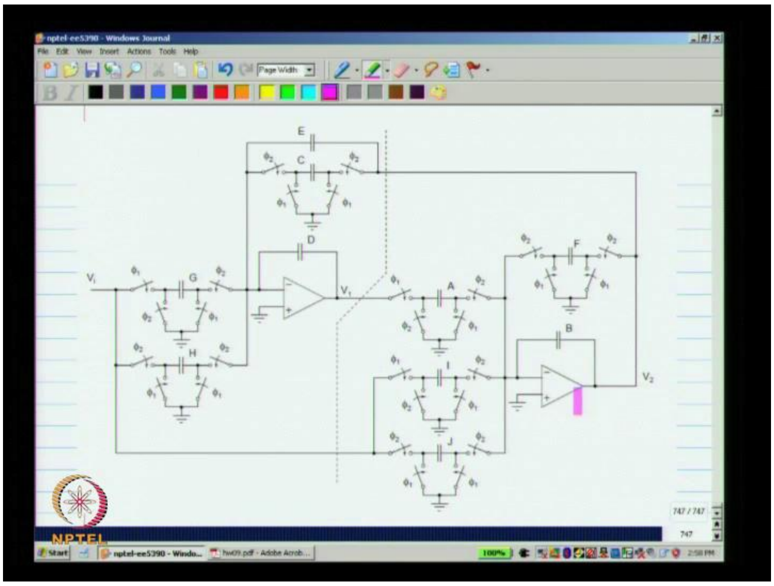

This is a topology of switched capacitor filter which was finding out by fleischer and laker long long ago and you can identify two integrators this integrator with this feedback capacitor D and this integrator with the feedback capacitor B and we can also identify these switched capacitor elements this capacitor B which tampers the input in phi 1 and dumps it to D in phi 2 and this one which samples the input in phi 2 and dumps it also in phi 2.

So, this forms the delayed integration and this forms the part for delay less integration similarly I is for delayed and J is for delay less and this is delayed and that is delay less and you also see the direct capacitor that is sometimes useful in switched capacitor circuits. So, I will not go into the details of this also can be analysed by writing b one in terms of V i and V 2 in appropriate phases and similarly writing V 2 in terms of V 1 and V i and also V 2 in the appropriate phases and finely solving the set of equations to get the final differential final difference equation.

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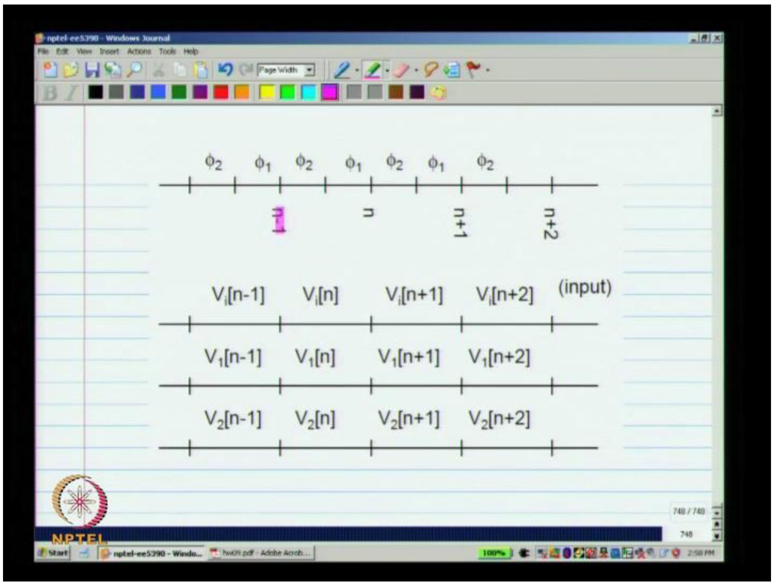

Now, let us assume that the timing of the filter is like this you have the n minus 1 th sampling instant n sampling instant and so on, and the change of output occurs on phase phi 2 you see here that in phi two the output b one will change. And also in phi 2 the output V 2 changes. So, defining the input voltage V 1 and V 2 like this you can write out the difference equations please take this as an exercise and do it for yourself and see that it is consistent with the answer that I am going to show you.

