Signal Processing Techniques and its Applications
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Lecture - 52
Fractional Rate Conversion

Now, I come to sampling or interpolation. So, what is upsampling?

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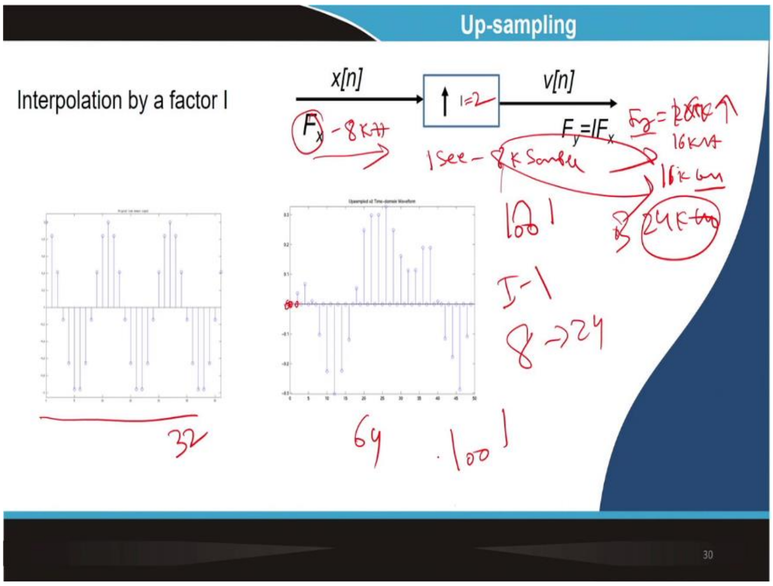

.So, upsampling means that Fx, let us say Fx is equal to 8-kilo hertz. I want to make Fy equal to I is equal to 2, then Fy will be 2 into 8-kilo hertz; that means 16 kilohertz. So, that factor I is equal to 2. So, let us say I have a signal Fx is equal to 8 kilohertz. So, within 1 second of the signal, there is an 8 k sample.

Now, once I make it 2, Fy equals 16 kilohertz. So, within 1 second, I have a 16 k sample, 16 kilo sample. So, I have to double that number of samples. In the same example, there is a 32 sample. So, if I want to make it double frequency, the number of samples should be 64. So, what will we do? In between 2 samples, I put a 0.

Now, if I want to make it 3 times, then, in between 2 samples, I have to put two 0. So, a number of 0 will be I minus 1. If it is factor 2, then 1 sample one 0. Suppose it is factor 3, then two 0. If it is factor 4, then three 0. I have to put three 0s because factor 2 means I have to double the number of samples. Factor 3 means I have to 3 times that; when it is 3, then it will be 8 into 3 24 k samples in 1 second.

So, I have to make it 24K. So, 8k to 24k. So, in between 2 samples, I have to put two 0 OKs. So, if I minus 1 number of 0, I can put 0. So, there is a sample value, and there is a sample value I put 0. So, what I am doing is interpolating the curve.

So, upsampling is nothing, but an interpolation down sampling is nothing but a discard. I have discarded the sample. So, dissemination or this disseminated the sample and discarded the sample. Here, I am inserting the sample. So, that is why it is called interpolation ok.

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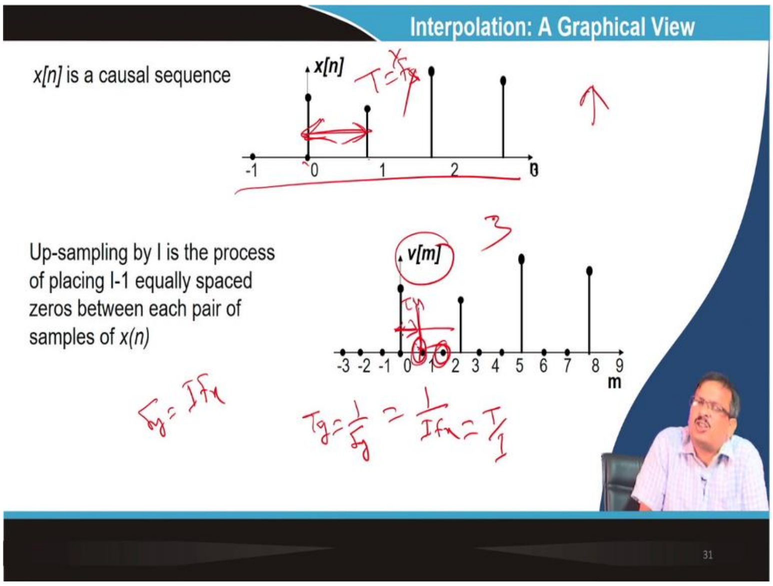

So, I know that up-sampling means interpolation. So, if the x[n] is the sequence, then the v[m] is the up sample by 2, 3 times, then I know in between 2 samples, I have to put to another sample. Because I know the distance between the 2 samples here, T is equal to 1 by Fx.

But here, the distance between the 2 sample this is also a sample is also a sample. So, this is my T y. So, Ty is equal to 1 by Fy. I know the Fy is equal to I into Fx. So, I put 1 by I into Fx. So, I can say Ty is equal to T by I. So, I have divided the duration, but factor I.

So, that is why you know that if the sampling frequency increases, then the distance between the 2 samples decreases, which is why it decreases by I. That is why I got a larger sample.

