Signal Processing Techniques and its Applications
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Lecture - 51
Analysis of Decimation and Interpolation

Let us say so; I said that the sampling rate converter is in the system and has a transfer. You can say impulse response h is a h[n] and m. And we know the upsampling factor. We know the downsampling factor.

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Let us talk about down sampling or dissemination only by factor D. D is called the downsampling factor or dissemination factor. So, what do you mean by dissemination factor D or what is downsampling? What is downsampling?

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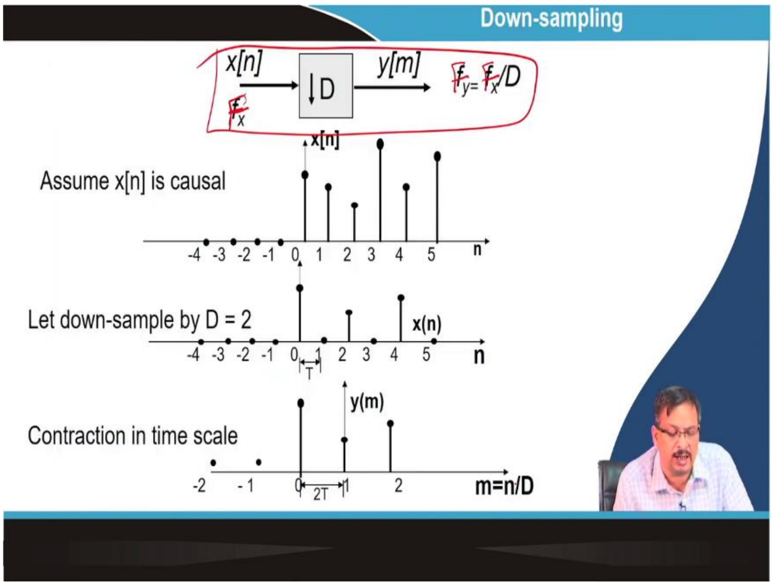

Let us say I have a signal. Let us say I think there is a let us say there is a 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, let us say 32. So, there are 32 samples in this signal.

Let us say the signal frequency is Fs. Sampling Frequency is Fs, let us say. Now, I want a system that will downsample the arrow downside indicated by downsample factor D. So, this is my x[n], this is my downsample, and this is my y[n]. So, this is the diagram for the downsampling block diagram. D is called downsampling factors. I think I have the diagram.

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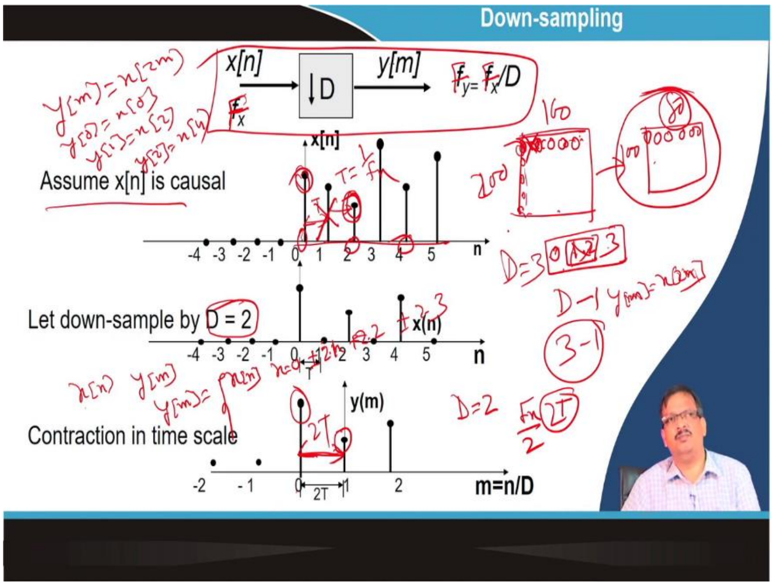

So, this is the diagram for downsampling. You can write capital F because this is the analogue frequency; I write capital F, Y, and F x. So, instead of writing small, I write capitals. So, this is a down sampler. So, I have 32 samples. Let us say I choose D equal to 2. So, this x[n] is applied here, and I want y[n], which is downsampling by factor 2.

So, that means I know if there are 32 samples, then the downsampling factor 2 will reduce the number of the samples by 2. So, there will be 16 samples here. So, out of 32 samples, I have to choose 16 samples. So, how can I choose? I can Instead of choosing every sample, I can take every alternate sample. So, I take this sample, skip this sample, take this sample, skip this sample.

