Signal Processing Techniques and Its Applications
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Lecture - 06
Concept of frequency in Continuous-time and Discrete-time signal

So, in the last class, we discussed analog-to-digital conversion and digital-to-analog conversion. Basically, I am interested in the concept of sampling frequency and quantization. We also discussed the memory requirement for storing a digital signal, which may be a one-dimensional signal or maybe a multidimensional signal. So, in both cases, we have done that.

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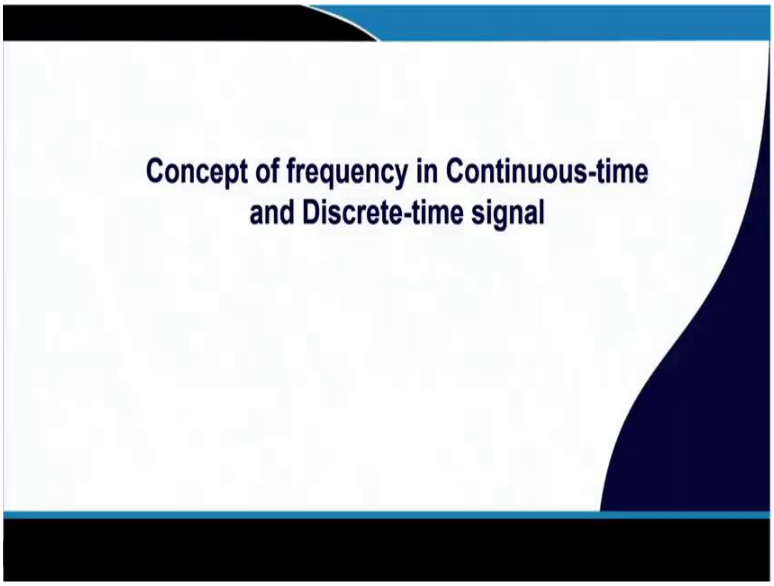

Today we will discuss the Concept of frequency in Continuous time and Discrete time signal. So, continuous time and discrete time signals basically talk about sinusoidal signals. So, what are the concepts of frequency, discrete frequency, and analogue frequency of all kinds of things we will talk about.

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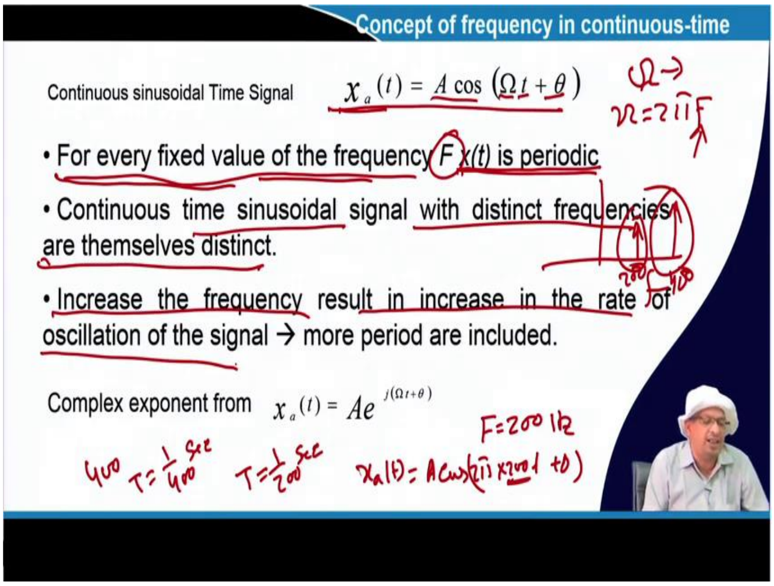

Let us consider the analogue sinusoidal. So, I can write down continuous sinusoidal time signal $x_a(t)$. So, $x_a(t)$ is nothing but $A\cos(\Omega t+\theta)$. This ω is analogue frequency; that means the radian per second, θ is the phase, and t is the continuous time. So, if this is my signal, then I can say that for every fixed value of the frequency F, $x_a(t)$ is periodic. So, what is Ω? Ω is nothing but a $2\pi f$. This F is analog frequency, which is radian per second. So, F x(t) is periodic for the fixed value of the frequency. What is the meaning?

