Analog Integrated Circuit Design
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Lecture No - 10
Nyquist Criterion; Phase Margin

Hello and welcome to another lecture of analog integrated circuit design, in the previous class what we did was to look at what happens to a negative feedback amplifier. When you have multiple parasitic poles and as you saw the analysis was somewhat complicated even with identical poles. And towards the end of the lecture we said we will look at more general criteria which is more easily applicable for the obituary case of multiple poles in different locations and that criterion is known as the Nyquist criterion.

What it does is to change the problem of finding whether the poles are in the right half plane to evaluating the loop gain as a function of frequency, of some sinusoidal input and making some statements about the plot of the imaginary plot versus real plot of the loop gain. Now, last time we stated the criteria rather widely now we will make it more precise and see how to use it for the case when you have multiple parasitic poles.

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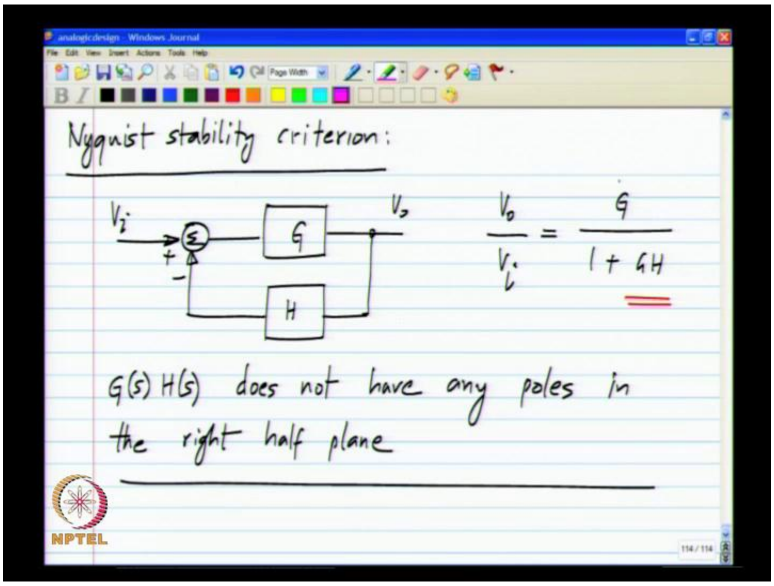

And we will assume a system like this our system confirms to this as does any other negative feedback system and what is important is not so, much the G and H, but the product GH. So, V naught by V i is G by 1 plus G H and everything literally revolves around this GH. And the wage basis of the criteria can be thought of as avoiding G H being minus 1. So, we will also make some other assumptions that G of s, H of s does not have any poles in the right half plane.

Now of course, the generalised Nyquist criterion can be applied even in that case, but for us typically what happens is you design the loop gains itself to be stable. It is not that you will start with an unstable system here, unstable system there and then try to make a stable amplifier around it, you will start with G of s, H of s which does not have poles in the right half plane. So, we will make that simplifying assumption because then the condition is simpler to state.

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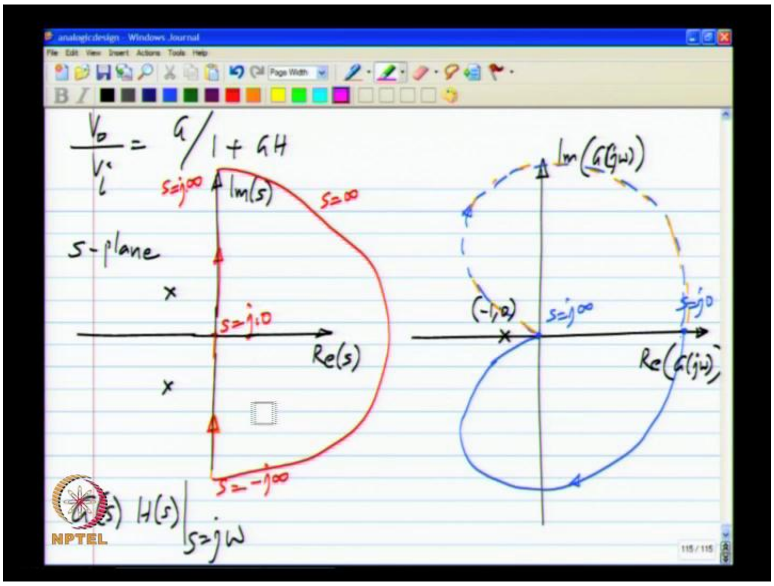

So, given that what we can say is so, we know that V naught by V i is G by 1 plus GH, where everything is a function of s and this is the s plane. And there will be some poles which could be in the left half plane or the right half plane we do not know yet and this is what we would like to avoid evaluating indirectly, this is the real part of s and that is the imaginary part of s.

So, instead of this we go to a plot of imaginary part of G of j omega versus real part of G of j omega and we will also mark the critical point minus 1, 0 and essentially what is done is that this G of s, H of s is evaluated along the imaginary axis. So, we will evaluate G of s, H of s for s equals j omega; that means, we will start from s equals j 0 which corresponds to dc go all the way to a s equals j infinity and then again redo it from s equals minus j infinity and all the way back to s equals j 0. So, then will get some complex number GH as we traverse this for every frequency point the complex number will be different.