So, I keep the 0th sample, skip the larger one, skip the second one, skip this one, skip this one, skip this one, skip this one, skip this one, skip this one, skip this one, and take this one. So, in that way, I have to choose. So, that means I am every alternative sample I am choosing. So, because D is equal to 2. So, how many samples do I discard D minus 1? Let us I can give an example from the image. Let us say I have an image size of 1 6th. Let us say 200, and this side is 160.

So, I have 160 pixels on this side and 200 pixels on this side, which I want to reduce by 80 by 100 sample images. So, how do I do that? I am not saying I am cutting that image. One portion of the image is cut; the whole image is there, but I am reducing the size of the image. So, the number of pixels will reduce if I want to make it half-size. So, basically, I am reducing the number of pixels. So, instead of 160 samples, I have to choose 80 samples.

So, instead of choosing every pixel in this direction, I will take only the 0th pixel I have taken, the next pixel I drop and then the next pixel I have taken. So, out of 160 samples, I will choose 80 pixels for every alternative pixel I have chosen and here also, every alternate pixel I have chosen. Then, I get the dissemination factor 2 in the case of image processing.

So, whether it is an image, whether it is the audio signal, 1-dimensional signal, 2-dimensional, or multi-dimensional, it does not matter. So, when I say disseminating, when I sample the image or sampling the signal, that means if it is D equal to 2, then every alternative.

Suppose I said instead of D equal to true 2, I want to reduce by 3 times D is equal to 3; that means I have to skip 2 pixels in between. So, pixel number 0 1 2 3, in between 0 1 2 3. So, in between that, I will skip these 2 pixels. So, 1 and 2 will be skipped. So, out of 4 pixels, I kept 2 pixels.

So, how many pixels are I discarding? D minus 1, 3 minus 1 is equal to 2. Let us give another example, which is given here. So, xn is the causal signal, this is the xn signal, so this is T, and the duration is T. So, T is equal to 1 by f x. Now, I am saying instead of taking that, let us say D is equal to 2 decimation by 2 times.

So, instead of taking the 0 1 2 signal I am taking 0th signal, I am taking the 0th signal I am taking. So, I am not making 0; I am only taking this index and this index. So, I am increasing the distance between the samples because the dissemination factor is equal to 2; that means F x is decreased by 2. So, F x is down by 2.

So, F x is down by 2, which means T is increased by 2 times. So, instead of T, I will get 2T. So, I am increasing the distance, which is equal to 2T. So, if I want to write down a mathematical equation, what are the relations between the x[n] and y[m], or do we say both indices are n y[m]? So, what is that y[m] is equal to a signal is equal to x[n], when n is equal to 0 plus minus 2 into n plus minus 2 2 into n.

So, that means 2 into 1 plus minus 2 into 2 plus minus 2 into 3; that means 2 2. So, I am saying 0th sample, the second sample, and then the fourth sample. I then take the sixth sample. So, I can say it is 2 n. So, y[m] basically is equal to x[2m] 2 n x[2n] n equal to 0 y 0 n equal to 1 x[2] is equal to y 1.

So, I can say if both the index is same, I can say y[m] is equal to x[2m] or n whatever. So, I can say that y[m] is equal to x[2m]. So, y[0] is equal to x[0], y[1] is equal to x[2] y[2] is equal to x[4]. So, I am discarding others for those samples I am taking, okay? So, that is my motto.

Now, what will happen in the frequency domain? So, I am discarding every alternative sample, which is why it is called dissemination, but what will happen in the frequency domain if I discard the alternative samples? So, what will the problem be in the frequency domain?

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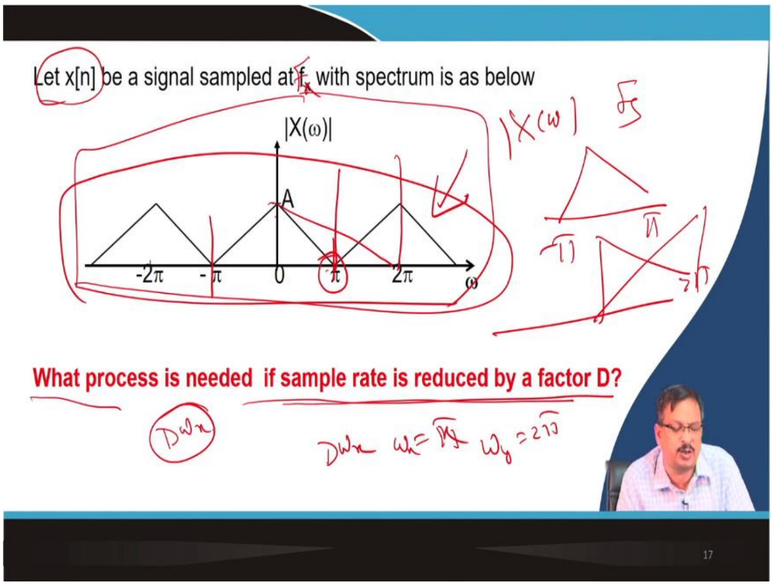

So, let us say I have a signal x[n], and F x is my sampling frequency. So, I can say let us say x[n], the spectrum of x[n] looks like this. So, the maximum frequency is π to minus π. So, this is twice Fs. So, this will be the sampled version of, or I can say, x[ω] mod of x[ω] when the sampling frequency is Fs.