Suppose I said I want to generate a 200 Hz sinusoidal signal. So, small f is equal to 200 Hz. Now, what should be the analog frequency? What should the signal be? So, $x_a(t)$ should be $A\cos(2\pi*200t+\theta)$. So, I can say in that case, if the frequency F is 200, then what is the period T? T is equal to 1 by 200 seconds.

Now, if I change the frequency F to 400, the period will also be 1 by 400 second. So, for every fixed value of the frequency F, the x(t) is periodic, whose period is 1 by F. So, continuous time sinusoidal signals with distinct frequencies are themselves distinct. So, if I said a cos 200 Hz cosine wave and a 400 Hz cosine wave, they are two distinct.

So, they are distinct. So, if I say you have already learned about the Fourier transform. So, when I say cos 200 Hz and turn the frequency transform, if this axis is my F, then I get a peak at 200 Hz. When I converted the 400 Hz, I got a peak at 400 Hz. And those two peaks are distinct themselves. They do not overlap in nature. So, continuous time sinusoidal with distinct frequencies are themselves distinct.

So, $x_a(t)$ is 200 Hz, and let's say $x1_a(t)$ is 400 Hz. They are not equal. They are distinct. Increasing the frequency results in an increased rate of oscillation.

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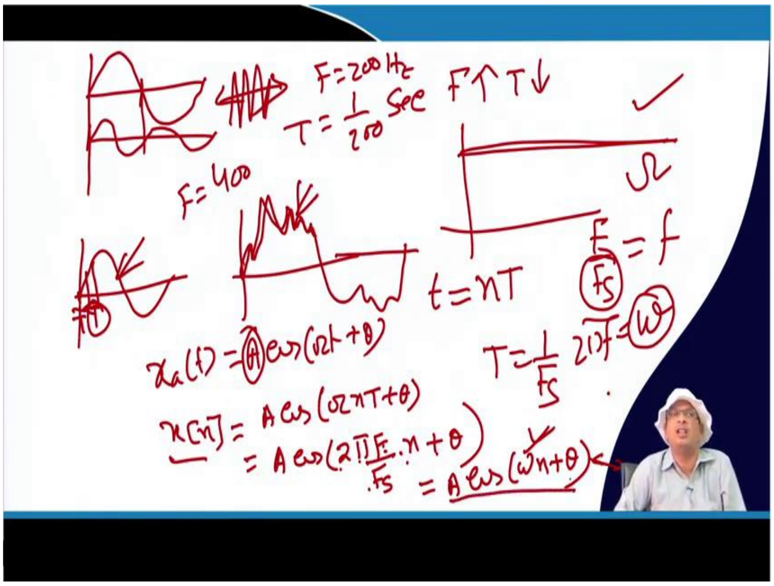

So, if I say I take a 200 Hz sinusoidal signal, it will look like this. Now, when I take the 400 Hz sinusoidal signal, then within a half period, there will be a complete sinusoidal signal. So, if I see, if I increase the frequency, the number of oscillations will increase per second because T decreases.

So, how many oscillations will be in 1 second if the frequency F equals 200 Hz? So, 200 Hz frequency means T equals 1 by 200 Hz second. So, in 1 second, how many oscillations will be there? I can calculate the number of frequencies, which is 200. Similarly, when I say F equals 400, then 400 complete periods will be there within 1 second.

So, if I increase the frequency, the rate of oscillation will increase, and the number of periods will increase within 1 second, or I can say the period will be less. T will be decreasing. F is increasing, and T is decreasing. So, there are three properties for a continuous time sinusoidal signal, the frequency concept. One is that every fixed value of the frequency x(t) is periodic.

Yes, I can calculate. Continuous time sinusoidal signals with distinct frequencies are themselves distinct, yes. 200 Hz signal does not overlap with 400 Hz signals. They are distinct. An increase in the frequency means the rate of oscillation will increase. So, that is the concept of frequency in a sinusoidal signal. When I say these two signals, one signal looks like this.

Another signal looks like this. If I ask you which one contains a high-frequency component? So, what is a frequency? With more variation, more oscillation will increase. So, if I see more oscillations here, that is why these contain the high frequency component compared to this signal. So, that is the concept of frequency in a sinusoidal signal.

Similarly, if I told you the concept of frequency in an image. Suppose I say black whiteboard like this PPT white portion. If this is completely white, there is no change in the pixel; then I said it is totally DC, with no frequency component. So, when I say no oscillation, it means my signal will look like this. No oscillation signal is constant. So, it is a DC signal.