Then we will be we will plot the imaginary part of that complex number versus the real part of the complex number and we of course, are not taking a specific function now so, I will draw something arbitrary. So, let us say this corresponds to s equals j 0 and then the plot in general will do something like this and I will assume that for s equals j infinity, it goes to 0. And this is typically the case also because all real systems will have some band limiting. However, you choose to implement your loop gain as you increase the input frequency the loop gain diminishes to 0 you cannot have substantial loop gain at infinite frequencies. So, at infinite frequencies regardless of how you build your system the loop gain will eventually go to 0.

So, I will assume that it goes to 0 like that now, we also have the remaining part of the curve which is from s equals minus j infinity to s equals j 0 and it turns out that, this will be exactly the mirror image of this curve about the x axis it will look like that. Let me show this in a slightly different colour, so this part corresponds to the upper half of it. So, now it turns out that what an Nyquist criterion states that is the number of poles in the right half plane, in the right half s plane here equals the number of times this plot this weird plot that you drew encloses minus 1, 0.

That is if you have poles in the right half plane you evaluate the function along this line and as you know I mean from a theory of complex numbers basically, this line plus a semi-circular contour at s equal to infinity closes the loop. Whether, there are poles of the system inside this can be mapped to whether this particular map of the complex number encloses minus 1, 0 and the way I have drawn it there is no enclosure of minus 1, 0. And I forgot to mention one thing that the number of poles in the right half plane equals number of clockwise encirclements of this curve, that is the curve of the loop gain of minus 1 0.0 and the way I have shown it, it is not enclosing it when it goes around in the clockwise direction so, actually this system is stable.

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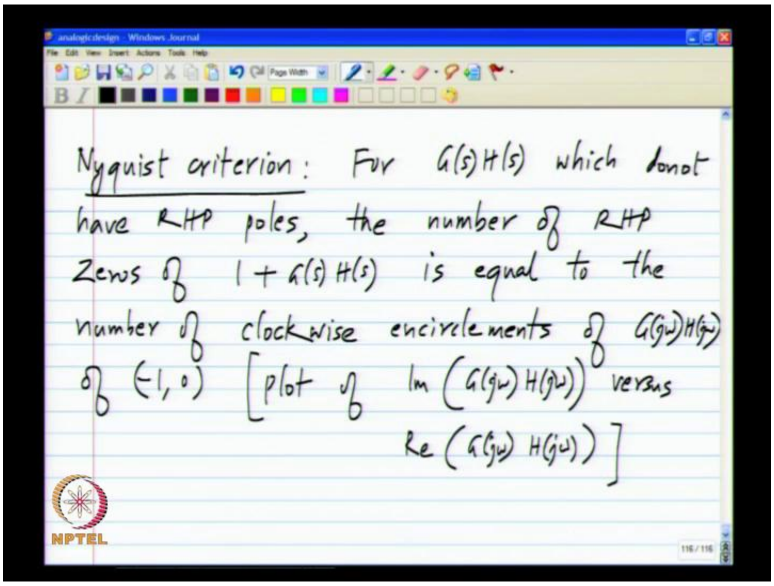

The number of let us state the Nyquist criterion properly, if GH do not have right half plane poles number of right half plane poles of rather the number of right half plane 0's of 1 plus G of s, H of s. Which corresponds to the number of right of plane poles of G by 1 plus G of s, H of s is equal to number of clockwise encirclements of G of j omega, H of j omega of minus 1, 0 of this point.

So, what I mean by this is in the plot of imaginary part of G of j omega, H of j omega versus real part of G of j omega, H of j omega. And this can be proven with little bit of theory of a complex numbers and it can be found in many standard control system textbooks. So, we are not going to prove it here, but this is a criterion that we will use to come up with the stability criteria, which does not involve finding out roots of polynomials.

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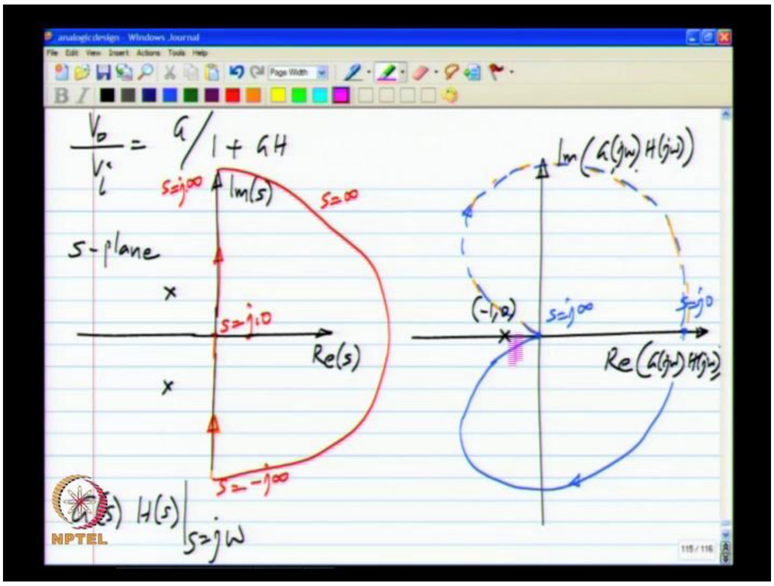

And we already looked at an example here which corresponds to a stable system. So, in this case the Nyquist plot, the plot of imaginary part of sorry I seem to have made a mistake here. This is imaginary part of G of j omega, H of j omega and this is real part of G of j omega, H of j omega, that plot the number of times it encloses minus 1, 0 in the clockwise direction corresponds to number of right half planes, of number of right half plane poles of G by 1 plus GH. Let us quickly take an example another example.

