Signal Processing Techniques and Its Applications
Dr. Shyamal Kumar Das Mandal
Advanced Technology Development Centre
Indian Institute of Technology, Kharagpur

Lecture - 56
Power Spectrum Estimation (Contd.)

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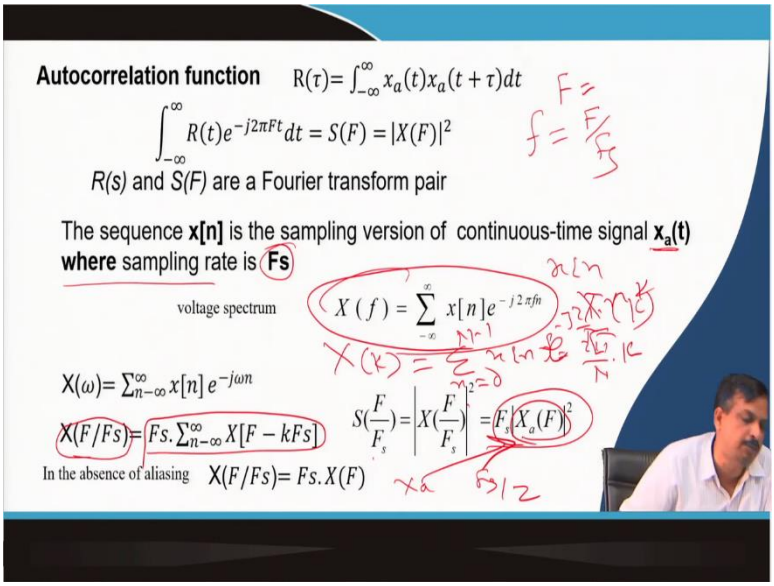

So, what I said is that I can calculate that spectrum. Let us say I have a sequence x[n], which is the sampling version of xa(t) ok, where the sampling rate is Fs. Now, what is the voltage spectrum? How do I calculate it? I calculate x[n], and I can calculate X(k). So, this is nothing but a voltage spectrum. So, this f is nothing but a twice π by you can say the N or Fs by N into k, f is called normalised discrete frequency.

Normalized discrete frequency ok. So, X(F) is nothing, but this is the; this is the, or I can say that like say this is X(k) instead of X(k) I say X(k).

$$X(k) = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}nk}$$

Now, what is small f? So, I have already given you the notation in lecture week number 1 that capital F represents analogue frequency.

Small f represents this normalized discrete frequency, which is nothing but an F by Fs. So, X(F) by Fs is nothing but an Fs into this one. So, the spectrum s F by Fs is equal to X(F) by Fs whole square. So, it is nothing but an Fs into Xa(F) whole square, understand or not. So, since it is restricted that X of a is restricted by Fs by 2, then in that case, it is nothing but an X of s into X(F) whole square. So, that is the spectrum of the signal voltage spectra.

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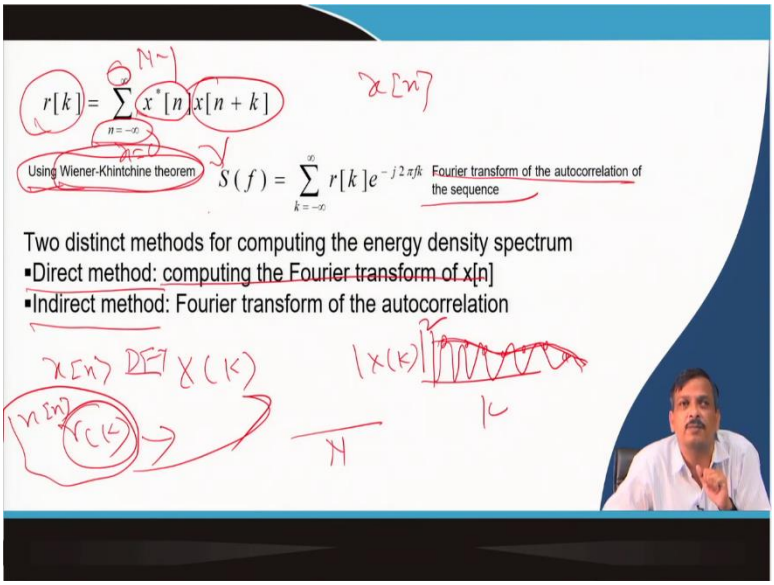