Once I include variation, only oscillation will come. So, in the case of an image, if it is completely white or completely black, there is no only DC. Now, once I oscillate, suppose here when there is a boundary change. So, there is a change in pixel colours. So, here, some frequency is introduced. So, high frequency means more variation and low frequency means less variation. So, that is the concept of frequency in the analogue signals. Is it clear?

So, now I go for the digital signal or discrete signal. So, what is a discrete signal? Suppose I said I have a signal $x_a(t)$; let the same example, which is A $\cos(\Omega t+\theta)$. I want to discretize. So, say discrete time. Discrete value means A will be quantized. Discrete-time means t is quantized. So, what should be the discrete signal of $x_a(t)$? Which is nothing but a represented by a $x[n]$.

So, once I want to do the discretization what parameters I required? I required a sampling frequency. So, what is sampling frequency? This means after what interval I am collecting the evidence of the analogue signal, or I can say what is the resolution of the time axis that I have to know.

So, I can say continuous time is represented by nT. Capital T is the difference or distance between the two samples. So, suppose I take this sample and this sample. So, this distance is T, and the next distance is also T. So, who determines the T? Sampling frequency determines T. So, I know T is equal to 1 by Fs. So, when I discretize this analogue signal, I can write down A cos(ωnT+θ).

So, when I write down nT, what is capital T? It is nothing but an A cos what is ω? ω is nothing but the 2πF. So, F denotes the analogue frequency in T.

$$T = \left(\frac{1}{Fs}\right) * n + \theta$$

So, this F by Fs is called normalized discrete frequency denoted by small f.

Now, this 2πf we write down by this ω, not this. This ω is analogue; this ω is normalized discrete frequency. So, when I say normalized discrete frequency, that means radian per sample. Since it is divided by Fs, it is called radian per sample. So, I write this as Acos(ωn+θ). There will be no T; n is the sample index.

So, this ω is a discrete normalized discrete frequency, which is nothing but an F by Fs. Is it clear? So, due to the sampling, when I write down a discrete signal, it is nothing but a (2πF/Fs)* n + θ. So, it is nothing but Acos(ωn+θ).

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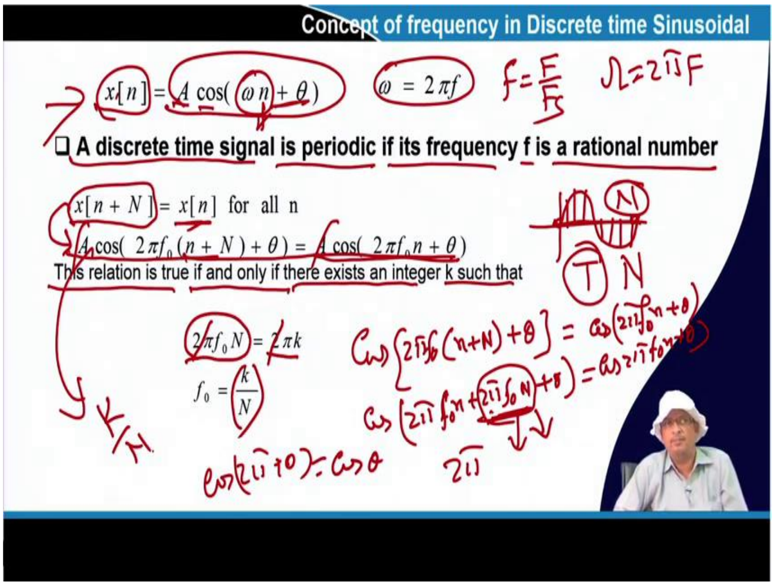

So, when I say discrete frequency, x[n] discrete signal. So, discrete sinusoidal signal. So, I said x[n] = Acos(ωn+θ), where A is the amplitude, ω is nothing but a 2πf. And this ω is nothing but a radian per sample. Is it clear? So, this small f is nothing but a F divided by Fs. So, it is a ratio ok.

And when I write capital ω, it is nothing but a 2πF. Now, what is the concept of frequency in discrete sinusoidal signals? It said a discrete-time signal is periodic if the frequency f is a rational number. So, if it is periodic, f must be a rational number. So, how do I prove it? Let us say what a period in a digital signal is.