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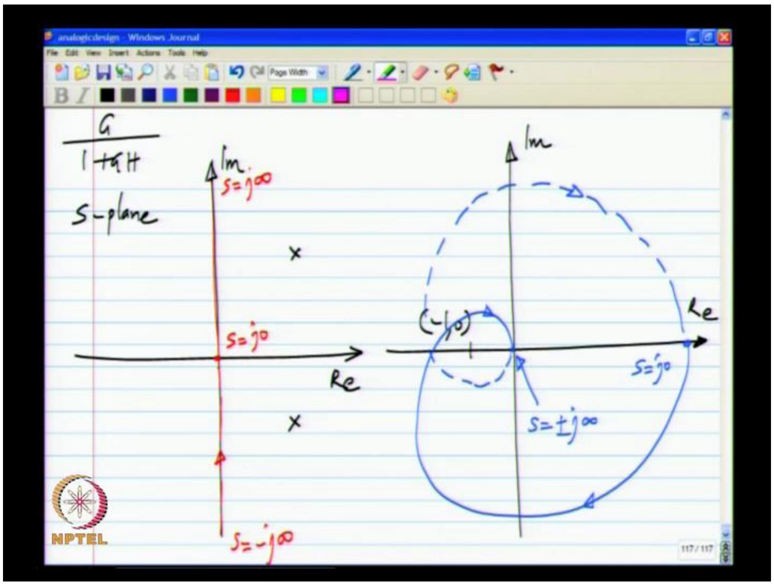

So, this is the s plane and this is the real part and the imaginary part of s and what you do is you evaluate j th starting from 0 to infinity, and infinity back to 0 you evaluate GH along this and then plot it is imaginary versus real parts. So, you will get something and as always the plot for the upper half of this is the mirror image of the plot of the lower half. And let us say the plot comes out like that and the remaining part of the plot is simply a mirror image of that one.

So, now, you can see that as you go from here which corresponds to s equals j 0 to this which is s equals plus minus j infinity, you have gone around this minus 1, 0 point twice. So, what this means is that this function in this case G by 1 plus GH has two poles in the right half plane. We have not evaluated the poles, we have not do not know where they are, but we do know that they are in the right half plane and that is a significant thing because once it is in the right half plane the system is unstable and therefore, useless.

So, whatever system we design has to have left half plane poles and that is equivalent to saying, whatever system we design should have an acquiesce plot which does not encircle the minus 1, 0 point. So, that is the stability criteria and the reason it is popular is that it is rather much easier to evaluate this compared to evaluating the poles as I have said repeatedly.

Now, our amplifier design needs not only stability, but also good behaviour meaning we not only need the response not to blow up; that means, instability it is not enough we cannot have a lot of ringing either we should have a good behaviour. So, what we will do then is to look at the Nyquist plot for something, we can calculate the results analytically for the ideal delay case and the second order case and then extended it to all the other cases.

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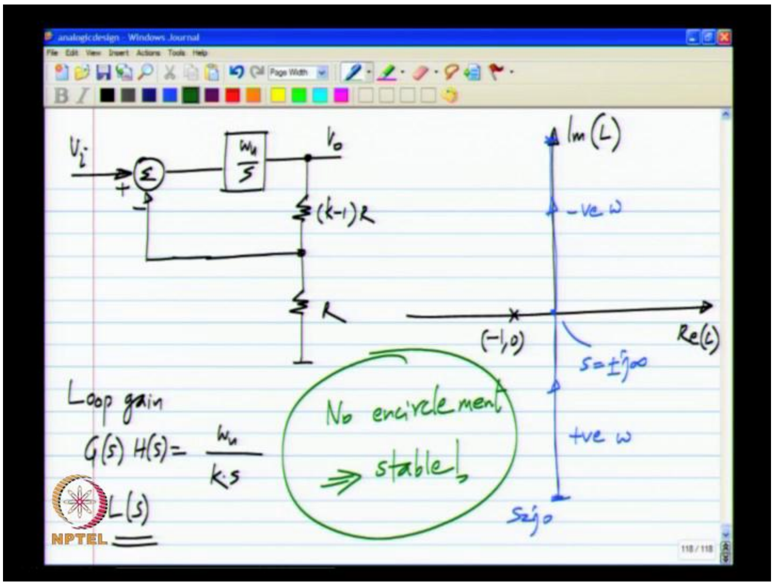

So, now let us go back to our amplifier and I will take initially let me take simply the ideal case with no extra parasitic poles. And we already know that this is unconditionally stable and there is no ringing and we will see what it corresponds to in the Nyquist plot. What is the loop gain in this case if I break the system here make input equal to 0 I will easily evaluate the loop gain to be omega U divided by K s. So, I will use a shorthand for a loop gain L of s so, that it is easier to write and I would have plot imaginary part of L versus real part of L. So, what happens here if I start from s equals to j 0 this is equal to infinity and it has a phase angle of minus 90 degree. In fact, this function has a phase angle of minus 90 degrees for whatever value of s equals j omega, whatever value omega and s equals j omega.