Now, said I want what a process that I want to reduce the sample rate is reduced by factor D, then what will happen to this spectrum.

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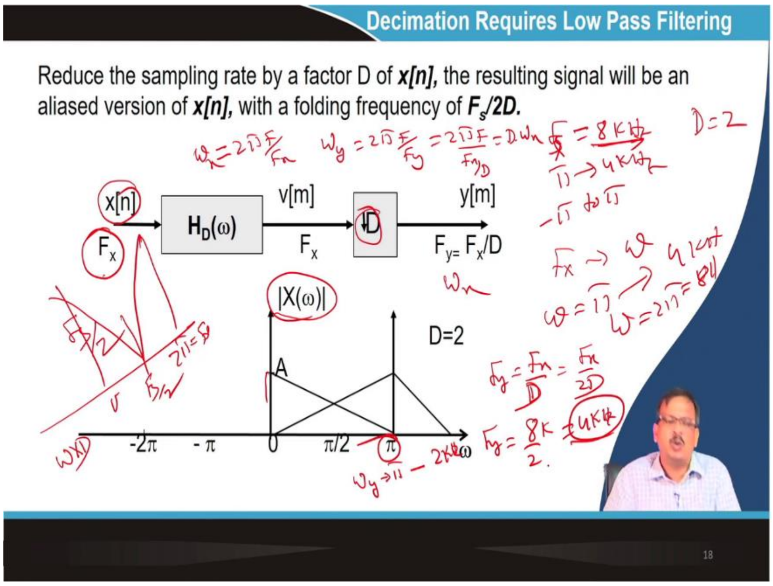

So, what is the meaning? This means that I have a signal x[n] sampling frequency of F x Fs or F x. Let us say F x is equal to 8 kilohertz. Then what is the maximum value of the signal, or I can say π is related to 4 kilo hertz in a digital signal? Maximum oscillation frequency varies from minus π to π, and 2π equals F x.

So, I can say if my F x[n] is sampled at 8-kilo hertz, then I know the π in F when I say Fs F x is related when I converted to the ω. So, ω is equal to π, which is related to the 4 kilohertz. So, a 4-kilo hertz component will create the π radian per sample in the ω axis. Now, let us say I down-sample it by factor D.

So, what is the new sampling frequency F y? It is equal to F x by D. Then what is the folding frequency? Folding frequency is nothing but an Fs by 2. So, you know that if this is equal to 0, this is equal to twice Fs, and twice π is equal to Fs, then you know this is folding frequency, which is Fs by 2.

So, I can say that if my sampling frequency is down by the D factor, then the folding frequency is F x by 2 D; for example, let us say that F x is equal to 8 kilohertz and my D is equal to 2 then I know F y is equal to 8K divided by 2 is equal to 4K. Now, the sampling frequency is 4K when I say ω y, so the maximum frequency of the signal, which is ωy related to the π, is equal to 2 kilohertz. Because the sampling frequency is 4 kilohertz.

So, the folding frequency is 2 kilohertz. So, the maximum signal content can be 2 kilohertz, so when I say F x, I say ωx. So, ωx representing π means 4 kilohertz. Now, when I down-sampled it, ωx ωy representing π is 2 kilohertz. Then, if I thought in frequency, given what is happening?

So, I can say that the ω is multiplying by the factor D why? Now, the ω you know the discrete frequency 2π f by F x is the ωx and what is ω y? ωy is equal to 2π F by F y. Now, F y is equal to F x by D. So, I can say it is nothing but the 2π F by F x divided by D. So, I nothing but a D into ωx D into ωx. So, I can say that when I draw the spectrum, this x ω is related, so this 2π now instead of 2π ω is equal to 2π is equal to 8 kilohertz.

Now, if ω equal to π becomes my sampling frequency, ω equal to π becomes my sampling frequency because it is Fs by 2. So, I can say that my signal, which is there π 2 minus π will be stretched to 2π whole 2π will be stretched because it is D into ωx. So, when ωx is equal to π in the downsampling, the signal that ω will ωy will be 2π.