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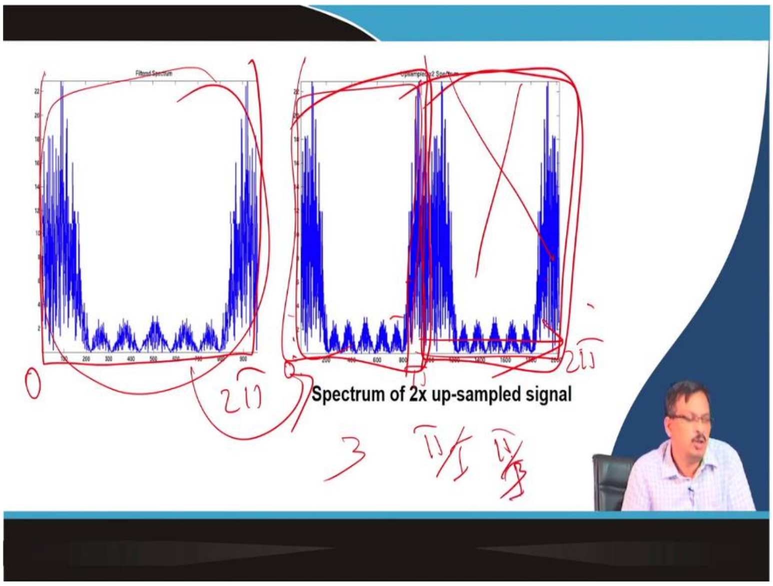

So, if I say that, what will happen in the frequency domain? Now, if this is my, let us say, spectrum of a signal. So, this is 0, and this is 2π. Let us say when I up the sample by factor 2, and I get 2 mirror images. This is 0, and this is 2π.

So, a number of mirror images will be. So, if it is, if it is factor 3, I get 3 mirror images using 0 to 2π. If it is factor 4, I get 4 mirror images between 0 to 2π.

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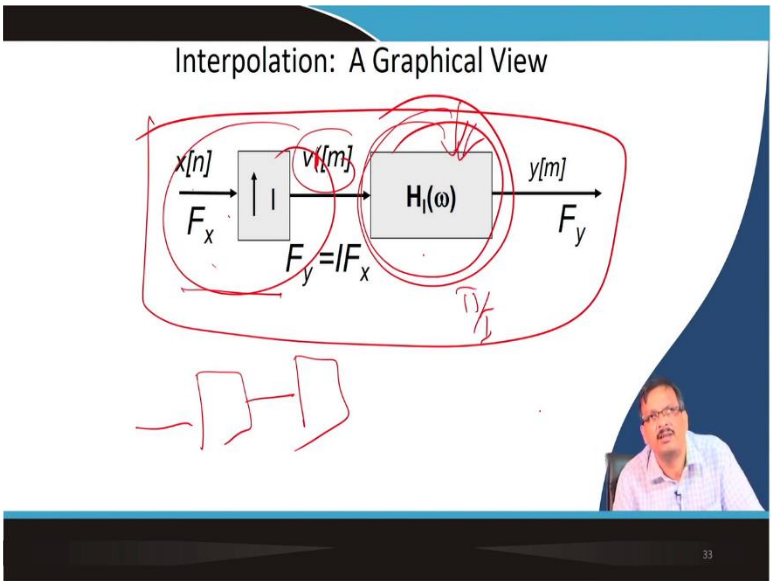

So, I can say I have to analyze in the frequency domain and what? I have to restrict it. So, that means I have a lot of images I have given. I am getting 2 images. So, when I say the 2 images, I am getting, but what do I want? I want a single image.

So, somehow, after sampling, I have to remove these multiple images. So, what do we have to do after half-sampling? I have to pass through a low pass filter again. Because 0 to π I am interested, not this portion I am interested.

So, again, I required a filter low pass filter whose cut-off frequency is π by I. Because it is 2 times, it is π by I π by 2. If it is 3 times, that will be π by 3. So, that means π by I. So, after that sampling, I required a filter whose cut-off frequency is π by I. So, I discarded all the other images and only kept the single image. So, that is an up-sampling diagram, okay?

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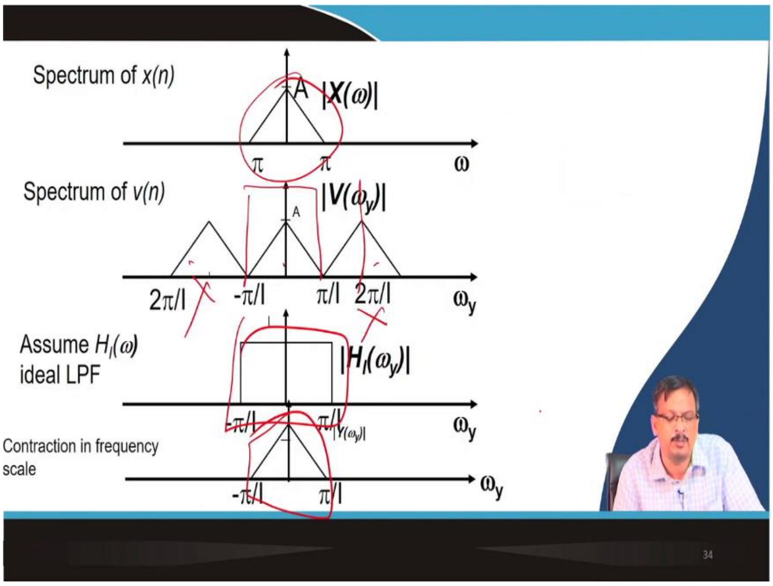

Now, let us say what will be given. So, the spectrum of x[n] is equal to this one. So, the spectrum of v[m] will be multiple. So, this line will be, you know, v[m] and will be a multiple of x[n]. So, a mirror image of x[n] will be there. So, I only want this portion. So, I designed a low pass filter and kept this portion's rest discarded.

So, that is my requirement. That is why upsampling is also followed by a low pass filter. In the case of downsampling, first low pass filtering then downsampling. In the case of upsampling, after upsampling, I have to reduce that mirror image. That is why I have to put a low pass filter to get the signal ok. So, that is the up-sampling diagram.