So, the phase is minus 90 degrees s equals to j 0 the magnitude is infinity I will show infinity as some far out point on the imaginary axis and as the frequency increases it comes down and for s equals j infinity for infinite frequencies the loop gain is 0. Now, if you go to negative frequencies, that is you start from s equals j infinity and go back to s equal to j 0 through negative values, you will get mirror image of the same plot. And this corresponds to both plus and minus infinity and the lower part is for a positive omega and this is for negative omega.

And actually this point and this point are the same that corresponds to s equals j 0. So, you can complete the contour in a circle of in a semicircle of infinite radius and you have the point minus 1, 0 here, and it is absolutely clear that our blue curve our Nyquist plot is not encircling this point at all. So, the system is unconditionally stable and what I mean by unconditionally stable is sometimes you can have some systems which do not encircle this, but if you change some parameter in the system it may encircle. But in this case we know that first of all there is only one parameter here omega U or omega U by K if you count both of them, but regardless of what you change the Nyquist plot will remain here and it is not going to encircle it. So, the verdict is so, it is unconditionally stable.

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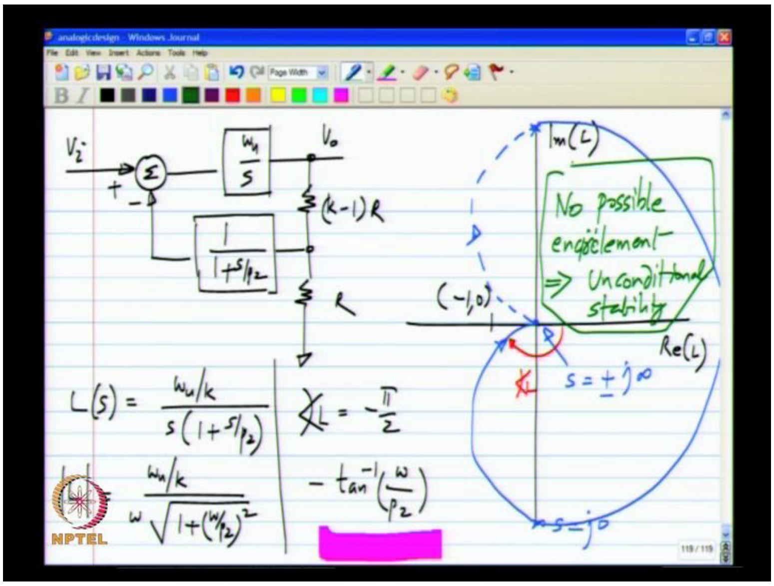

So, now let us take the next case where we have and in this case the loop gain L equals omega U by K S 1 plus S by p 2. And the magnitude of omega can be calculated to be omega U by K divided by omega and square root of 1 plus omega by P 2 square and the angle of L can be calculated to be minus pi by 2 because of this S minus tan inverse omega by P 2. Now, we can draw the Nyquist plot, the plot of imaginary plot of L versus real part L and we have this point minus 1, 0 as always.

So, for very low frequencies for very low values of omega the magnitude is close to infinity and the phase angle is close to minus pi by 2. So, it starts of in about the same point as the ideal case the only difference is that in the ideal case we had a phase angle which was always minus pi by 2 and the magnitude was decreasing here the magnitude is decreasing, but the phase angle is also the phase lag is also increasing. So, as omega increases this keeps increasing and for omega equals infinity this second term also becomes minus pi by 2 and the angle of l asymptotically reaches minus pi.

So, it will do something of this sort we do not have to worry about the details, but it will do something like that or this corresponds to S equals j infinity and you said repeatedly plus minus j infinity. And the rest of the curve is a mirror image of that one and this point and that point are the same. Now, the key point here to note is that this angle, the angle will become minus 90 degree the angle is nothing but the angle from the positive real axis, the phase lag is this particular angle this is the angle of L for any particular frequency.

So, you can see that it can become 180 degrees for omega equals infinity so that means, that this part of the curve for positive omega will always stay below the negative real axis. So, that means that again there can be no possible encirclement of minus 1, 0 point. That means, that I can change the parameter omega U I can change the parameter K, I can change the parameter p 2 regardless this maximum phase lag is pi. So, there can be no possible encirclement of this right.

So, again the verdict is no possible encirclement and which means unconditional stability and this is something we also knew from our direct analysis of writing down the expression for the close loop gain and finding out the roots they could not be in the right half plane. So, this cannot encircle the minus 1, 0 point which also means that the roots cannot be in the right half plane, the roots of the poles of the close loop system cannot be in the right half plane.