Now, what is autocorrelation? Let us say r k is the autocorrelation factor. So, I know if the x[n] is in finite duration, then it is minus n by infinity. If it is finite duration, then n is equal to 0 to N minus 1, which is nothing but an x star n multiplied by x[n+k]. This is autocorrelation. So, using the Wiener-Khinchin theorem, they said the same spectrum can be calculated from the Fourier transform of the autocorrelation sequence.

So, it is said that I can calculate the spectrum of a signal using 2 methods. One method is called the direct method, computing the Fourier transform of x[n]. So, the direct method is that I know x[n] I using discrete Fourier transform I compute X(k), and then I can draw the k versus mod of X(k) curve. So, if it is the energy power spectrum, it is a square. I know that this square, which I discussed.

The second one is called the indirect method, which uses the Wiener-Khinchin theorem, which states that the power spectrum will be the same as the Fourier transform of the autocorrelation sequence. So, I have to calculate the autocorrelation sequence r k of a given

signal x[n]. Step 1 and then step 2 is the Fourier transform of the autocorrelation coefficient, called the indirect method.

During LPC analysis, I also said that. So, if I calculate the spectrum, I will get this kind of value for every k, but there will be an overall variation that is called an envelope. So, who represents the envelope? The envelope is represented by the spectral or I can, the system parameters, which are nothing but the formant frequency. So, who is represented by the formant frequency, nothing but an LPC coefficient from LPC analysis?

So, the autocorrelation coefficient basically represents that envelope. So, I can also calculate the spectrum of a signal using the autocorrelation coefficient. So, for a deterministic signal, I can say I can use direct method x[n], I can take the discrete Fourier transform, I get that, but from a non-deterministic signal, random process signal.

So, what is the basic assumption when I say length Fourier length is N? We said the signal is periodic with length N, but a random signal is not periodic with length N deterministic signal may be. So, that is why we go for autocorrelation of the signal, and from that autocorrelation, we try to find an estimate of the signal's power spectral density.

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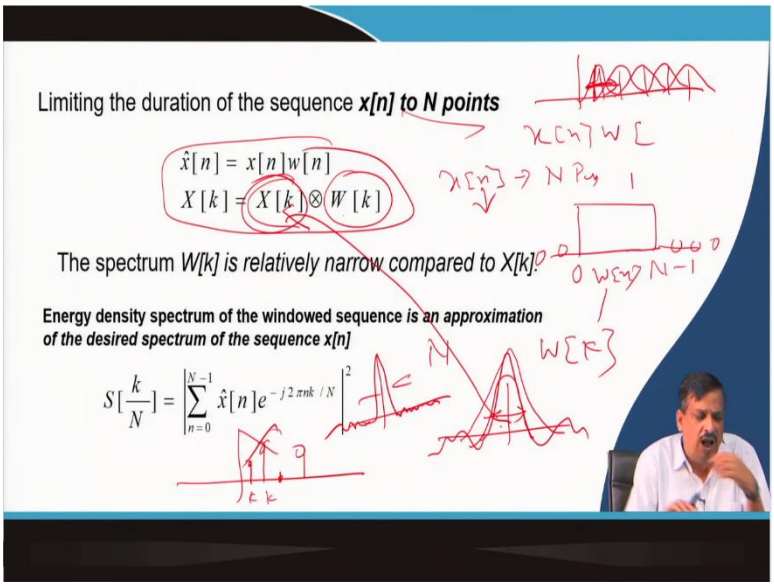

So, in that case. So, the duration of the sequence is limited to N point. So, I already explained that once I said that limiting that x[n] by a window function, the spectrum is nothing but a convolution of window frequency response multiplied by the signal

frequency response. Let us, for example, suppose I have an x[n], and I take an N point DFT. So, x[n] is restricted by the N point. How do I restrict it by multiplying a rectangular window? So, the rectangular window is 1 for 0 to N minus 1; elsewhere, it is 0.