So, if I say a sinusoidal signal, I say that two points to this point is a period. In terms of samples, let's say there is an N number of samples, and T is time. Now, time is converted to index in the discrete signal. So, let us say there are N samples within this period. So, instead of period T, I write capital N, which is the number of samples after which signals repeat themselves.

So, if N is the period, then x[n+N] = x[n], then only N is a period because the signal has to repeat itself for all n. N ow if it is that if I put that, if you change the index n to n plus capital N, it should be equal to A cos $2\pi f_0$ n+ θ.

So, from this equation, I put the value of this one, and I get this one. Now, when will this equation be true? So, this relation is true if and only if there exists an integer k such that because this is nothing but A. So, I have to prove

$$A\cos(2\pi f_0 (n+N) + \theta) = A\cos(2\pi f_0 n + \theta)$$

When is it possible?

Now, when will this be true? When this part must be equal to the integer multiple of 2π. You know that cos (2π+θ) = cos θ. So, only when $2\pi f_0$ N should equal the integer multiple of 2π. So, I can say $2\pi f_0 N = 2\pi * k$, and k is an integer. It may be 2π, it may be 4π, it may be 6π does not matter. So, in that case, I can say 2π, 2π cancel. So, f_0 is equal to k/N, What is k? k is an integer? What is N? N is also an integer.

Because that is the period index of that period. So, integer by integer is nothing but a rational number. So, I said a discrete time sinusoidal signal is periodic if f is a rational number, clear.

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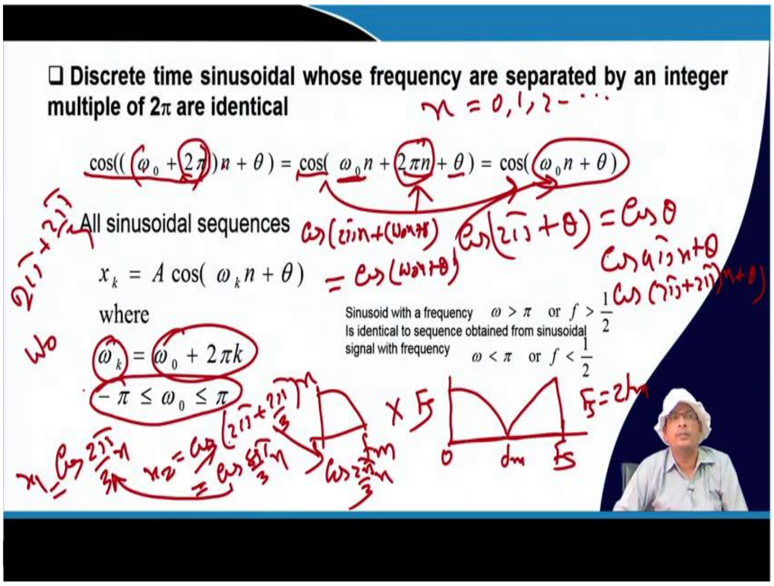

The next one is a discrete-time sinusoidal whose frequencies are identical and separated by an integer multiple of 2π. Before that, let me give you the back concept in the sampling; what is the concept? Suppose I have a baseband signal like fm, this kind of structure. If I sample this signal with sampling frequency Fs. So, what is the frequency response we have drawn for the frequency?

So, this is Fs. So, this is fm, and this is fm. This is 0 to fm and this is fm to Fs. So, that is why Fs is equal to twice fm. If it is not, there will be the aliasing effect that we have discussed during the sampling. So, I said a discrete time sinusoidal whose frequencies are separated by integer multiple 2π are identical. Why? Because I know $\cos(2\pi+\theta) = \cos\theta$. So, whatever is written so for $\cos(4\pi n + \theta)$ that can be written as $\cos((2\pi + 2\pi)n + \theta)$.

So, I can say if it is cos 2π separated by 2π, then the cos signal will be again returned. So, how do I prove it? Let us say $\cos(\omega_0 + 2\pi)$. So, this signal is 2π is separation is 2π. So, I added the separation 2π here. So, I changed the ω_0 and added 2π. So, $\cos((\omega_0 + 2\pi)n + \theta)$. so

$$\cos((\omega_0 + 2\pi)n + \theta) = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta)$$

n is an integer, and n varies from 0, 1, 2, 3 dot dot dot number of sample.