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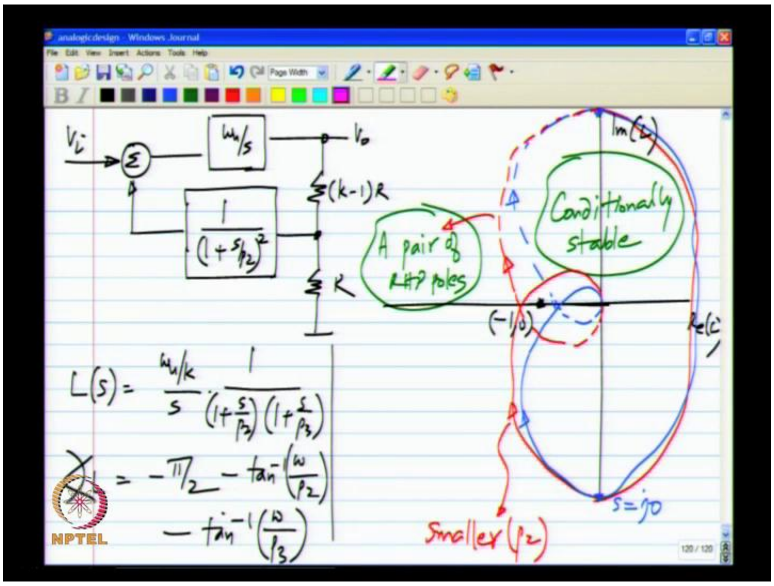

So, let us repeat this for a third and higher orders and see what happens, I will take the system we had analysed earlier where I will take two identical parasitic poles the rest of the system of course, remains the same and the loop gain is that one. So, it is obvious that the magnitude of L keeps on decreasing with omega the magnitude of L is omega U by K divided by omega 1 plus omega by p 2 square, but what is more interesting is the angle of L which is minus pi by 2, because of this s term minus 2 tan inverse omega by p 2.

So, here is the interesting part that is omega becomes very very large this part becomes pi so, the total phase lag can be minus 3 pi by 2. So, again we plot our usual stuff let me move the y axis a little to the right imaginary part of L and real part of L and as usual we start from S equals 0 which starts at basically infinity. At 0 frequency this function has an infinite gain which I show at the bottom of the imaginary axis here, and then the magnitude keeps on decreasing and the phase lag keeps on increasing, but the point is for very high frequencies it will do that it will asymptotically reach minus 3 pi by 2. From here to here is minus pi by 2, from here to there is minus pi and from here all the way to there is minus 3 pi by 2. So, this is what happens and the rest of the plot as usually is the mirror image so, that is what happens and this point and this point are the same. So, I will just show some connection there.

Now, does this enclose minus 1, 0 or not now we cannot tell because the way the shape is it could enclose or it could not depends on the parameter. If minus 1, 0 happened to be there it is not enclosing it and if the minus 1, 0 happened to be in that region it is enclosing it. So, what determines whether it encloses it or not, so let us say for a particular set of parameters the Nyquist plot is like this and it is not enclosing the minus 1, 0 point which means that the system is stable.

Now, let us see what happens if I use a lower value of P 2 if I reduce a lower value of p 2, what happens is the phase lag increases because the argument inside here increases. And if I use a lower value of P 2 the magnitude also decreases, but may be the effect on the phase is more prominent than the effect on the magnitude. So, for a smaller value of P 2 the curve will look something like that it still starts from infinity for DC and the phase changes more rapidly than the magnitude and it does something like that and the remainder of the curve will be something like that one.

So, clearly here the blue Nyquist plot is not enclosing minus 1, 0 and the red Nyquist plot is enclosing minus 1, 0. And this is for a smaller p 2 a smaller value of p 2 means a larger value of parasitic delay due to p 2 and the system tends to be unstable. This again we have evaluated the condition for instability exactly and that is that when P 2 is half of unity loop gain the system will have poles on the imaginary axis. And if the pole value P 2 is smaller than half of omega U loop that is half of omega U by K then it will have roots in the right half plane.

So, you can see that the red curve encloses the minus 1, 0 point twice so, this corresponds to a pair of RSP poles because it encloses it twice. And also the conclusion here is that it is only conditionally stable this again we had evaluated exactly, but that is what comes out from the Nyquist plot as well. So, we had evaluated that for certain values of P 2 it can be stable, but there are also values of P 2 for which it is unstable and here the key point in the Nyquist plot is that the total phase lag can be minus 270 degrees minus 3 pi by 2.

So, if you have a overall second order system the overall phase lag can be minus pi. So, in that case the Nyquist plot can approach the origin only from this angle it cannot cross the negative real axis. But once the total phase lag exceeds minus pi it can cross the negative real axis and only if it crosses the negative real axis then it encircle the minus 1, 0 point.

So, the bottom line is that if you have a second order system it is unconditionally stable and if you have third and higher order systems it is only conditionally stable because if you take now for instance, let me take 3 extra poles let me take 3 extra poles. So, what happens sorry this is not correct this is basically, square to the power of 3 half's, but the key point is in the phase that. Now, the total phase lag is minus pi by 2 minus 3 pi by 2 which is minus 2 pi. So, this can go all the way round and then approach it from the positive real axis side. So, it can enclose minus 1, 0 so, any order of 3 or higher can enclose minus 1, 0 with the right combination of parameters so, all of them are only conditionally stable.

And this Nyquist plot also helps us to treat some cases which we were not able to do earlier that is we I said that we will use 2 identical poles P 2 when we are considering 2 extra poles, but here we can also look what can happen when you have 2 non-identical poles. So, let me consider them to be p 2 and p 3 so, what happens now, the angle of L is minus pi by 2 because of the integration term and minus tan inverse omega by P 2 minus tan inverse omega by P 3.