So, it is nothing but a gate function. So, if I take that w n, this is w n, and what should be the frequency response of the W k, which is nothing but a sink function? So, you know the main lobe width depends on the window's length; if the window's length is very wide, then either the mainland or the main lobe will be very sharp. If the length of the window is very narrow, then the width of the main lobe will be very wide.

So, I can say. So, when it is convoluted, this response will be convoluted with the original signal. So, if I say the original signal has a frequency response like this one for k equal to 1, that is there; for k equal to 2, it is there; k equal to 3, it is 0; k equal to 4, it is 0 for every ki get a frequency response frequency value, let us I know it, but basically when I do the Fourier transform I will not get this one.

I will get every k, and for every k, I will get a Gaussian. I can. I will get this kind of Windows frequency Windows convolution if the main lobe width is less. So, the power of the particular frequency will be distributed with the nearby component because of the side lobe, and also, if the main lobe is very wide, then there will be an overlap in nature. So, that is called DFT leakage.

So, due to the DFT leakage, even the signal is deterministic, but once I make it a finite length signal, then the spectral estimation is nothing but the spectrum is nothing but the convolution of the window function and frequency response of the window function and the signal. So, I want that window response to be very neglected. So, it will be neglected when it is a narrow main lobe and a very small side lobe.

So, if the side lobe's power is very high, it also leaks into the other component. So, I want a window function whose frequency response main lobe is very, very narrow and also side lobe attenuation is very high; that means the side lobe does not disturb the next component of the signal, but getting that kind of window is very difficult. So, how do I get the narrow main lobe by increasing N? Once I increase the N, I know if the signal is non-stationary, and then my time resolution is lost.

So, I cannot increase the length of the signal many times. So basically, I am estimating the power spectrum; what the power spectrum sees is not an original spectrum of the signal; it is in the window and is also in there. So, how do I eliminate that window error due to the finite length or due to the window function that is introduced? That is why it is called estimation of the power spectrum. So, now, this is a case of a deterministic signal.

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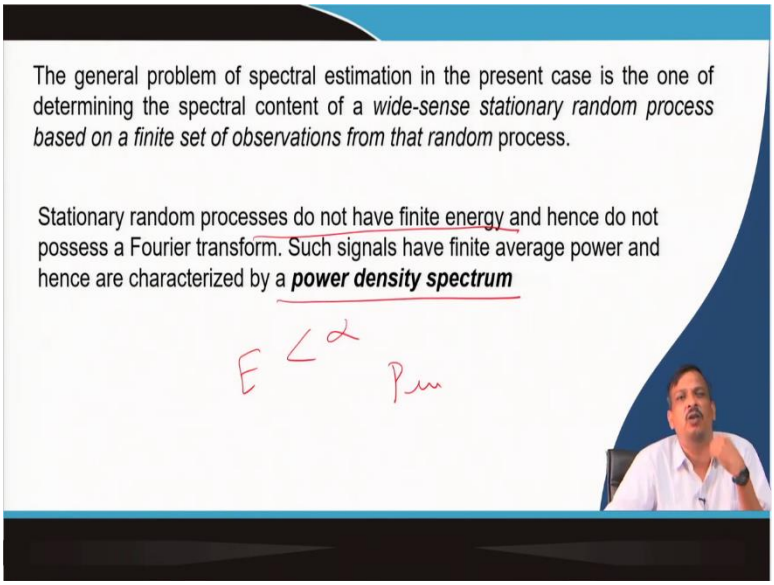

Now, let us discuss the general problem of spectral estimation wide sense stationary random process. So, wide stationary random processes do not have that finite energy. So, E is not finite. As I said, there are 2 kinds of signals: one is a power signal, and one is an energy signal. If the energy is finite, then we call the energy signal; if the power is finite, then we call the power signal. So, here, energy is not finite, but power is finite.