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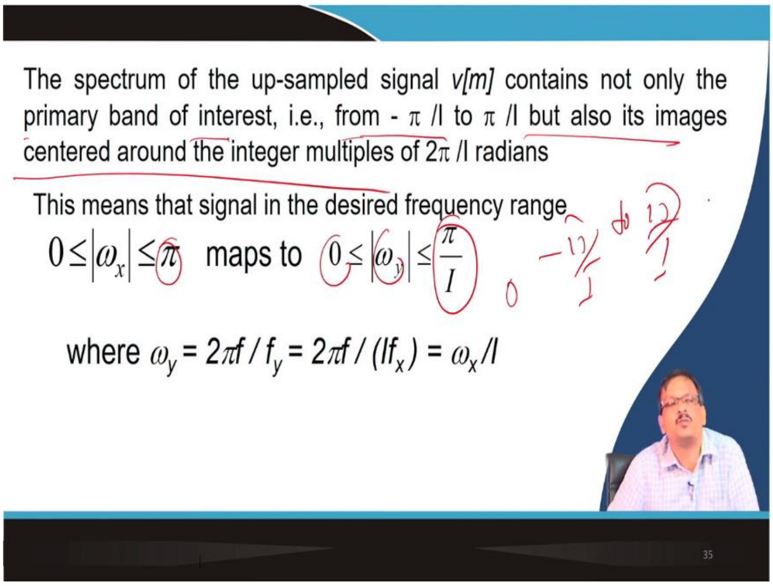

Now, the spectrum of the up-sample signal contains not only the primary band of interest but also the image. So, I have to reduce those images by a low pass filter 0 to the mod of x is equal to π 0 to the mod of y equal to π by n; that means minus π by I 2π by I ok.

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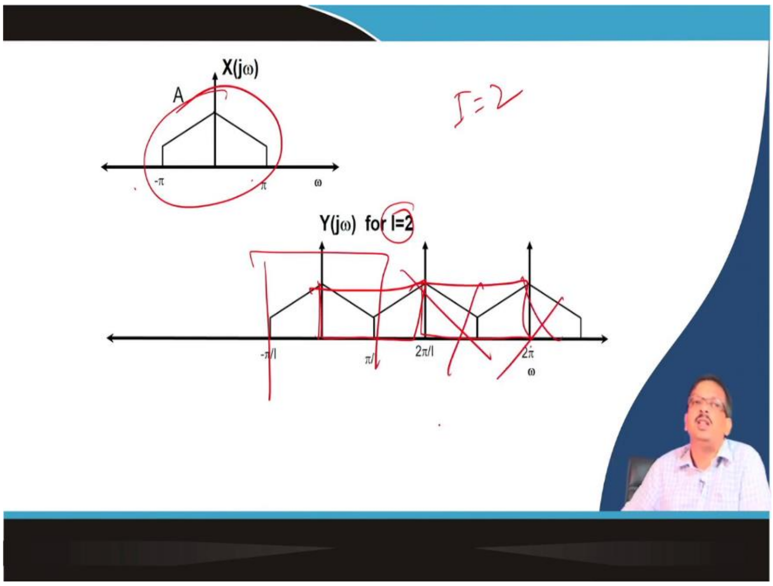

Now, let us say here let us say this is my j ωx minus π to π. So, after I equal to 2, I will get 2 that, what I said? If I equal to 2, I get 2 mirror images.

So, this to this is 1, and this to this is another one. So, within 2π, I get 2 mirror images. So, instead of that, I want only a single one. So, I will keep this portion, and this portion will be discarded using a low pass filter.

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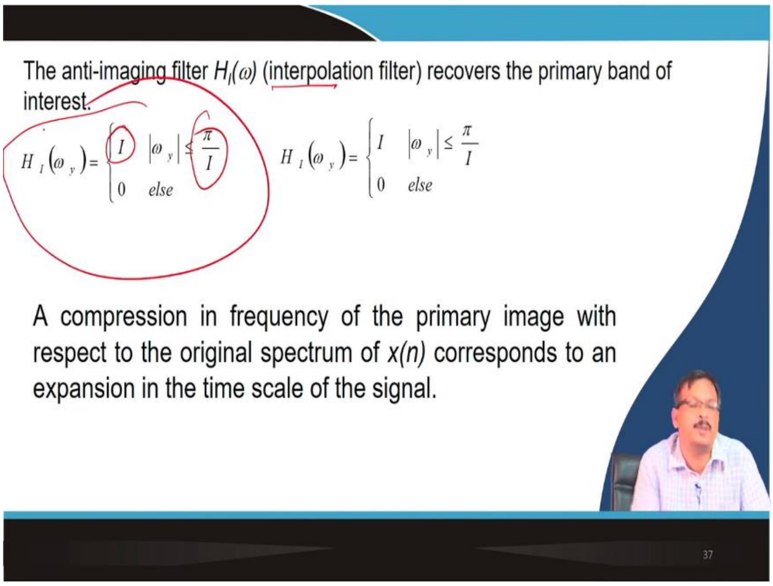

So, again, it is called an anti-aliasing filter or interpolation filter to recover the primary band of interest instead of all other bands being discarded. So, I know it is equal to I. So, it is I. So, this is equal to π by I. So, how is this coming?

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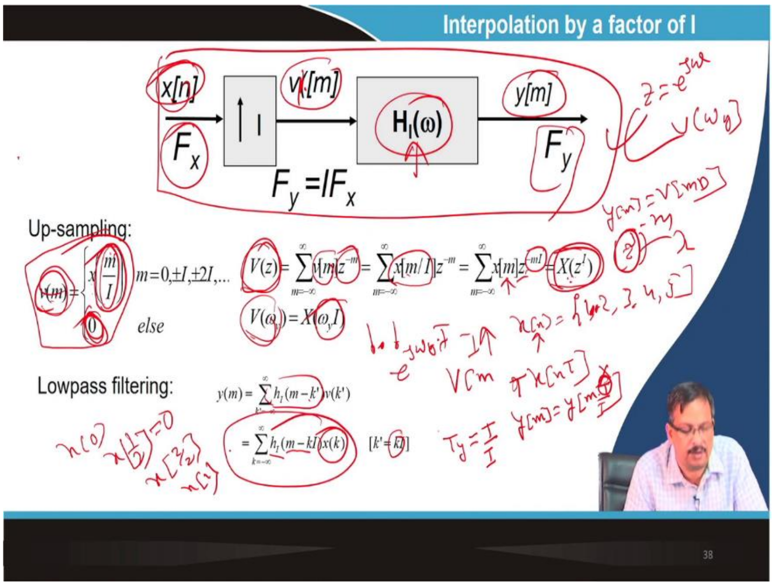

Let us say. So, I know that my interpolation filter in the upsampling block diagram looks like this x[n] Fx is the input sampling frequency v[m] and then, H(ω)and y[m] Fy. This is my upsampling block diagram.