So, while this looks more complicated the main point is that for very high values of omega this contributes phase lag of minus pi by 2 and this also contributes a phase lag of minus pi by 2. So, the overall phase lag at very high frequencies is still minus 3 pi by 2 for a third order system. So, the Nyquist plot still approaches the origin from the top, from the positive imaginary axis side which means that the Nyquist plot will cut the negative real axis which means it can enclose the minus 1, 0 point.

So, it has potential to enclose the minus 1, 0 point which means that it is only conditionally stable. So, this helps us treat other cases where the poles are not at the same position. To summarise the phase angle of the loop gain can only be minus pi by 2 if you do not have any extra poles and minus pi if you have a single extra pole in either case the Nyquist plot cannot cut the negative real axis. Which means that it simply cannot enclose the minus 1, 0 point which is sitting on the negative real axis, but when you have 2 extra poles or more the total phase lag can be minus 3 pi by 2 or higher minus 2 pi minus 5 pi by 2 etcetera, etcetera.

So, now with the right combination of parameters the Nyquist plot can enclose the minus 1, 0 point because it cuts the negative real axis. So, any system with order higher than or equal to 3 is only conditionally stable. So, this is what we can conclude from the Nyquist plot and so, far we have not done much new we have introduced the Nyquist plot and we have evaluated the same things that we evaluated earlier. That is we have evaluated instability for 2 extra poles, 3 extra poles, 1 extra pole and so on.

And the same thing we verified with the Nyquist plot, what does is to show that what you get from a Nyquist plot is consistent with what you got earlier which is the exact analysis. At the very end we did a small generalisation to multiple poles which are not at the same frequency, but the real value of this is for amplifier design where we are not looking for instability. We want the amplifier to be stable absolutely no doubt about it, but it cannot ring a lot right. So, that is a condition that we want to satisfy you cannot have bad behaviour in terms of a lot of ringing you have to have the amplifier to be well behaved which means a limited amount of ringing. Now, how does the Nyquist plot help us to design an amplifier like that we will see?

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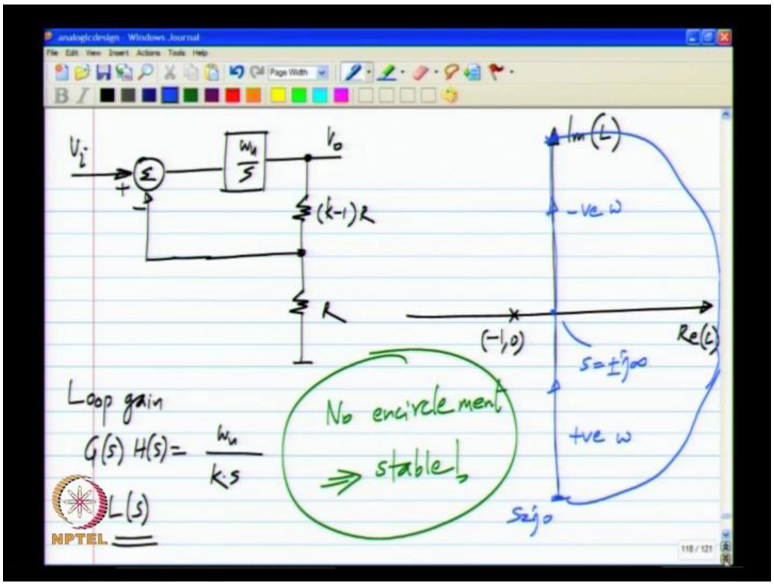

We will go back to the earlier cases this is the first order system the Nyquist plot is simply a straight line along the imaginary axis. And then finally, you can close the loop here because they are the same thing and this clearly does not enclose it and in the first order case there was absolutely no ringing either. Now, we can get our hints from the second order case where there can be ringing, but no instability and to limit the amount of ringing you have to set the parasitic pole P 2 to be at least 4 times greater than the unity loop gain frequency.

So, we will see how to do that now before I go there just one point that basically this minus 1, 0 is the danger point so, to speak you do not want to enclose it. Now, if you want to be absolutely sure that the system is stable if you want to leave some margin what it means is you should not even come close to minus 1, 0 right you do not even want to do something like that. Now, it is coming close to minus 1, 0 it is not enclosing it, but it is coming close and this corresponds to a case with higher ringing. I will show that in more detail with the case of one extra pole.

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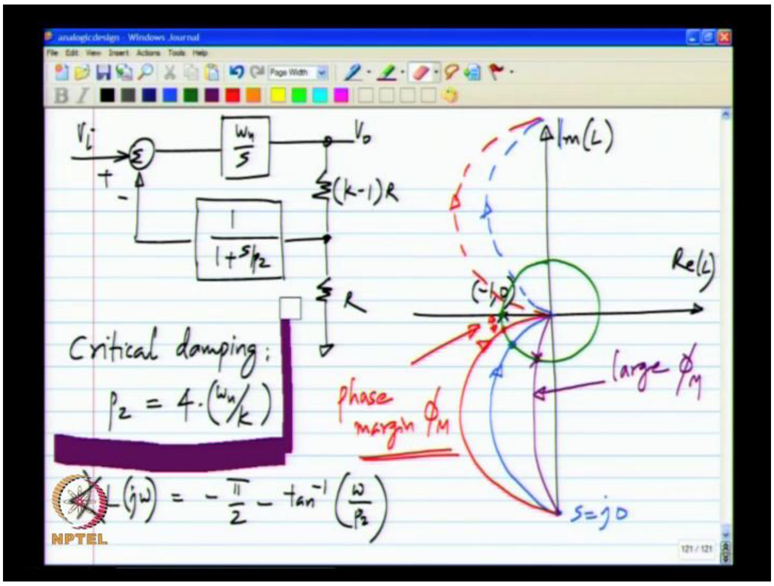

And our loop gain is known and the angle of the loop gain is minus pi by 2 minus tan inverse omega by P 2 it is minus pi at high frequencies. So, the Nyquist plot starts from minus infinity on the imaginary axis and this is minus 1, 0. And what happens is that it comes around like that asymptotically reaching an angle of minus pi. Now, it turns out that if you have a lower value of P 2 it does that a phase angle increases and it comes closer and closer to the negative real axis.