So, at the π, it is 2 π, which means the new sampling frequency is here. So, I can say this one will be stretched this way, which means I am saying what is happening that D into ωx is ok happening. So, D into ωx D is multiplying then. So, my Fs becomes π so, I can say what is the band limitation of the signal.

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So, let us say another example I have given. So, let us say this is ωx. So, this is 0.4 π; let us say I up sample this D equal to factor D. So, what is ωy is equal to D into ωx. So, 2 into 0.4pi is equal to 0.8 π. So, I can say that within 0 to 2π so, it is 0 to 2π so, this is stretched the. So, I can say the up sample version is a stretched version of the downsample version, and the spectrum is stretched.

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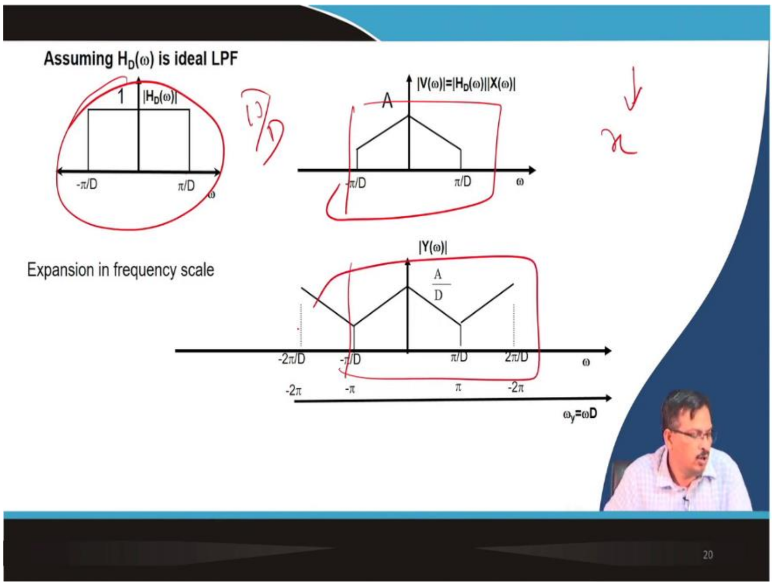

So, due to downsampling, the spectrum is stretched. So, if it is stretched, then what will happen? If it is stretched much more, then aliasing will happen. Let us say this: instead of 0.4 π, let us say it is 0.8 π, then it will be 0.8 π into 2, 1.65, so that means it will go there. So, this side will also be going there, so aliasing will happen.

So, if I want to avoid this aliasing, what do I have to avoid? Although it is D multiplied, ω is equal to D into ωx. Now, if I want to make it ω x, then it has to be divided by D. So as if π act in the transpose plane is equal to π by D. So, π is equal to π by D. π is the maximum frequency content in a sampling frequency 8-kilo hertz, but when I downsample by 2, the sampling frequency is 4-kilo hertz; that means, π by 2. So, π is analogous to π by 2.

So, I can say that I have to restrict the baseband signal to π by D. So, if I say that here I restrict it to π by D, that means this spectrum cannot go 0.4 8 π, so it will always be π by D. So, if I want to restrict it, then this aliasing will not happen. So, if I want to do that, I must require a filter whose cut-off frequency is π by D.

So, I designed a filter with a cut-off frequency of π by D. So, and then, I can say that before downsampling the signal, I have to pass the signal to a filter whose band limitation is π by D. So, now, my baseband signal looks like this, and then after downsampling I get this one. So, there is no aliasing happening.

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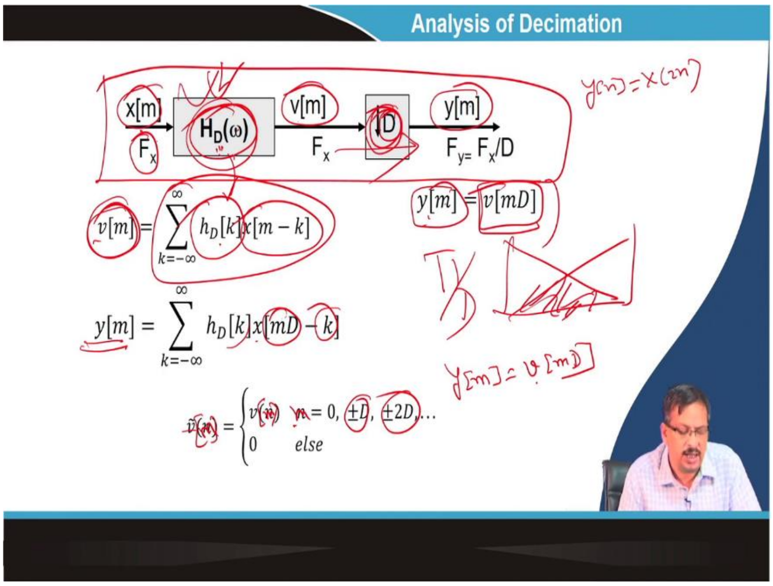

So, what is the new dissemination circuit that I require an F x? I want to disseminate it by D factor D, and then I have to require a digital filter whose cut-off frequency or low pass filter, whose cut-off frequency is π by D. So, I have to design and so, before downsampling, I have to band-limited the signal with π by D unless what will happen the signal will be stretched and then the aliasing will happen.