So, x[n] is the input. So, I know v[m] is nothing but a x[m] by I. Because if x[n] is equal to 1, 2, 3, 4, 5, then I know if d is, I is the upsampling factor, then I have to know I minus 1 number of the samples I have to insert in between that. That means that you know that in the case of x[n], the distance between the 2 samples is T. So, it is nothing, but I can say that when I write x[n], it is. Basically, it is x[n] into T. Now, my Ty is nothing but a T by I.

So, my y[m], when I say it is nothing, but a y[m] into T by I. So, t I am not writing; when we write discrete signals, we do not write this T. But I know that the index is divided by I. So, I know v[m] is nothing but a x[m] by I. So, m equals 0; I know it is x[0] then x[1], I cannot get, but 1 by let us say I equal to 2.

So, can I get it in x[1/2]? That is why x[1/2] is nothing but a 0. Then I get the sample at x[2] by 2, which is nothing but a x[1]. So, that is why I said in between 2 samples, I put a 0. So, how do I put a 0? Using this function.

But in the case of downsampling, we said y[m] is equal to v[m] into d. We have discarded that; we have not taken it up. I am not discarding; I am putting 0 between 2 samples. So, if I say I equal to 3. So, I am putting two 0, I equal to 3. So, what will happen? So, if I say I equal to 3, I will take another slide here.

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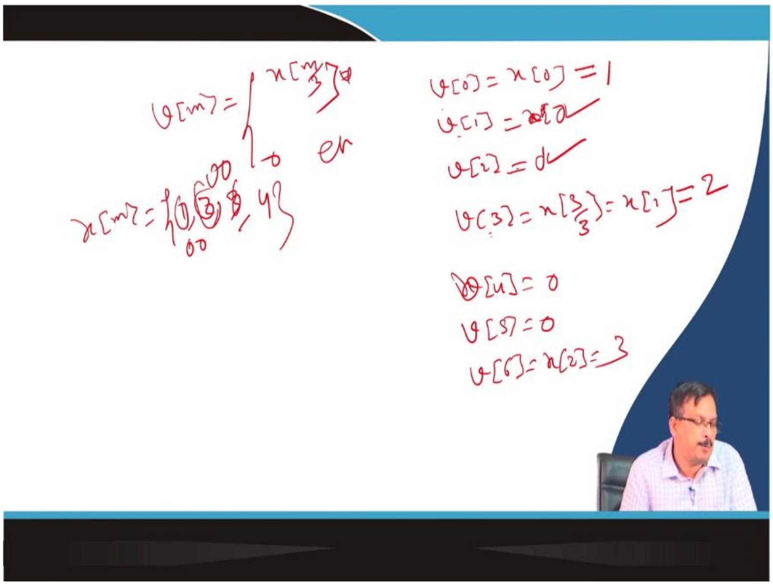

So, when I say I equal to 3. So, I said, v[m] is equal to x[m] by 3 else 0. So, that means I am selecting v[0] is equal to x[0], v[1] is equal to 3 is equal to x[1] by 3. So, this is not else. So, this is 0.

Then, v[2] 2 by x[2] by 3 0 v[3] x[3] by 3 is equal to x[1]. So, let us say I have a signal x[m] is equal to 1, 2, 3, 4. So, I can say, v[0] is equal to x[0] is equal to 1, v[1] is 0, v[2] is 0, v[3] is equal to 2. So, in between the sample 1 and 2, I put two 0 understand. Similarly, x[4] v[4] is 4 by 3 is 0, v[5] 5 by 3 0, v[6] 6 by 3, x[2] is equal to 3.

So, in between 3 and 3 samples, I put another two 0. So, upsampling means the meaning of the upsampling is v[m] sequence is generated from x[n] selecting the signal index, which is an integer multiple of I, and elsewhere it is 0. Now I can say, let us say, what is V(z)? V(z) = v[m]* z^{-m}.

So, what is v[m] x[m] by I elsewhere it is 0, z^{-m} into 0 is nothing, but a 0. So, x[m] by I into z^{-m}. So, again, if I change the index. So, it will be x[m], and this will be z^{-m} I. So, I can say this is nothing but an $X[z^I]$. Because z I whole^{-m}. So, I act as a z.

Now, x[z] I. Now, I can know the V(z). V(z) is equal to x instead of X[z]; it is $X[z^I]$. So, if I say z is equal to ω. So, V(ωy). So, I know z is equal to $e^{j\omega}$ ok. So, V(z) is V(ωy) is equal to x, x(ωx).

So, ωx into I. So, it is $e^{j\omega}$y into I. So, I said ωy into I z is equal to $e^{j\omega y}$. So, this side is ωy, which is z^I. So, it is ωy into I.

Now, this signal will be low pass filter by a filter. So, the filter impulse response is m k dash. So, k dash is equal to k again. It is nothing but an index changing k l. So, it is nothing but a h 1 m minus k l into x k. Because this is ω, this is $x[z^I]$ or I can say, v[m] is equal to x[m] by I ok.

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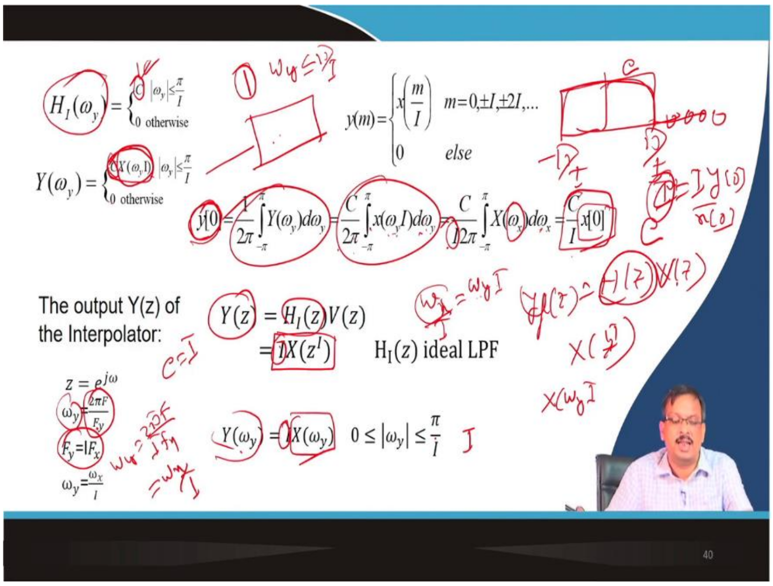

Now, I can say what the value of this constant is. So, when I say that, what is the output of this filter? So, what is the filter? So, HI(ω). So, I require a filter which is instead of 1 ωy less than equal to π by I equal to instead of 1; I want this to be C; C is a constant; otherwise, it will be 0. So, I can say I required a low pass filter, which will be C up to π by I. After that, it will be 0.