So, this product is nothing but an integer multiple of 2π. So, I know if the cos signal is separated by an integer multiple of 2π, the cos signal again comes back. So, this is nothing, but again cos ($\omega_0 n + \theta$). Because I can write down this one cos ($\omega_0 n + 2\pi n + \theta$), which equals nothing but a cos ($\omega_0 n + \theta$).

So, a sinusoidal whose frequency is separated by 2π, ω value is separated by 2π. So, I have $\omega 1$ and $\omega 2$. If the difference is 2π, then they are identical signals. So, I give an example. Let us say

$$x1 = \cos(\frac{2\pi}{3})n,$$

$$x2 = \cos(2\pi + \left(\frac{2\pi}{3}\right))n = \cos(\frac{8\pi}{3})n$$

So, this signal and this signal are identical because they can be written like this. So, if it is written like this, then cos ($2\pi + \theta$) is nothing but a $\cos\theta$, again $\cos(\frac{2\pi}{3})n$. So, I am saying if signals are separated by 2π, the signal will repeat itself ok. So, if the signal is separated by 2π, then the integer multiple of 2π, then the signal will repeat itself. Then what is the variation of the ω_0? ω_0 varies from $-\pi$ to $+\pi$, how? How is it possible?

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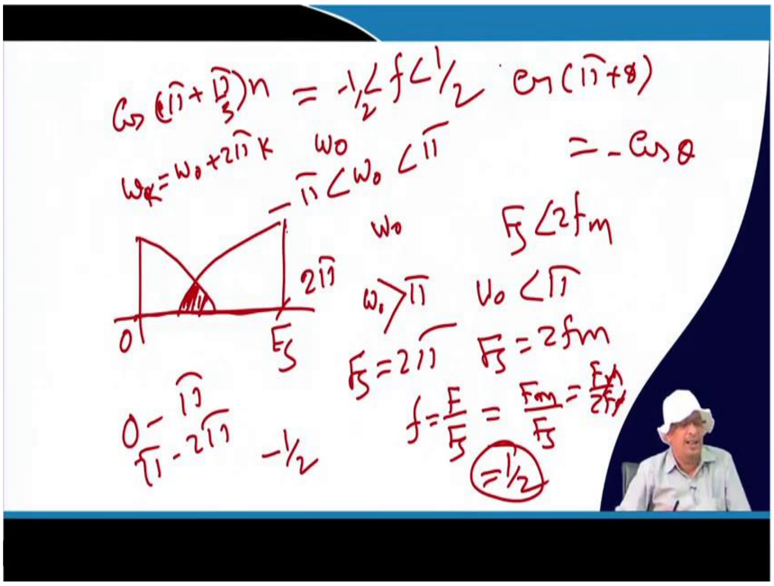

If I told you, let say cos ($\pi + (\pi/3)$)n is equal to how much?. So, cos $\pi + \theta$ is equal to how much? - $\cos\theta$. So, I am saying that ω_0, if it is separated by an integer multiple of ωk, is equal to $\omega_0 + 2\pi k$, then it is ω_0. The signal will be repeated back, but what is the variation of the ω_0? ω_0 varies from $-\pi$ to 2π, $-\pi$ to $+\pi$. How is it possible?

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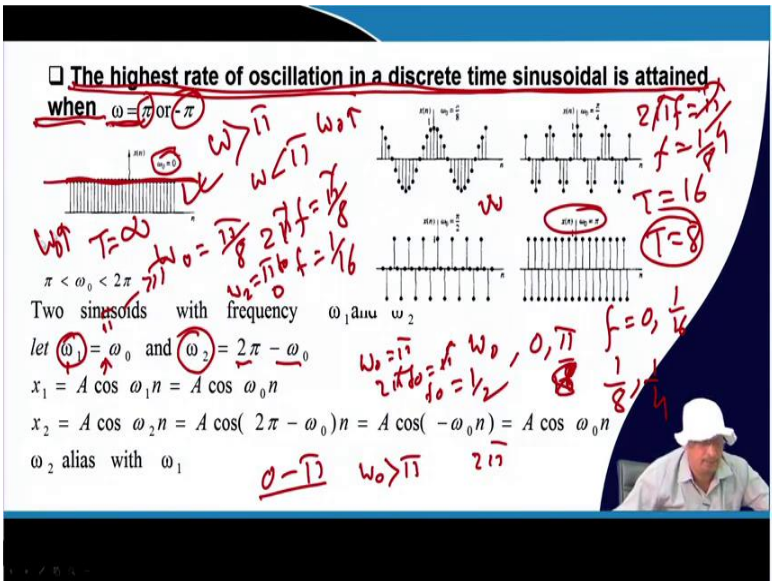