So, this is my now downsampling system diagram. So, x[n] passes through a low pass filter, I get V[m] , pass through a down sampler, and I get y[m]. So, what is v[m] ? v[m] is nothing but the convolution of x[n] with this filter. Let the filter impulse response is hD[k]. So, hD[k] into x of m minus k. And downsampling, you know that when I say that y[n] downsample by factor 2, then I said x[2n].

So, here, the down sample factor is D. So, I can say y[n] is nothing but a v into m D, D is the down sampling factor, so I can say y[m]. Now, I change the notation so, y m, then this v[m] is passed through a down sampler, and I get that y[m]. So, y[m] is equal to v[m] D and v[m] is equal to the convolution of the input with the filter transfer function so, I combine this together. So, I can say k equal to 0 to infinity hD[k] x of instead of m I put m D minus k.

Instead of m, I put m D. Now if you see this, instead of n, you write v[m]; let us say you write v[m], v[m] is equal to n or m equal to 0 plus minus D plus minus 2D else it is 0 ok.

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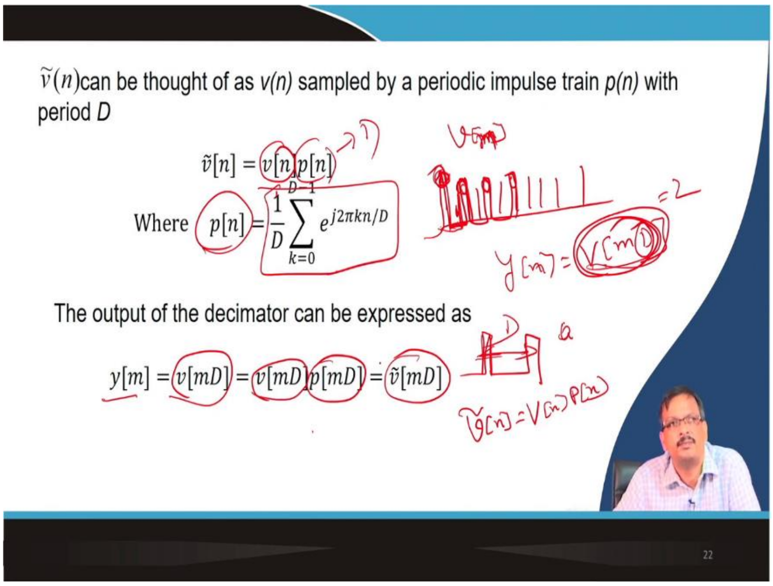

So, that means what are we doing? Let us say I want a down sampler so that v[m] y[m] is equal to v[m] into D. So, what is required? I require that I have a v[m] or, let us say, v of n n or m, whatever v[m] I have. So, v[m] samples are there, I want to create I want to choose y[m] is equal to m, m is equal to V[m] into D. So, I have to make v[m] into D from v[m] so, how do I make it.

So, what is D? D is nothing but a downsampling factor. So, what I have to do is let us say D is equal to 2; that means I am selecting this signal, this sample, this sample, this sample, this sample. So, what am I doing? I am basically multiplying an impulse. So, when the impulse exists, the sample exists; when the impulse is not there, it is 0. So, I am taking this signal, this signal, this signal. So, I am multiplying that v[m] with an impulse.

So, let us say v[n] or v[m] with an impulse p n, whose period is D 2 samples, and whether this differs by D only ok. So, I can say that let us say y[m] v[m] D is nothing but a represented by v cap n. So, v cap n is nothing but a v[n] multiplied by impulse whose period is D. So, how do I create a periodic impulse is D? So,

$$p(n) = \frac{1}{D}\sum_{k=0}^{D-1} e^{j2\pi kn/D}$$

Now, I can say y[m] is equal to v into m D, which is nothing but a v[m] 2 m D into p into m D, which is nothing but a v cap m D ok.