So, minus π by I to minus π by I to plus π by I, it will be C that I want. So, how do I get the value of the C? I know y y if you see that y y is nothing but a C into x of. So, what is y y? y y(z) I can say y(z) is nothing but a H(z) multiplied by x[z] v[z]. So, v[z] is nothing but an $X[z^I]$. So, I know that is nothing but an x[ωy] into I and H(z). I want a constant. So, H(z) is constant and x(ωi) I know.

Now, what is the initial value theorem? So, I can say y[0] power of the output. So, output power and input power will be the same. I have a sampling rate conversion system input power, but I cannot generate the power. So, power will be the same. So, I can say y[0]. So, d c power y[0] is nothing but a 1 by 2π summation of all the spectra. So, I summing all the spectra. So, it is nothing, but the C by 2π x of y i I by else are 0 into d y.

So, I can say that C by I, because you know d y. So, it is x(ωi) ωy into I. So, ωx is equal to ωy into I. Because you know ωy is equal to 2π f by Fy, Fy is equal to I into x. So, I know ωy is equal to 2π f by I into Fx. So, it is nothing but a ωx by I. So, I know ωy into I is equal to ωx and ωy into I; that means divided by I. So, ωx divided by I will become here. So, it is C by I x 0.

So, I know that the constant in this C is nothing but a C by I into x 0, which must be equal to I. So, for better, I can I can say that I is equal, or C is equal to I into y[0] divided by x 0. So, in normalized cases, the C must be equal to I. So, if the C is I, then HI(z) is nothing but an I. So, y(z) is equal to I into $X[z^I]$ or I can say the frequency response Y(ωy) is equal to I into x(ωy). So, is the interpolation gain I am called interpolation gain? Is it clear? So, it is called interpolation gain.

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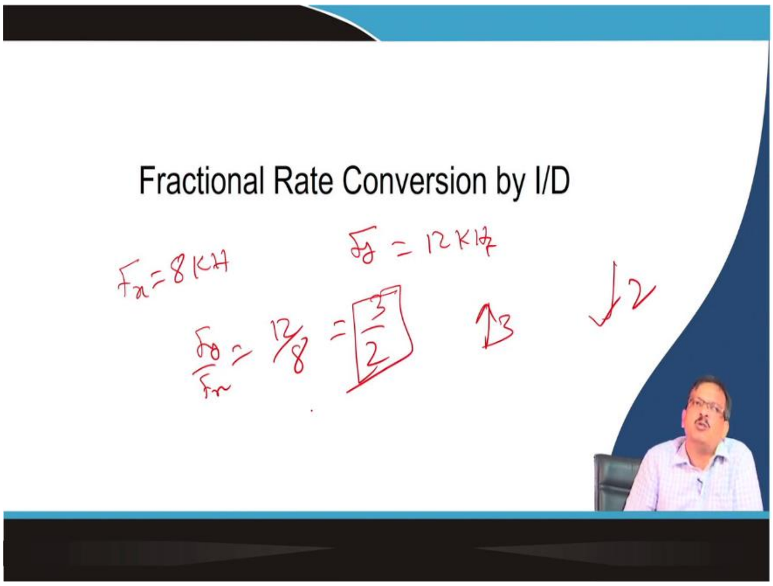

Now, I said up sampling down sampling I know. Now I said fractional rate conversion. That means, suppose I have an Fx is equal to 8 kilohertz I want Fy is equal to 12 kilo hertz. That means, I know that Fy by Fx is equal to 12 by 8 is equal to nothing, but a 3 by 2. So, I have to up sample by 3 times and down sample by 2 times. So, this is called fractional rate converter.

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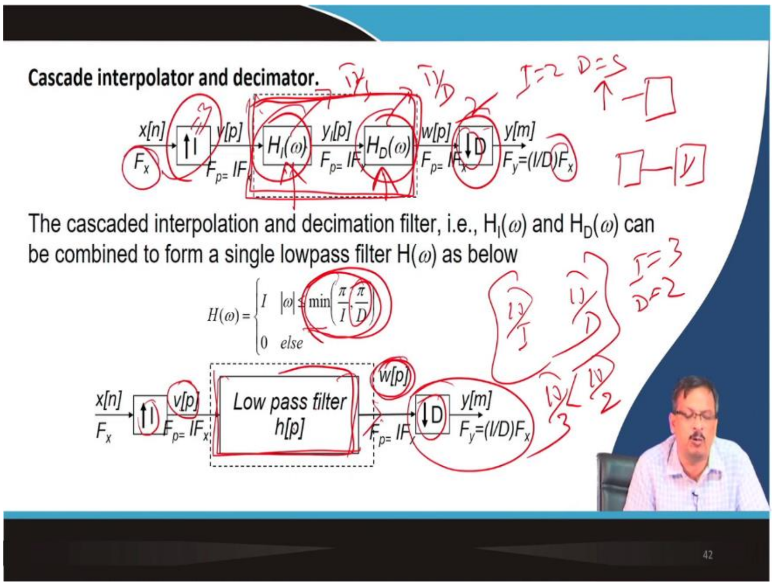

So, how do I do the fractional rate conversion? So, in this case, the sample was increased by 3 times. So, here I equal to 3 and downsample by 2 times. So, you know that upsampling means upsampling followed by a low pass filter, and down-sampling means a low pass filter followed by a down sampler. So, upsampling followed by a low pass filter down-sampling means a low pass filter followed by a down-sampling.