That is why I call it power density spectrum or power spectrum density. The energy spectrum is when the energy is infinite, and the power spectrum is when the power is finite.

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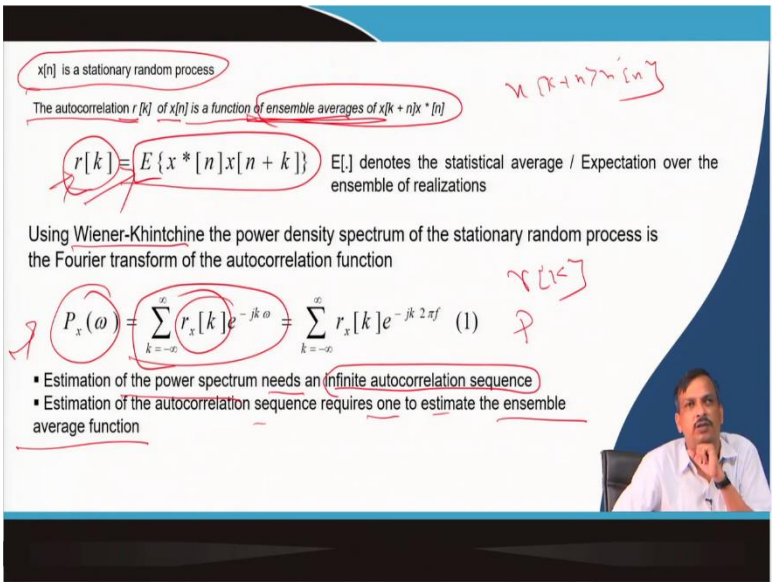

So, what is the mathematics of that? So, let us say x[n] is a random stationary random process signal digital signal. So, the autocorrelation x of r k of x[n] is a function of the ensemble average of x[n] k plus n and x[n]. So, I have been non-deterministic infinite energy. I said the power is finite, but I cannot take the infinite length of the signal; I have to take the finite length of the signal.

So, once I take the finite length, it automatically becomes finite energy, but a random process is infinite energy. So, what I require is an ensemble average. So, let us say autocorrelation represents the ensemble average of the random process signal. Then, using the Wiener-Khinchin theorem, I can say that the power spectrum is nothing but a Fourier transform of the autocorrelation function.

Now, if I want to do that then what I require. I require the estimation of the power spectrum, which needs a finite autocorrelation sequence. If my rk is infinite length, then how do I get the power spectrum because I cannot get that infinite length rk? Estimation of the autocorrelation sequence required one to estimate the ensemble average function. So, if this rk is the finite autocorrelation sequence, r represents the ensemble version of the signal, then only I can say the power spectrum is nothing but a Fourier transform of that autocorrelation sequence.

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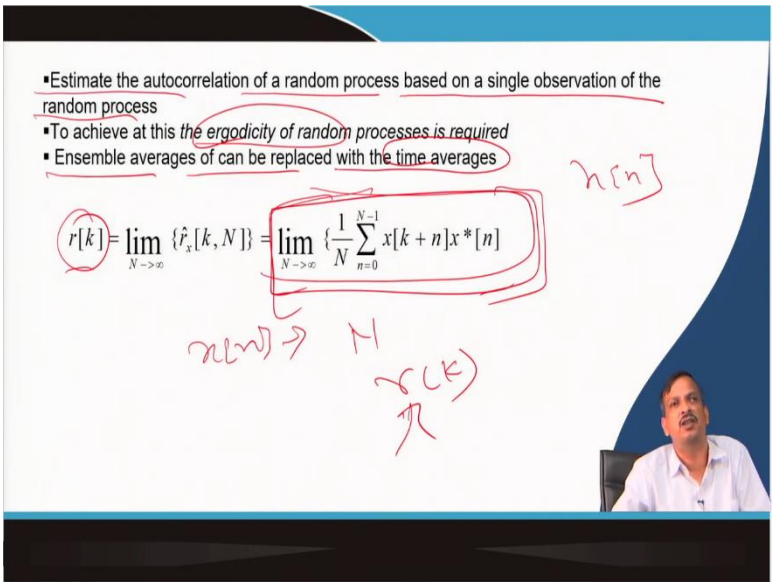

So, that is used for the power spectral estimation. So, what would we use? Estimate the autocorrelation of a random process based on a single observation of the random process. So, I have a single observation from there, and I have to estimate the autocorrelation. So, to achieve that, we use the ergodicity of the random process, which says that the time average can replace the ensemble average.