So, the highest rate of oscillation in a discrete-time sinusoidal is attained when ω is equal to π or $-\pi$. I have to prove it. So, if we remember the concept of sampling, there is an aliasing. So, what I said? What is aliasing? So, this is 0, this is my fm; when Fs is less than twice fm, less than this was Fs, then there will be aliasing overlapping, aliasing that is called aliasing. So, if the ω0 value is greater than π. Let us say ω1 ω0 is greater than π, which will be aliased with ω0, which is less than π.

So, why is it possible? As you know, the highest rate of oscillation is 2π because, after 2π, the signal will repeat itself separated by 2π. So, 2π is nothing but an Fs. So, I said Fs is equal to 2π. According to the sampling theory, the Fs must equal twice the fm. So, without aliasing, what should be the discrete frequency normalized discrete frequency f is equal to analog frequency divided by Fs.

So, I can say Fm divided by Fs which is nothing but a Fm divided by twice Fm. So, it is nothing but a half. So, if I consider from 0 to π, it is half; if I consider π to 2π. So, 0 to π and π to 2π, if I consider so, that means it will start from 0 to -π. So, max, then it will come at minus half. So, ω0 varies between -π to π, or I can say small f varies from minus half to half ok.

So, now let us say that it says the highest rate of oscillation in a discrete-time sinusoidal is attained when ω is equal to. So, I said the highest rate of oscillation would only be attained when ω is equal to π, or ω is equal to -π. If the ω is greater than π, then it will be aliased with ω less than π. How would It be possible? So, at ω = 0, there is no oscillation then which means it is a uniform signal. Then let's say ω is equal to π/ 8.

So, what is the period? So, ω is 2πf = π/8. So, I can say the f is equal to 1/16. So, T =16, 16 samples in one period. Now if ω = π/4, then 2πf = π/4. So π, π cancel f =1/8, so T = 8. So, the value of ω_0 can be 0, π/8, π/16, I can say the value of small f is 0, 1/16, 1/8, and if it is π/ 2, it will be 1/4 like that or T will be 16, 8, 4.

So, if I say if I increase ω, ω0 increases within π 0 to π, then the rate of oscillation is increasing because T is decreasing. So, the rate of oscillation is increasing; when ω0 is equal to 0, the T is equal to infinite because t is nothing but a 1 by f, equal to 0. So, 1 by 0 is infinite. So, it is an infinite period.

So, when the ω is increasing from 0 to π the oscillation rate of oscillation is increasing. What is the highest rate of oscillation when it is possible? So, when ω0 is equal toπ ω0 is equal toπ this case, ω0 is equal toπ; that means 2πf 0 is equal to π. So, I can say the f 0 is equal to half.

Let us say I am increasing further; I am increasing ω0 further. So, I am increasing 0 to π is ok. Let us say ω0 is greater than π; that means, let us say I take a sinusoidal signal. So, ω ω1, so ω0 is greater than π and less than 2π. So, I am increasing the ω0, not 0 to π from π to 2π.

So, I am increasing ω0 from π to 2π ok. So, I am taking a signal. So, ω1 is nothing but ω0. So, ω0 varies, taking from π to 2π. Let us say ω2 is another frequency of 2π minus ω0. So, I can say that ω2 varies from π to 0 when ω0 is equal to π, that is π, when ω0 is equal to 2π ω2 will be 0. I am saying I am taking ω2. Let us take another slide.