Now, this is also a low-pass filter. This low pass filter cut-off frequency is π by I. This low pass filter cut-off frequency is π by D. So, can I combine these two low pass filters? Yes, I can combine these two low-pass filters. So, let us say I combine this low pass filter, then. What should be the cutoff frequency?

So, whichever will be the minima, π by I and π by D. So. Because I have to rest. So, let us say I is equal to 3 and D is equal to 2. So, if it is π by 3 then, then it supports π by 2. Because π by 3 is less than π by 2. So, I can say if I restricted the output signal of the up sampler to π by 3, it would support the restriction of π by 2. That is why I said a minimum of this is required. in case of, let us say, and I equal to 2 and D equal to 5, then the output minima will π by 5 only. Because I have to the input, which the down sampler can accept, is only π by 5. So, I cannot produce an output that is π by 2.

So, the cut-off frequency is the minimum, whichever is the minimum, but yes, I can combine these two low-pass filters in a single low-pass filter. So, the input is vp, and the output is wp. And then it passes through a disseminator. Let us analyze the system's first-time domain then, the frequency domain

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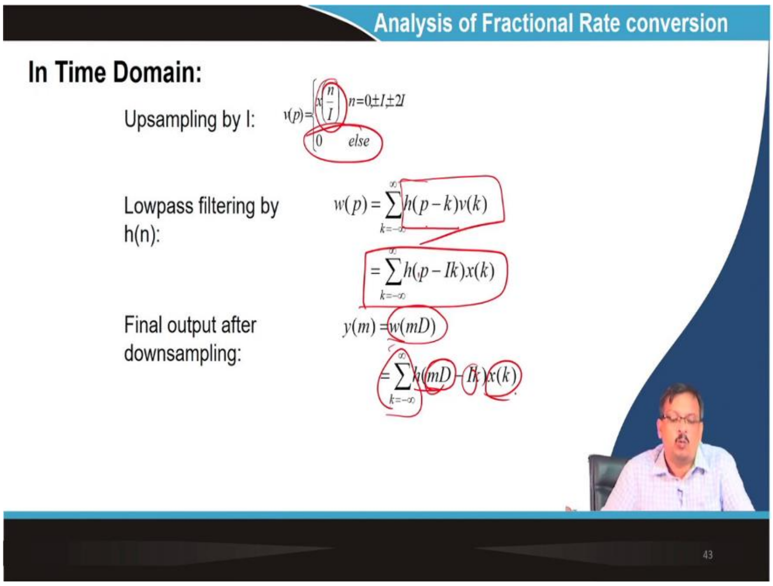

So, time domain analysis. So, I know what v(p) is? What is v(p)? v(p) is nothing but an up sample by I. So, I can say v(p) is equal to x[n] by I else it is 0 unless the low pass filtering is h(p).

So, once the h(p) is, what is the output of the low pass filter v(p) as an input and the low pass filter and output as a convolution? So, I convoluted the v(p) with low pass filter wp. So, this is the convolution, ok? Now, I have the final output after downsampling. So then, w(p) is down sample by D.

So, now that w(p) is downsampling, I know y[m] is equal to w(mD). So, I can say it is nothing k equal to infinity h(mD). So, p is changed to m D I k into x k. So, if I explain in pictorially, what will be the time domain factors? Let us say pictorial, and I will explain it. So, I will take a slide here.

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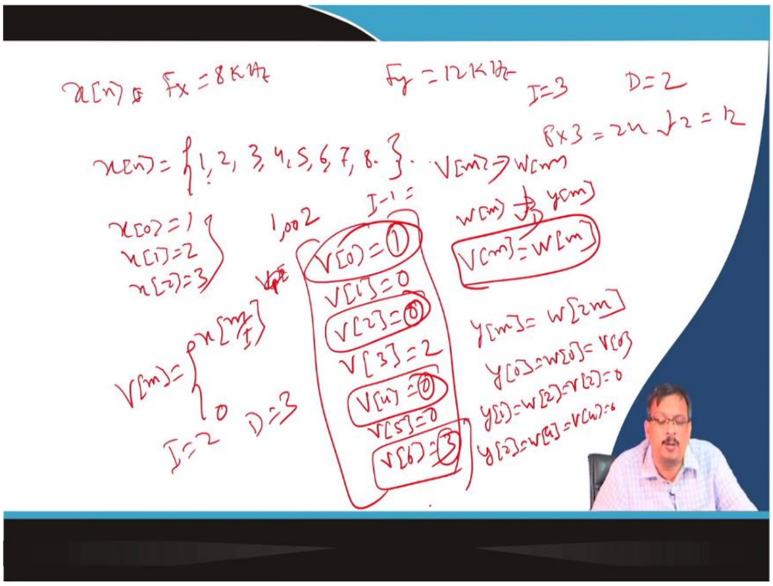

So, I have a signal x[n] x[n] whose sampling frequency is Fx is equal to 8 kilohertz. Now, I want to make it Fy, which is equal to 12 kilohertz. That means the up sample is 3 and the down sample is 2, with I equal to 3 and D equal to 2. An up sample is measured by 3; that means 8 are divided into 3, and a down sample is measured by 2. So, it is 24, and downsampling by 2 means 12. Ok, done.

What is the or suppose now, not less x[n], equal to this? Let us say 1,2,3,4,5,6,7,8 are all the sample values. So, x[0] is equal to 1, x[1] x[1] is equal to 2, x[2] is equal to 3. So, I know this.

Now, what is up? Sampling upsampling by 3. That means, in between 1 and 2, I put two 0, minus 1. So, 3 minus 1 means two 0. So, if I say v(p) is my up sample signal, then V is my up sample signal. So, I know V[0] is equal to 1, V[1] is equal to 0, V[2] is equal to 0, and V[3] is equal to 2. Because V[m] is equal to x[m] by I else 0, V[4] is equal to 0, V[5] is equal to 0, and V[6] is equal to 3.

So, in between 2 samples, I put 0, two 0. Then, I did the filtering. So, filtering does not change the index of the signal. So, I get the same V b signal v(p) V[m] is passed through a filter, filter only changes the characteristics of the signal the length-wise there will change no change.