So, this is the time average x[n], which is a random signal process signal. From the x[n], I can calculate r k using the time average of the signal. So, the time average is along the length of the signal. So, x[n] is the length. So, I have evidence of random process observation of N number of samples. So, the ensemble average is replaced by the time average. Once I get r k, I can take the Fourier transform of this r k to get the power spectral density of the signal ok.

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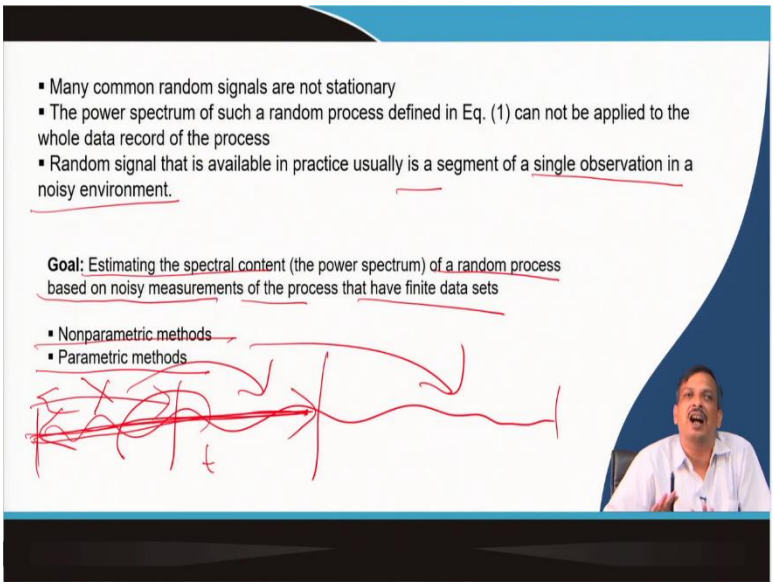

So, that process. So, many common random signals are not stationary. The main problem is not stationary. So, suppose I have a very long signal length. I take only this observation if this signal is non-stationary. So, this observation is not true for all the signal problems. The next problem is if the observation length is very high. So, what is the assumption of the Fourier transform?

So, I assume the signal is stationary, but if the time length, the length of the signal, is very high during that time, the signal also changes, and the colour of the properties or properties of the signal will be changed. So, in that case, that change of property will not reflect my spectral analysis. So, I have to consider that the signal is stationary during this timeline. So, I cannot take a large amount of signal.

That is why the finite length of the observation is very important. So, random signals that are available in practice usually segment a single observation in a noisy environment. So, from there, my goal is to estimate the spectral content of the random process based on noisy measurements of the process that has a finite set of data.

So, noise is a finite data set; I have to find the spectral estimation. There are 2 methods; one is called a non-parametric method, and one is called a parametric method. Non-parametric methods mean not designing any model; parametric methods mean I am designing a model, and then I want to estimate the power spectra.