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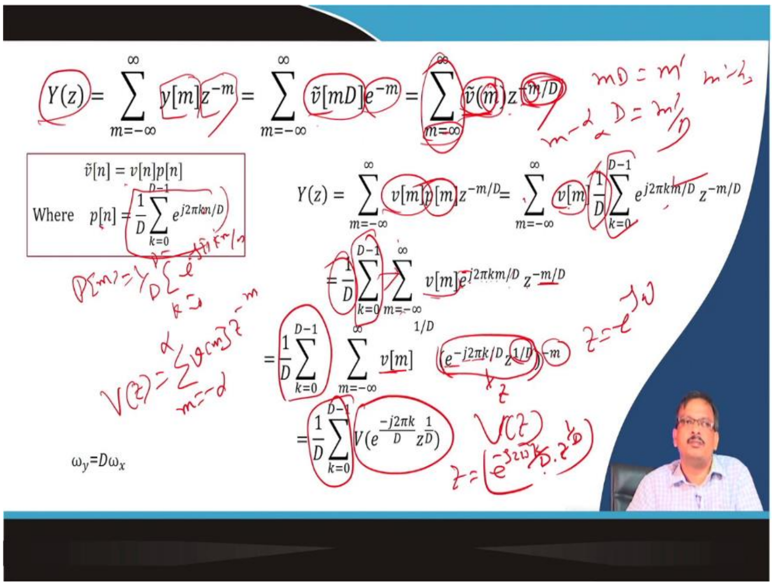

So, now v cap m D. if I want to do the z z domain. So, what will be the? So, once I get the z domain, I get the frequency property. So, I have explained what will happen in the frequency domain, but now I have to explain it mathematically and mathematically show that this is happening.

So, now I am showing in mathematically using Z-transform. So, what is Y(z)?

$$Y(z) = \sum_{m=-\infty}^{\infty} y[m]z^{-m} = \sum_{m=-\infty}^{\infty} \tilde{v}(mD)z^{-m}$$

So, m D is equal to m, so m is infinite. So, the summation infinity will remain the same, so, D is equal to so, the nu less than m dash. So, it is nothing but an m dash by D, so the m dash and m are the same. So, that is why I said this is m dash. Let us say this is m dash. This is m by D.

So, since m is minus in v to infinity, m dash t is also minus infinity to infinity, that is why the index remains m, and this is instead of z^m, it is $z^{-m/D}$. Now, I put v cap m value. v cap m is nothing but a v[m] multiplied by f p m.

So, now v[m] and what is p m? Instead of n, p[n] is

$$p(n) = \frac{1}{D}\sum_{k=0}^{D-1} e^{j2\pi kn/D}$$

So, now, if you see that 1 by D is a constant term kept outside, let this sum I kept it here, and this sum is taken inside m sum.

So, m is minus infinity to infinity v[m] $e^{j2\pi km/D}$ z equal to minus m by D. So, I can say v[m] $e^{j2\pi k/D}$ into $z^{1/D}$ whole^{-m}. Since it is minus m, this becomes negative, and this becomes positive. Now, what is that? So, you know that V(z) is nothing but a summation of m equal to minus m a to infinity v[m] into z^{-m} ok or not?

So, as if this is act as in z, this whole thing is a Z. So, z^{-m} so, which is nothing but a V z. So, I can say k equal this sum into v z, where z is equal to, you know the z equal to $e^{j\omega}$ instead of that z is equal to $e^{-j2\pi k/D}$ into $z^{1/D}$ that whole thing is z ok.

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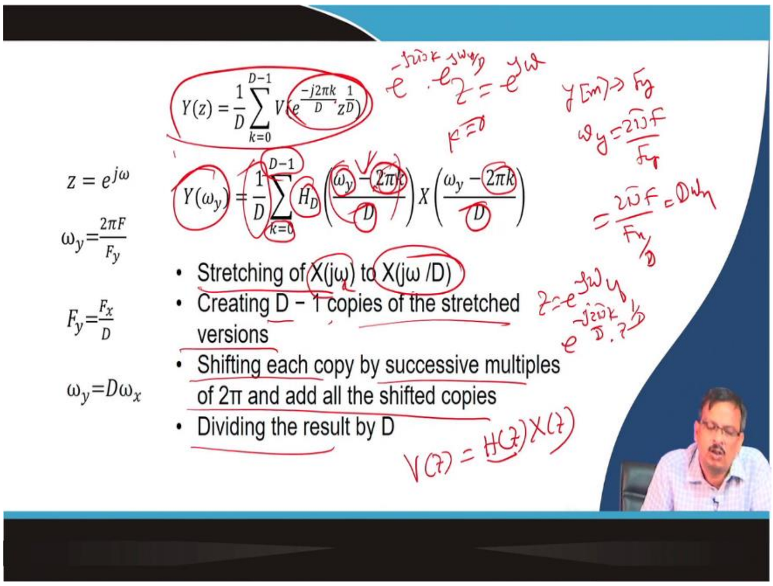

If I show that so, Y(z) is equal to this we have derived. Now, z is equal to $e^{j\omega}$. So, if I say y[m] y m has a sampling frequency F y, ωy. So, what is ω y? Is nothing but a twice π F by F y, which is nothing but a twice π F by F x by D, which is nothing but a D into ωx.