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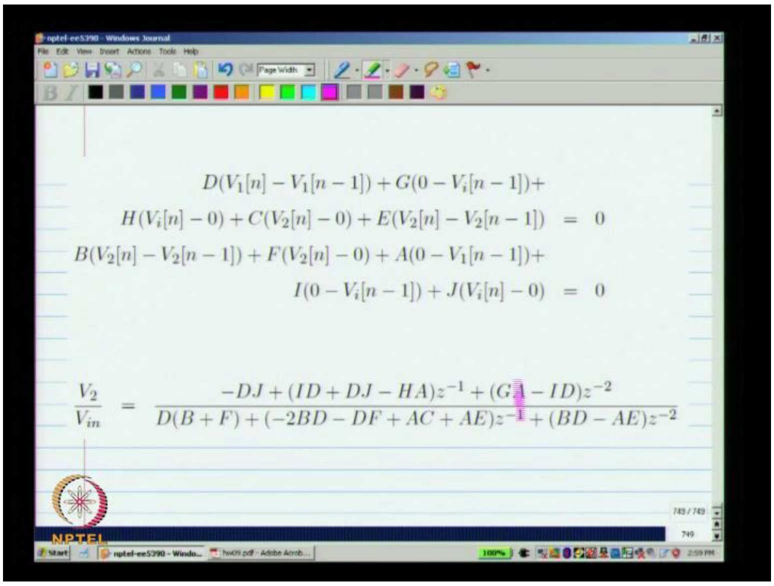

So, it is a valuable exercise in trying to analyse switched capacitor circuits. Now it turns out that these are the difference equations this is for the first integrator, and this is for the second integrator and then if you solve for V two by V in the z domain you will get this you see that you will have a second order polynomial in z inverse in the numerator and a second order polynomial in z inverse in the denominator. So, you can realize any second order polynomial that you wish to have now it turns out that this. In fact, has more degrees of freedom that then you need there are. So, many capacitors in many cases you can set some of them to zero and constrain some of them to be equal to each other.

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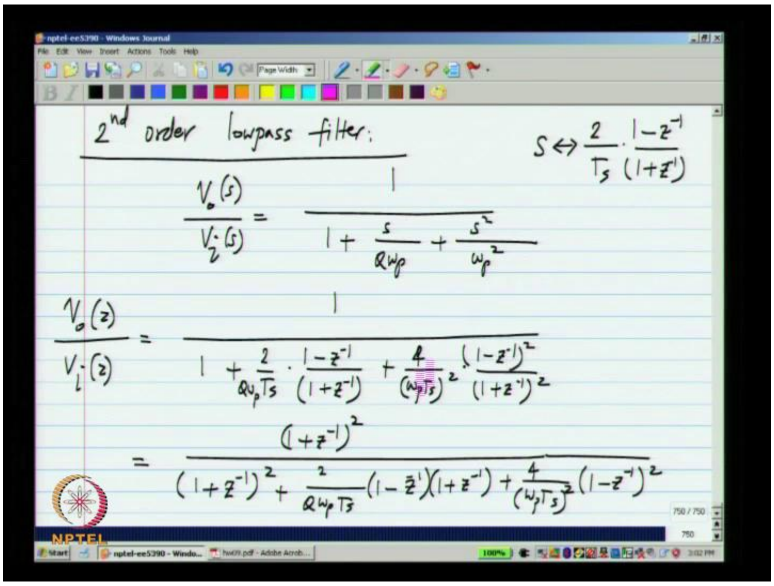

So, what I am going to do is just take a second order low pass filter in the continuous time second order low pass filter the transform function would be one by one plus s by q omega p plus s square by omega p square and if I substitute s with two by t s one minus z inverse by one plus z inverse, what I am going to get here is the discrete time counterpart which will be one by one plus two over t s one minus z inverse by one plus z inverse plus four over two maker p t s square one minus z inverse square by one plus z inverse square, and this basically will be one plus z inverse square divided by this whole thing . So, the bottom line here is the continuous time low pass filter transforms to a discrete time transform function which has a second order numerator and a second order denominator. Now as we saw from our filter the resulting transform function is also something that has a second order numerator and a second order denominator.

So, by adjusting the values of these capacitors you will be able to get any discrete time transform functions that you wish to have. So, you will be able to realize band pass and other types of filters again I will leave it as an exercise to you to transform a band pass filter into its discrete time counterparts and compute the values.

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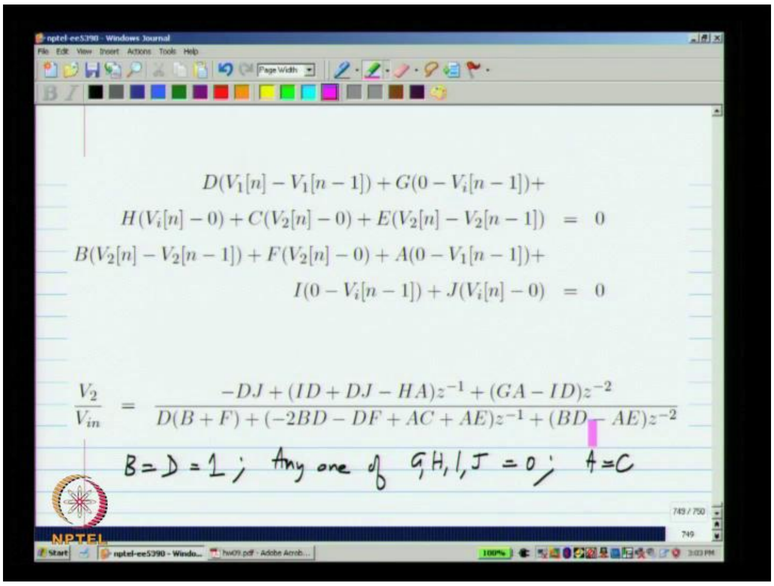