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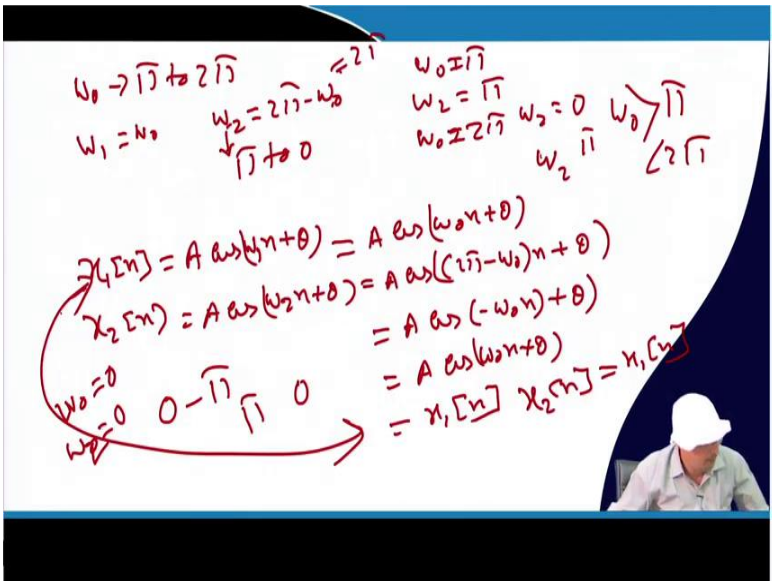

So, what I am saying is that I am saying ω0 varies from 2π. I take ω1 as a frequency that is equal to ω0, and I take ω2 as nothing but a 2π-ω_0. Because after 2π + ω_0 will be ω_0, that will be this thing that we already proved. So, within 2π. So, it varies from 2π - ω_0. So, when ω_0 = π, then ω2 = π. When ω_0 is equal to 2π, then ω2 is equal to 0. So, I can say ω2 varies from π to 0.

Now, if I write x 1 n equals A cos ω1 n let us say+ θ θ is you can write or not write. So, it is nothing but A cos (ω_0n+ θ). Because ω1 is equal to ω0. Now I say x2 is another signal: A cos (ω2 n+ θ). So, I can say it is nothing but an A cos ((2π - ω_0)n+ θ), which is nothing but an A cos (- ω_0n+ θ), which is nothing but an A cos (ω_0n+ θ), which is nothing but a x 1 n.

So, I am saying if my ω2 varies from π to. So, if it is ω0, it is greater than π and less than 2π. So, when I increase ω0 above the π, I see that it again comes back to the x 1. So, I am saying x signal x 2 n and x 1 n are aliasing. They are overlapping; they are the same signal, and they are overlapping; because they are the same, they are not distinct signals.

So, I cannot say that ω0 can vary from above theπ because they are not distinct; they come back again within the x 1. So, the value of sinusoidal, which varies from 0 toπ oscillation, is increasing. Now, once I say that it is greater thanπ. So, when it isπ, then I know the oscillation is maximum; when it is 0, it again comes back to 0. So, as at ω0 is equal to 0 and ω0 is equal to that 2 ω2 when ω2 is equal to ω0 is equal to 2π then also ω2 is equal to 0.

So, the concept is that a signal oscillation is increasing up toπ, then it starts decreasing, and it will become again 0 when ω0 is equal to 2π. Now, if I can represent it in a graphically. Let us say this one. Again, I draw the same graph.

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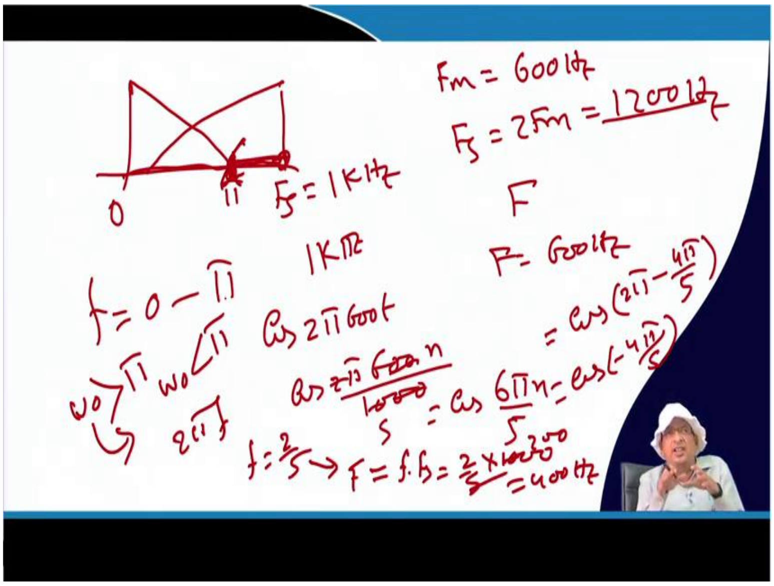

So, what I said is this is my 0, this is my Fs and if the signal is aliased. Let us say this is Fs is equal to 1 kHz. Let us say, my baseband signal is Fm is equal to, let us say, 600 Hz. So, what did the sampling frequency say? If Fs is equal to twice Fm, that means Hz, then there will be no aliasing.