So, if I say Z is e^{Z} is equal to $e^{j\omega y}$. So, it is y ωy y ωy is equal to 1 by D k equal to 0 to D minus 1. So, v what is V z? V(z) is nothing but a x z multiplied by H z H z. So, V(z) I can say that V(z) is nothing but a H z multiplied by X z. So, I replaced it with V(z) H z. So, it is H D z, and z is nothing but this one. So, this one why this one is coming $e^{j2\pi/k/D}$ into $z^{1/D}$.

So, I said e z is equal to $e^{j\omega y}$. So, it is nothing but an $e^{-j2\pi k}$ into $e^{j\omega y/D}$. So, nothing but a ωy minus 2π k divided by D. So, ωy minus 2π k divided by D. So, what is the meaning what it is expressing? Stretching of j ωx to j ω divided by D ok.

Create D minus 1 copy k equal to zero to D minus 1 copy of the stretched version. Shifting each copy by successive multiplication of 2π. And dividing the whole things by D. So, I can say that j ω is stretched to j ω by D ok and then I can say every stretched version will be 2π by shifting is 2π 2π k, K equal to 0 means 2π k equal to 1s means K equal to 0 means 0 shifting K equal to 1 means 2π k equal to 2 means 4 π like that.

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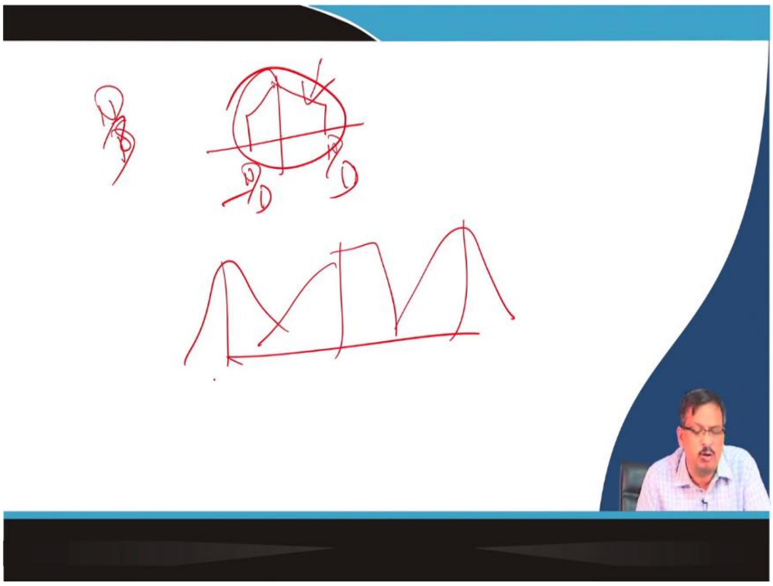

So, if I want to avoid aliasing, this is 0.4 π, and this is 0; stretching is done. So, if I want to avoid aliasing what I said, I am restricting the filter to π by 8 a π by D. So, instead of multiple copies of the stretched version, I only want a copy, which is π by D and this side also π by minus π by D.

So, this 1 copy I want to keep, others I do not want to take. If I take others, then another one will be aliased. So, to avoid aliasing, I want to take one copy. So, how do I use filtering because multiple copies are created 2π shifting? So, instead of multiple copies, I want 1 copy. So, what I will do instead of taking multiple copies is only take K equal to 0 terms.

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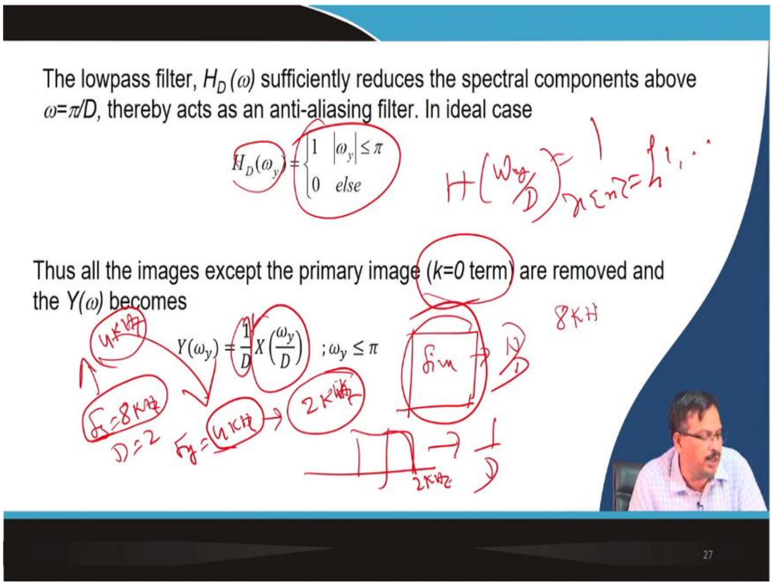