Now usually you it turns out that you can to start off with because there are too many degrees of freedom you set b and d to be one; obviously, we do not mean one farad here it is a unit capacitor whose absolute size you can choose later b and d to be one, and you can set any one of g h I and j to be zero it turns out you can realize all the transform functions even with one of these being zero and also you can constrain a and C to be equal to each other. So, this is just. So, that you do not have to fight with. So, many variables there are ten different variables here whereas, in this there are only six coefficients right. So, you can constraint some of them. So, that you can remove the degrees of freedom and later you can always do capacitor scaling to reduce noise et cetera. So, this is how you design discrete time filter.

Now, I have not gone into how to design a higher order filter and so on, but you also know that any higher order filter can be decomposed into second order and first order sections, and you can use these second order and first order sections to realize any higher order filter that you want these switched capacitor integrators are useful not only to make filters, but also for delta sigma modulators. Now delta sigma modulators are a class of a d c's that are very, very popular at certain frequency ranges, because of very useful properties that they have you can realize very high resolution without necessarily having components which are very accurately massed and so on.

So, whatever I showed now the integrators as well as the methods of determining the transform functions are also useful for switched capacitors delta sigma modulators, what I will do now is to just deal with a a couple of small aspects of switched capacitor circuits and then finish the lecture.

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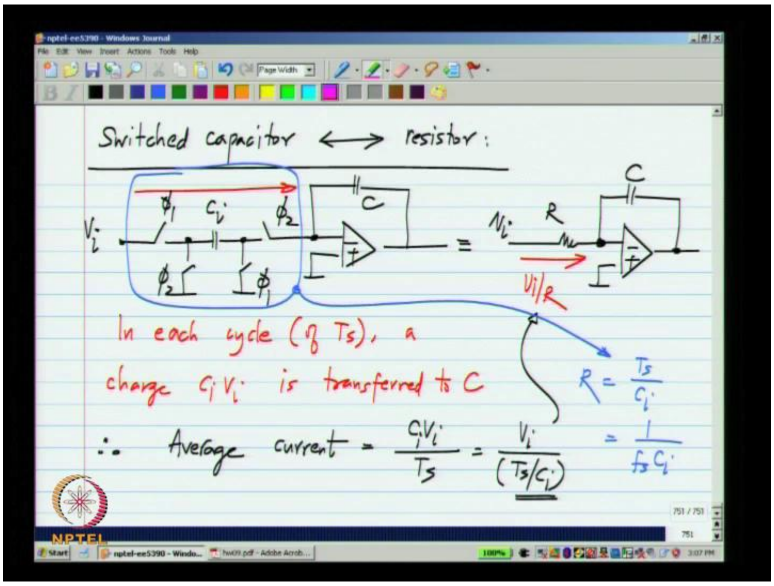

First of all many times you would see that and analogy is made between a switched capacitor and a resistor ok, that is this is considered to be equivalent to that, now clearly the equivalence is only the approximate because this is a continuous time system and this is a discrete time system and they cannot be exactly equal to each other, but that is why you have an input signal vi and let me choose these phases like that and also an input signal vi over there. We know that the current that flows here will be vi divided by r now if you look at the corresponding current that flows here what happens is in phi one this capacitor is charged to vi and in phi two that charges dumped on c.

So, what happens is there is not a steady current that is flowing here even if vi is a constant, but what happens is in each cycle of duration t s a charge ci times vi is transferred to C essentially it corresponds to some average current flowing that way, so the average current equals C I V I divided by t s or V I divided by t s by C i. So, if you make those two equivalent that is the average current here equal to the current over there you see that this circuit is equivalent to a resistor whose value is t s by C I or one over f s C I, where t s is the sampling period and f s is the sampling frequency.

So, if you have a continues time filter with resistors and capacitors that is you've you make them using these active r C integrators you can substitute the resistors with theses witched capacitors using this analogy now this is much simpler than actually deriving the discrete time transform function and doing it, but this works only on an average sense. So, this approximation is valid only for frequencies well below the sampling frequency when not much changes happening between one sample and the next. So, this can be used, but this analogy will give you accurately results only when you're making filters whose frequencies of interest are much lower than the sampling frequency. So, it is just something that is useful to know. So, what we will do in the next lecture is to discuss one other aspect of switched capacitor filters something that is related to practical implementation of switched capacitor filters.

Thank you, I will see you in the next lecture.