Then, after the filtering V[m] plus the V[m] after filtering, I get W[m]. So, this is equal to, let us say, W[m] filter is equal to 1 let us say. So, I get W[m]. I will get W[m]. Once I get the W[m] from the v[m], then I have to get y[m] downsampled by 2. So, that means I am not taking all the samples.

So, let us say y[m] is equal to V[m]. Let us say that y[m] v[m] or W[m] is equal to V m. Let us say that W[m] is equal to V[m]. Let us consider that the filter is equal to 1. Let us say the filter basically multiplies the v[m] with 1. So, now, I can say y[m] equals W into 2 m because v equals 2. So, I am selecting y. So, y[0] is equal to W 0, is equal to V 0, y 1 is equal to w 2, is equal to v 2, which is equal to 0.

So, I am selecting this sample then y 2 is equal to w, 4 is equal to v, and 4 is equal to 0. So, this sample is this sample. So, while I am doing multi-rate or fractional up-sampling downsampling, in that case, I am not required to compute all the interpolation. I can only compute this signal this V this V this V and this V others Vs are not important to me.

So, now, if I give you another signal, let us say I give you x[n], and then, I give you that I equal to 2 and D equal to 3; then, if the v[m] is equal to W[m] compute the sequence of y[m] you can do that ok.

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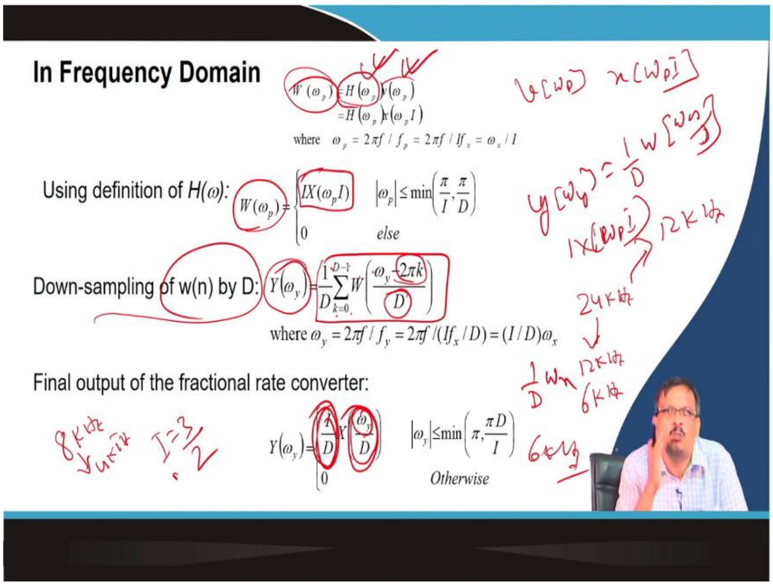

Now, what is happening in the frequency domain? So, in the frequency domain, I know w(p) that if you see the diagram, what is w(p) that it is nothing but a v(p) multiplied by the filter. So, w(p) is nothing, but a h(p) H w(p) w(ωp) is nothing but a filter frequency response multiplied by the v(p) frequency response. I know the v frequency response ωp is nothing, but an x(ωp) I because it is up the sample by I ok.

So, w(p) is equal to X I into Z w(p) I, which I know is used for up-sampling the filter. Now, what do I require for the downsampling filter? Y w is nothing, but a 1 by D into k equal to 0 to D minus 1 W into D y minus 2π k divided by D. Now, for imaging to reduce the image after filtering, what do I want? I want to, only w y by D and 1 by D. So, 1 by I can say (ωy) is equal to 1 is 1 by D into W ωy by D, where W ωy by D is nothing, but an I into X p the ωp by I ωp I.

So, I can say that it is nothing, but the I by D into x[ωy] divided by D, or if I say ωp, then ωy ωy is equal to ωp into ωy is equal to I by D into ωx. So, if I said the ωy instead of ωy ωx, then I have to put is I by D into ωx here. So, it will be I by D square.

So, the frequency response I by D is the amplification factor. So, I is equal to 3, D is equal to 2s, and is 3 by 2. Then, ωy by d is the output. So, in terms of frequency, let us say I have an 8-kilohertz signal. So, the maximum frequency content is 4 kilohertz. Now, once I say up the sample by 3 times, that means 24-kilo hertz. So, the maximum frequency content is 12 kilo hertz, then down the sample by 2 times; that means the 12 kilo hertz maximum frequency constant is 6 kilo hertz, understand or not.

So, it is nothing but a 3 by 2 into 4, 4 kilohertz is the input. So, 3 by 2 into 4, that is 6 kilo hertz ok. So, this is the frequency domain representation of the up sampler, up simpler and down simpler. Both combine ok.

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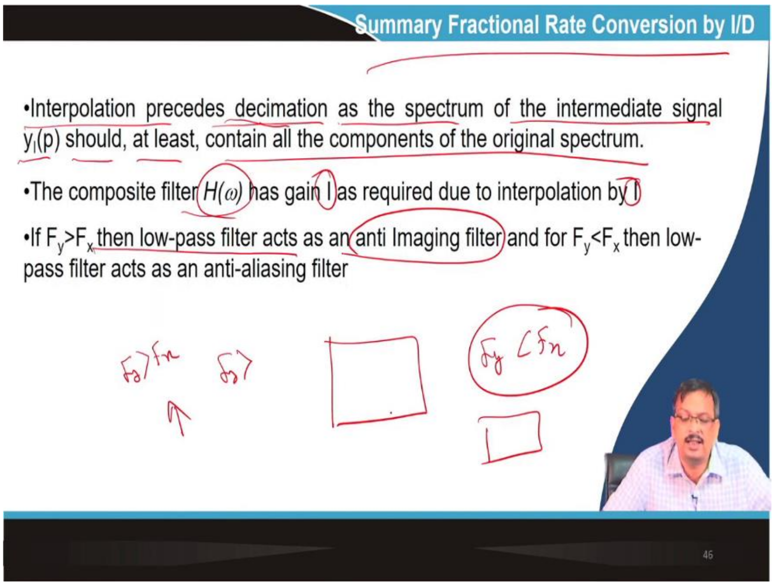

Now, so, the summary of the fractional rate conversion by I by D. Interpolation process, decimation, interpolation proceeds, decimation as the spectrum of the intermediate signal y 1 p should at least contain all the components of the original spectrum.