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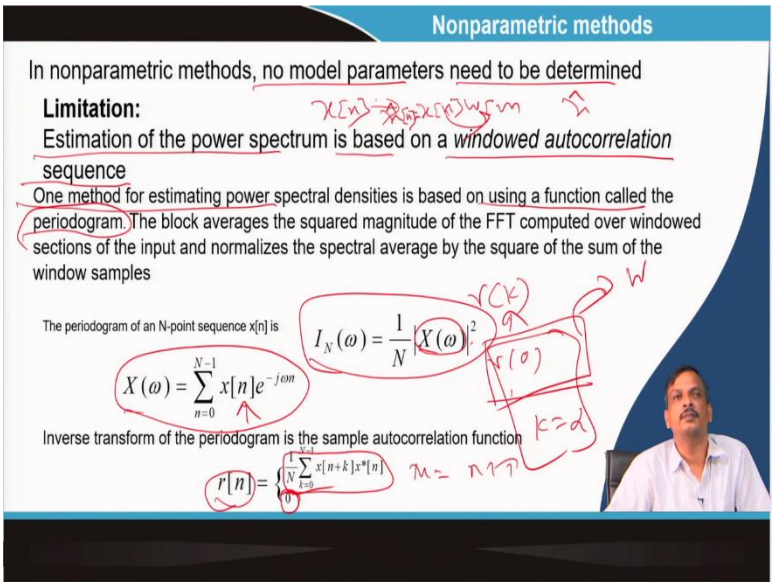

So, what are the non-parametric methods are there? So, I am using non-parametric methods, no model parameter, no model. So, what is the limitation estimation of the power spectrum based on the windowed autocorrelation sequence? So, what is the meaning? Meaning is that I have an x[n]. So, non-parametric, I am not modelling that x[n]. So, x[n] is there. I directly take evidence of x[n] using a window function. So, I was taking that measurement of x of finite length of x[n].

So, x cap n represents that finite version of finite length, or you can use a window function to use the sampled version of x[n]. So, windowed autocorrelation I am using means either I can take and then take the autocorrelation, or I can say, let us say I take the autocorrelation using ensemble time average, then I said r k has a k has an infinite value r 0, r 1 to r k equal to infinite.

From there, I am selecting only one portion of the rk value, the finite rk value, which is the windowed rk value. So, windowed autocorrelation means r[n] is nothing but a window version of N equal to 0 to N minus 1. It is there elsewhere. It is 0, and to estimate the power spectrum, we use a function called periodogram, which is defined this way.

$$I_N(\omega) = \frac{1}{N}|X(\omega)|^2$$

So, what is X(ω) is nothing but a Fourier transform of x[n] ok.

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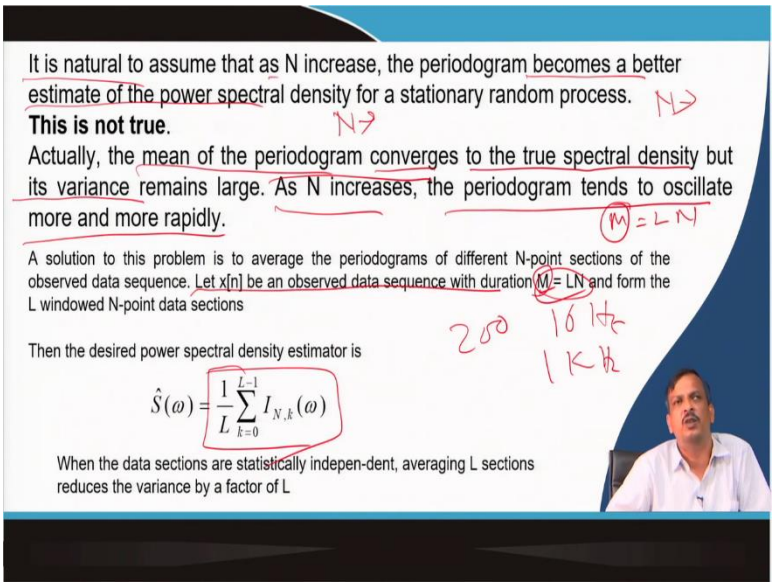

So that way, I can calculate the estimation estimate that periodogram methods for non-parametric estimation of the spectrum, and it is natural to assume that if the N length of the signal is increased, then what you said. The resolution frequency resolution is increased. So, it gives me the ability to become a better estimator of the power spectral density.

So, instead of, let us say, 200 hertz instead of 16-hertz resolution, if I have a 1-hertz resolution, then you can say I get every 1 hertz. So, the estimation is better, but it is not true. Actually, the mean of the periodogram averages to the true spectral density, but the variance remains large. So, I can say if I increase N, the mean converges, but the variance remains very large.