So, I can say this is 0 to 600 Hz. Normalized frequency f varies from 0 to π because Fs is when it is π; that means, it is 600 Hz. If it is Fs is equal to 2 kHz,1.2 kHz, then this will be the value of π. Now, if you see this value, it will be aliased with this value. So, this value is 600 Hz.

So, if I sample it. So, if the signal is sampled by 1 kHz, what will happen? Let us consider cos 2π 600 t as my analog signal. That is, F is equal to 600 Hz. This is sampled with the sampling frequency 1 kHz. This is nothing but a cos 6π n by 5. Which I can write; how can I write?

Cos 2π minus how much? 6π by 5, I have to do. So, 2π minus 4π by 5. Then, it is nothing but a 6π by 5. So, that is nothing but a cos minus 4π by 5. So, what is the f? If I say cos 2πf, then my f will be 2 by 5, and my f will be 2 by 5, ok or not. So, what is the analogue frequency if I convert? So, f is equal to f into F s. So, it is 2 by 5 into 1 kHz 400 Hz.

So, instead of 600 Hz, I get 400 Hz up. So, this is due to the aliasing. So, this is nothing but 400 Hz from this point to this point. So, instead of 600 Hz, I get a 400 Hz signal because it is aliased. So, that is why we said if ω0 is greater than π, that will be aliased with ω0 less than π; is it clear? So, this is the concept of frequency in discrete time sinusoidal.

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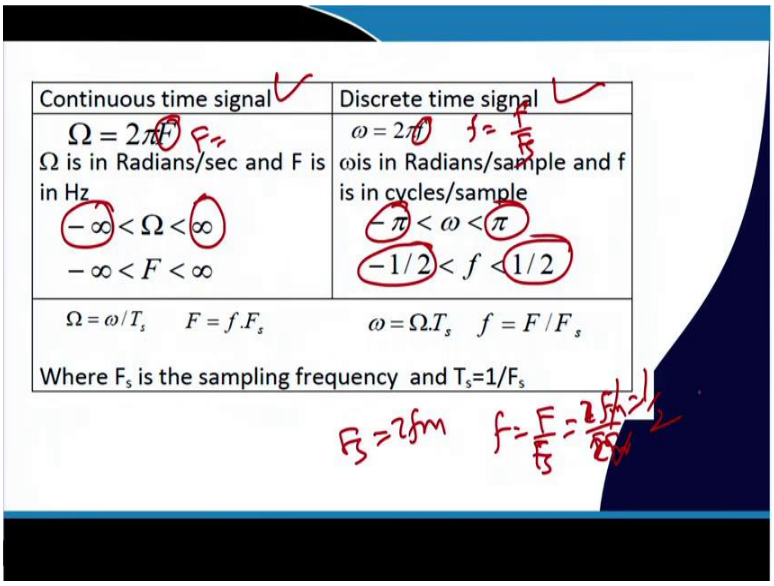

So, please remember this part: continuous-time signal, discrete time signal. Here, F is the analogue F in Hz; that means radian per second. Here is normalized discrete normal this normalized discrete frequency. This F is in radian per sample radian per sample. So, that F is nothing but a F divided by F s. So, the frequency can vary from minus infinity to infinity, but once it is sampled, it is restricted by to -π to π, or normalized frequency can vary from minus half to plus half because I know that F s is equal to twice f m.

So, I know f equals F by fm, F by Fs. So, it is nothing but a twice f m divided by F s sorry f m divided by twice f m, f m f m cancel it is half. Either it will be minus half, or it will be plus half, ok. So, please remember this part because we always use normalized discrete frequency in signal processing. So, even if it is given in Hz, you have to convert it into radians per sample.

So, this will be used in many ways; when you implement it, if you just consider the Hz, it will not come. You have to modify it to normalize discrete frequency. So, that concept will again take forward because this concept will be used in many places of signal processing.

Thank you.