The composite filter has a gain due to interpolation. Now, if you see that Fy is greater than Fx, then the low-pass filter acts as an anti-emerging imaging filter. So, Fy is greater than. So, the input is Fx, the input is Fx, and the output is Fy. So, if Fy is greater than F x, that means it is up-sampling. So, in the case of upsampling, ultimately, it is upsampling. So, in the case of upsampling, the job of the filter is to remove multiple images.

Now, when Fy is less than Fx, that is downsampling. The job of the filter is to act as an anti-aliasing filter. So, now, suppose I give you a problem developed. Write down the fractional sample, such as the conversion signal flow diagram or the block diagram, for converting sampling frequency 2.

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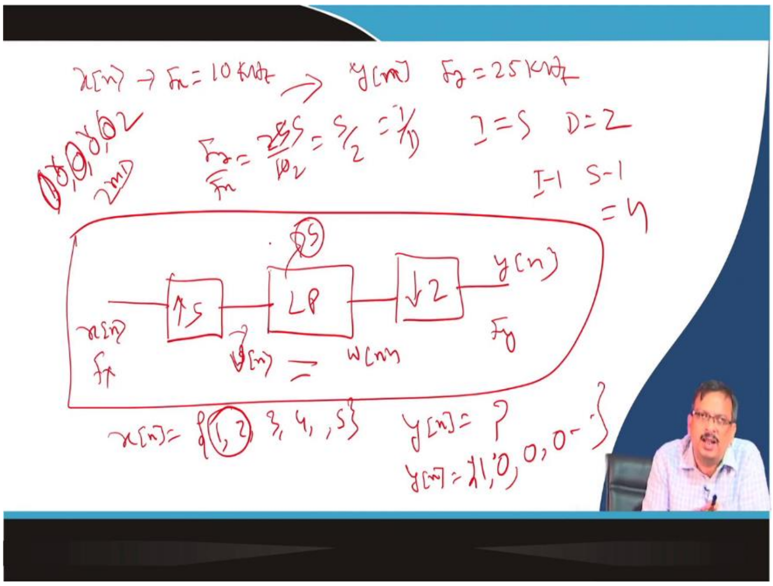

Let us say I have a signal x[n]. I have a signal x[n] sampling frequency of x[n] is equal to, let us say, 10-kilo hertz d. Write down the functional block diagram of a fractional signal sampling rate converter to convert to y[n] with the sampling frequency Fy is equal to, let us say, 25 kilohertz.

So, I want 10 kilo hertz input of 25 kilo hertz. So, in that case, what will I require? So, I know Fy by Fx is equal to 25 divided by 10, so, which is nothing but a 5, 2. So, 5 by 2. So, I by D. So, I is equal to 5, and D is equal to 2. So, I know this is my x[n] up sample upward array by 5 times then, I know low pass filter LP then, I know it has to be downsampled by 2 times, and I get y[n]. So, this is Fx, this is Fy up sample by 5 times, down sample by 2 times. Is it clear?

Similarly, I can give any arbitrary number and tell you to draw that signal flow diagram or block diagram of the fractional rate sampling rate converter. Now, if I told you that suppose x[n] is equal to 1, 2, 3, 4, 5, then what will be the y[n]? So, upsampling is 5 means I am equal to. So, minus one 5 minus 1 means four 0 between 2 samples. So, there will be a 1, 0, 0, 1, 0, 0, 0, 0,2 four 0 in between 2 samples.

Now, down sample by 2, down sample by 2 means, m D m into 2 m. So, this one, skip on this one. So, the y y n will be 1 0; then this skips this one 0, then 2 will be skipped 0, so like that, ok. Let us say LP is less if it is V[m], if it is V[m], and if it is W[m]. Let us say that V[m] is equal to W[m]. V[m] LP does not change that sampling that V[m] W[m] sample value, but basically, it will be changed because low pass filter interpolation will happen, and the samples will be changed.

So, I can say that it is nothing but a summation of 2 or 3 fields. Let us say LP is nothing but a summation of successive 5 samples. So, you can do it that way, ok. So, that is up sampling, down sampling and fractional sampling rate conversion.

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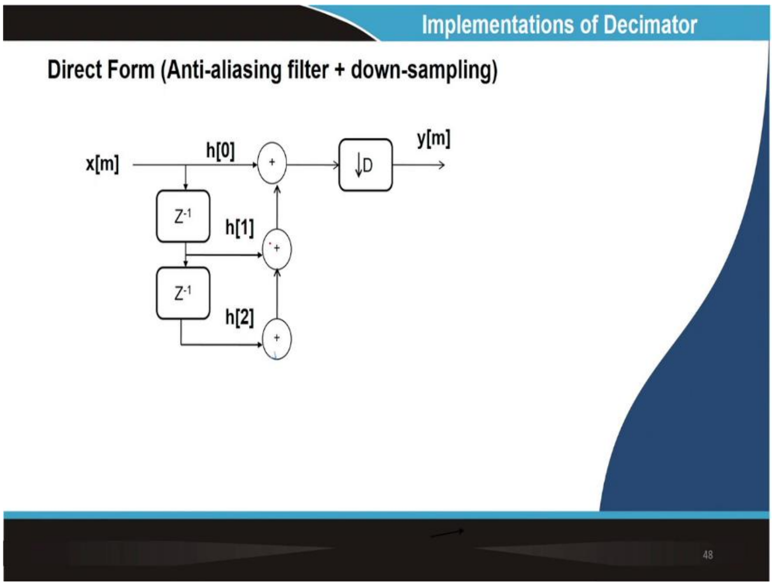

The next topic will be the implementation of that up sampler and down sampler in a digital scenario. How do I implement it? How do I implement the low pass filter? Ok. So, in the next class, I will talk about the implementation of those things.

Thank you.