Now, we know that for critical damping P 2 must be 4 times the unity loop gain frequency and if P 2 comes below this you will start to see ringing. Now, how do we quantify this on the Nyquist plot, what we do is to say that if it comes closer and closer to this minus 1, 0 point then there will be more ringing. And this is because this minus 1, 0 point is the critical point and if the loop gain comes very close to this what happens is 1 plus loop gain becomes a very small number and close loop gain V naught by V i which has 1 plus loop gain in the denominator becomes a very large number.

So, that signifies ringing at some frequency that means, you have a lot of gain at some particular frequency and there is ringing at some frequency. So, essentially we would like to stay clear at this point minus 1, 0. And to impose a quantitative criterion what we do is we draw the unit circle here and the point minus 1, 0 lies on the unit circle and we see where the Nyquist plot cuts the unit circle. So, the blue Nyquist plot is cutting here and the red Nyquist plot is cutting it there and the angle of that point where it cuts to the negative real axis is termed as the phase margin phi m.

So, the significance of this is as follows basically, if the phase angle here were this much greater then you would be touching the minus 1, 0 point and that would mean instability. So, this phi phase margin phi m signifies how safe you are how far you are from it when cutting the unit circle and the reason for taking the unit circle is that all the points on the unit circle, all of unit magnitude and minus 1, 0 also lies on it. So, this distance on the unit circle from the critical point signifies the margin that is the safety factor you have for stability.

So, this is the phase margin so, essentially if you look at a second order system it always intersects the unit circle below the negative real axis. So, the phase margin is always positive this is just another way of saying that the a system is unconditionally stable, but depending on the parameter P 2 the phase margin could be 1 degree the phase margin could be 50 degrees or 80 degrees. So, another curve could be something like that so, in this case you see that this corresponds to a large phase margin, because it cuts the unit circle at a distance quite a bit away from the critical point.

Now, how this is useful because we have already calculated for a second order system the condition for critical damping P 2 should be four times omega U by K. So, what we do is we know that this is a good case right so, what we do is we calculate the phase margin corresponding to this case. And impose it for higher order systems also because we cannot calculate the time domain response of higher order systems very conveniently.

We use the phase margin that we get from the second order system for a good case, for a critically damped case for instance and apply it to higher order systems also.

And the hope is that if the higher order system has the same phase margin it will also have a similar time domain response and it terms out that it is more or less true for reasonable systems. So, this is a very popular criterion for designing negative feedback amplifiers. So, the key to remember here is that a negative feedback amplifier has to be not merely stable that means, it is not enough if the response does not blow up, but it also has to be well behaved, and the criterion for well-behaved is inspired from the second order system.

We can also relate it to the system with the ideal delay and the criterion is that you look at the point on the unit circle where the Nyquist plot intersects the unit circle. And then calculate the phase difference between that point and the negative real axis and that is the distance you have left before you could become unstable and that is the phase margin, I hope that is clear. So, what we do is now we calculate the phase margin corresponding to the critically damped case P 2 is 4 times omega U by K.

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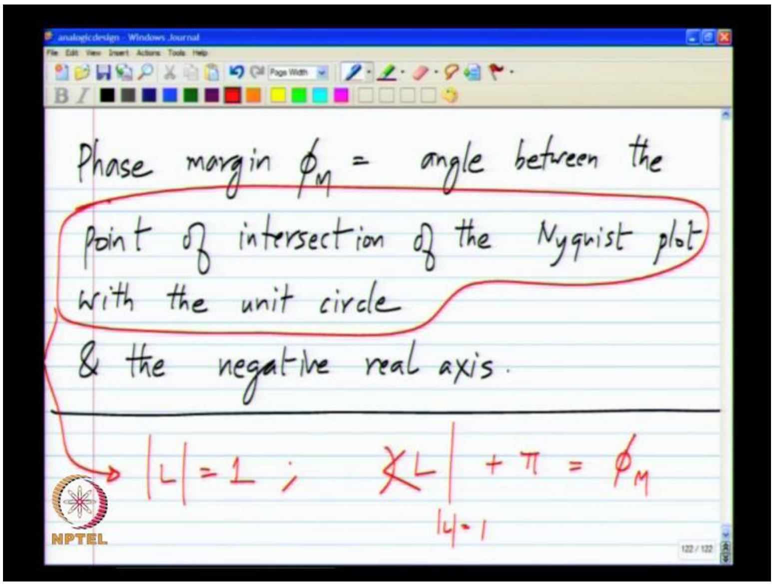

It is the angle of angle between the point of intersection of the Nyquist plot and the negative real axis, basically point of intersection of the Nyquist plot with the unit circle and the negative real axis. Now, what does that mean to say that the Nyquist plot intersects the unit circle this part here basically, means that the magnitude of L is unity right by definition that is what it means.