So, if N increases, the periodogram tends to oscillate more rapidly. So, what is the solution? They said they should forget about that increase of N if you have a large number of data to make it average. So, how do I make it average? Let us say x[n] has an observation data sequence M. M is equal to L into N; that means I have data which can be divided by L number of the windows of length N. So, the L number of window length is the N point data sequence.

Then, for every N point data sequence, I estimate the power spectra I N k, and then I take the average for all the windows. So, taking all the data at a time means that N is very large, which means the frequency resolution is very high, but that does not solve my problem. It

will give me the truth, but there will be some variance. So, to avoid that, if you have large data, then you can divide it into N number of windows and L number of windows.

So, suppose I have measured a long 2-second signal. I said within the 2-second signal, I divided two 200 windows, and for every window, I analysed, estimated the spectrum and then took the average, which gave me a better estimation of the signal's power spectrum using a non-parametric approach.

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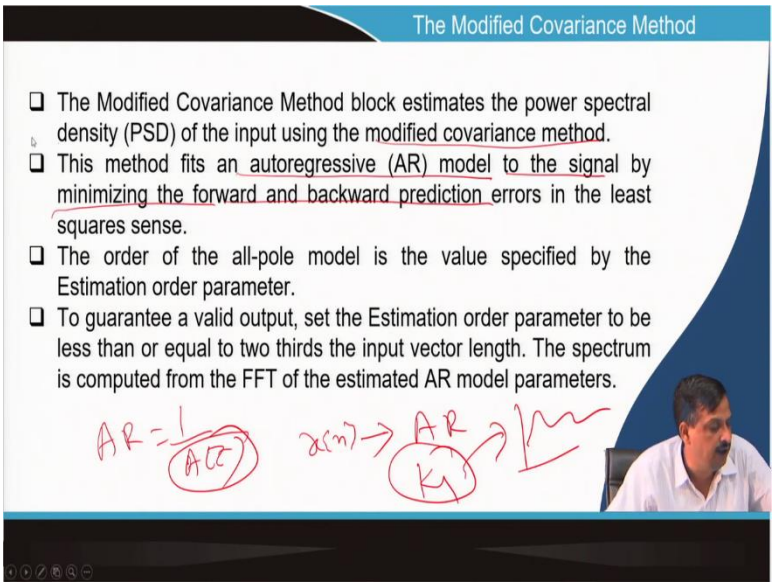

There is a parametric approach to the modified covariance method. So, the modified covariance method blocks the estimation of an input's power spectra density PSD using the modified covariance method. What is this method? This method fits an auto-regressive model to the signal by minimising the forward and backward prediction error in the least square sense. What is the meaning? I have already discussed it during the AR mode linear prediction.

AR model means only poles are there. A(z) all-pole model is an autoregressive model. So, that has a backward prediction and a forward prediction. So, I am saying that whatever the signal you get from the signal, you are estimating an AR model, which means you are estimating A(z). So, you are calculating that k value, ki value, and partial reflection coefficient value. So, it is an all-pole model. The value specifies the estimate. So, all ki basically specify the pole position of the model.

Now, from that k i, you estimate the power spectrum once I get the A R model parameter. So, basically, what k. So, if I say, suppose I have an; I have a signal x[n], ok, I will take a slide here, then I will explain it.