And you have to look at the angle of L at that point it will be some angle and you want to look at the angle of L that point. Where the magnitude of L is 1 and you should look at the angle between that and the negative real axis which means you have to subtract minus pi or you add pi to it and that is phi m that is the definition of phase margin in mathematical terms.

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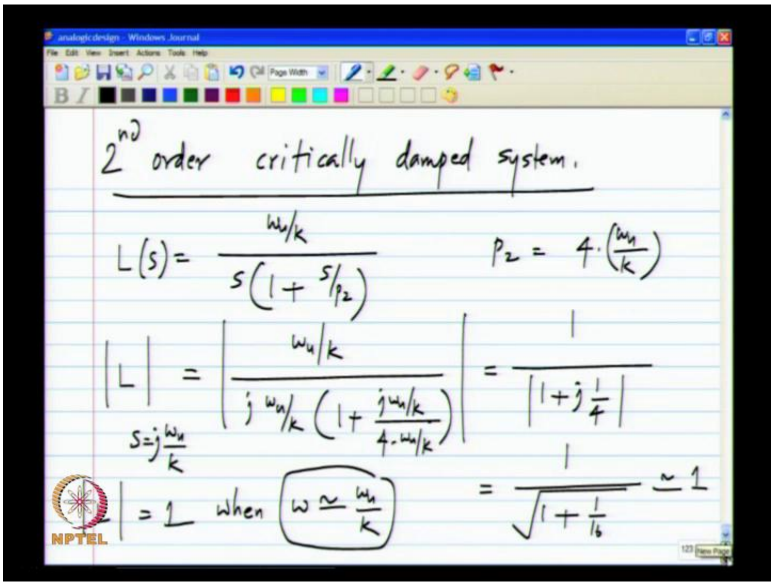

And for our second order critically damped system L of s is omega U by K s 1 plus S by P 2 and P 2 will be 4 times omega U by K. Now, we can calculate exactly where L the magnitude of L becomes 1, but just for simplicity I will do something slightly different I will just look at what happens at S equals j omega by U by K and at this point the magnitude is omega U by K divided by j omega U by k 1 plus the magnitude of 1 plus j omega U by K divided by P 2 and P 2 is nothing but 4 times omega U by K.

So, basically this is the magnitude of 1 by 1 plus j one-fourth, which is 1 by 1 plus one-sixteenth and the square root of that. And like I said we can calculate exactly where the magnitude of L goes to 1, but it is obvious that this number itself is rather close to 1 it is slightly less than 1, but it is close to 1. So, we will say that l becomes 1 when omega is approximately equal to omega U by K. So, this is almost the frequency where the magnitude of L becomes unity I did this because it is easier to calculate it if you try to calculate exactly where magnitude of L becomes 1 it becomes rather cumbersome, while doing it by hand calculations.

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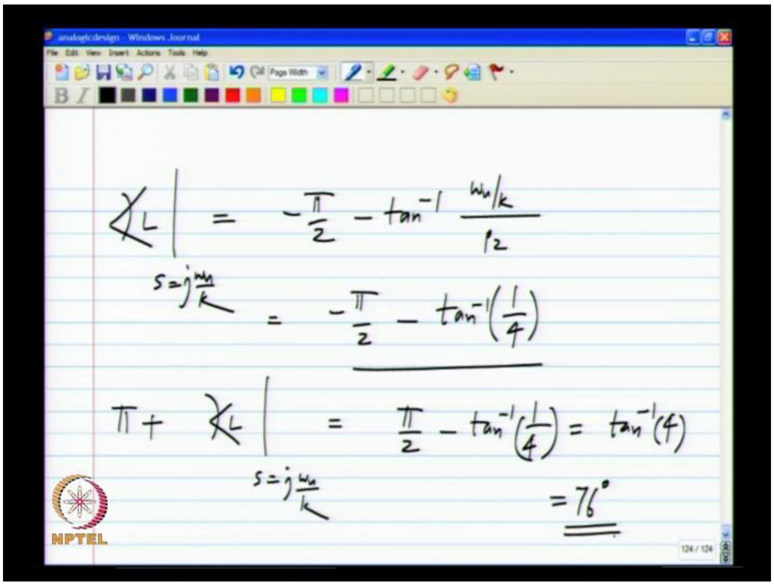

Now, we have to find the phase angle at omega equals omega U by K and that is quite simple. Angle of L where S is j omega U by K is given by minus pi by 2 minus tan inverse omega U by K divided by P 2, which is 4 omega U by K. And the phase margin is nothing but pi plus angle of L j omega U by K which is which turns out to be pi by 2 minus tan inverse 1 by 4, which is equal to tan inverse 4 which corresponds to 76 degrees.

So, that is the phase margin of critically damped second order system and we will apply that phase margin to all higher order systems and it turns out that they will be well behaved. We will see some details of this in the next class, but that is how the Nyquist criterion is very useful it helps us avoid calculating time domain response explicitly or the roots of the polynomial explicitly, while saying something about stability and good behaviour of amplifiers.

Thank you, and see you in the next class.