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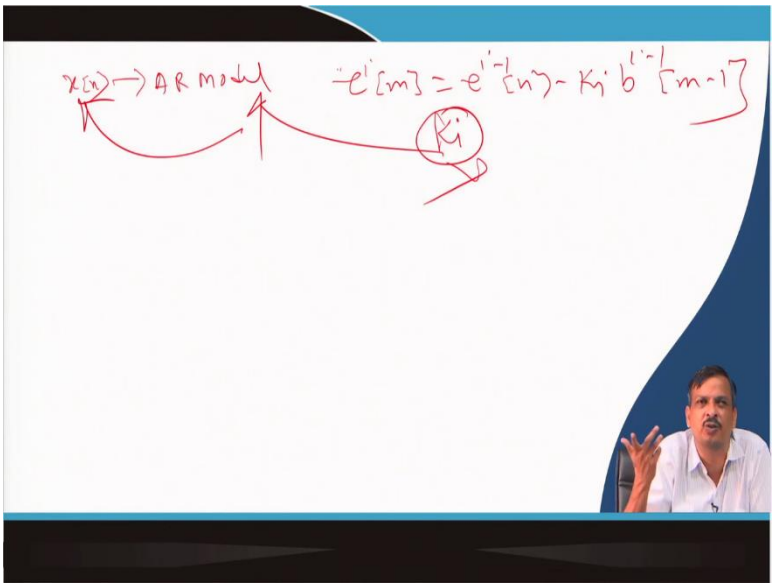

Let us know if I have a signal x[n]. So, I have a signal. So, what do I do is make an AR model, which is an all-pole model using linear prediction, autoregressive analysis, and millionaire prediction. So I can calculate the forward prediction and backward protection. So, $e^{i}(m)$, you know that $e^{i}(m)$ is equal to $e^{i-1}(m)$, m minus ki into b I minus 1 m minus 1 forward prediction error backward prediction error; from there, I can calculate the ki value so, if I take the Fourier transform of all the k.

So, ki is basically the model parameter ki represents the signal, and if I take the Fourier transform of the ki parameters, then I get the spectrum of the signal. So, that is called the parametric approach. So, I converted the signal. So, as if I am saying, I am estimating the system which produces that signal, and then I estimating the pole from the pole position of the system by estimating the spectrum of the signal system. So, that spectrum basically represents the signal. So, I get the signal spectral estimation.

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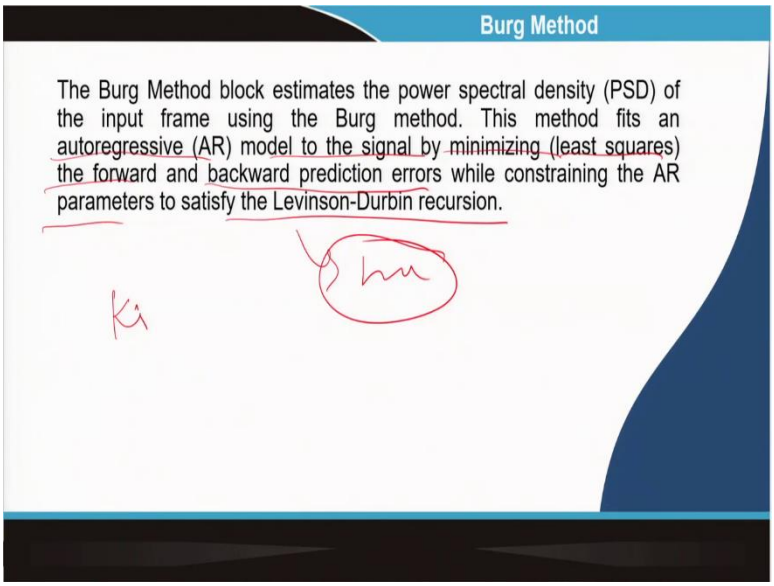

There is another method called the Burg method. So, I have already discussed the Burg method of calculating ki during the linear prediction calculation of the ki value using the Burg method. So, what is that estimate of the power spectral density of an input frame using a Burg method? This method fits an auto-regressive model to the signal by minimising the least square forward and backward prediction error. So, instead of k forward k backward geometric mean, they use the least square where AR parameters satisfy the Levinson Durbin recursion, which I discussed during your linear prediction analysis.

So, I will stop here with spectral estimation because if I continue, that will be a very long topic. So, again, it can take another 2 to 3 weeks time to complete the entire spectral estimation. So, I do not want to load that course so high. So, maybe I will offer another signal processing course. So, I can explain the details of this kind of spectral estimation, and they are using all kinds of things, ok. Thank you.

Thank you very much.