So, K equals 0 terms; that means, instead of this one, K equals this term will not be there, and this term will not be there. So, I can say it is nothing but a f H ω m e ωy by D is nothing but a this one. So, it is nothing but a 1. So, I put 1 1 by D x into ωy by D. So, my primary job for any dissemination, and this primary job is filter is to restrict that band to π by D, unless what I will be thinking about in mathematics term.

Let us say I have an Fs equal to 8 kilohertz. Let us make D equal to 2. So, F y is equal to 4 kilohertz. So, if you see Fs is equal to 8-kilo hertz maximum frequency component can take 4 kilohertz, but once it is reduced by 2 that F y is equal to 4-kilo hertz maximum frequency component baseband signal can exist only 2 kilohertz only 2 kilohertz. So, if this above 2 kilohertz is supposed to be 4 kilohertz present, then the aliasing will happen.

So, what I required was that instead of 4 kilohertz, I had to design a filter that would create a filter that was 2 kilohertz. Then I can reduce by D. What will happen if I do not do that?

These 2-kilo hertz signals will overlap each other, and aliasing will happen. So, to restrict that aliasing, I have required a digital filter whose cut-off frequency is π by D. Is it clear, ok?

So, for downsampling, I just gave a signal x[n] and chose the sample value that would not happen. I have given a signal x[n] 1, 2, 3, 4, 5, 6. So, alternative samples I store in a file it is not done downsampling. I have to ensure that before I choose the downsample, and the signal must be passed through a low pass filter, whose cut-off frequency is π by D. So, in the case of an 8-kilo hertz sampling frequency, let us say I was given a practical example.

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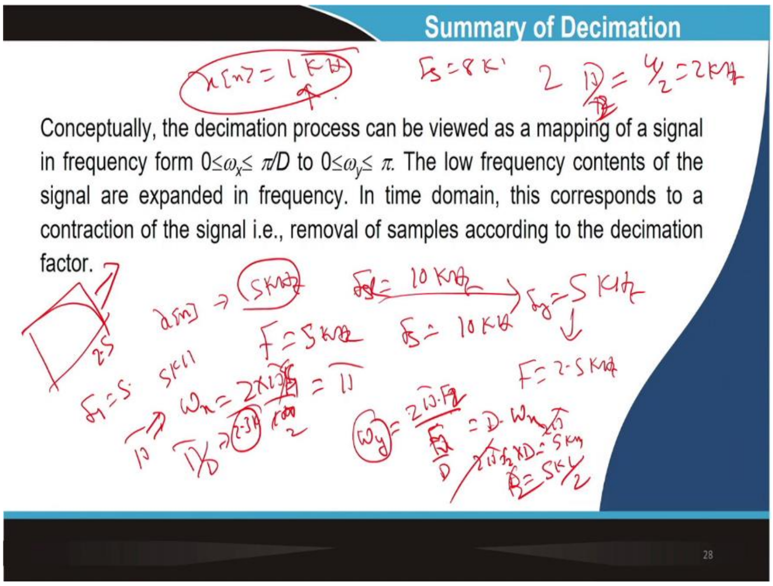

Let us say I have a signal x[n], in which the maximum signal frequency content is, let us say, 5 kilohertz, then the sampling frequency Fs is equal to 10 kilohertz, let us say. So, the maximum F value of F equals 5 kilohertz, and the value of Fs equals 10 kilohertz.

So, I know ωx is equal to 2π F 5 kilo hertz by 10 kilo hertz is nothing but a π so the maximum is π. Now, I change F x to 10 kilohertz to 5 kilohertz. So, once I change the F x, my 10 kilo hertz Fs is instead of sampling frequency F x is 10 kilo hertz instead of 10 kilohertz I make it F y is equal to 5 kilohertz.

That means, downsample D by 2, then what is the maximum frequency content F? 2.5 kilohertz. Now, if I say what is my F y ωy. So, ωy is equal to 2π into F divided by F y. Now, F y is equal to F x by D. So, I can say it is nothing but a D into ωx.

So, let us say this is F 1, and this is F, this is F 2, and this is F 1. So, F 1 is equal to 5 kilohertz. Now, what will be the F 2 value? So, F 2 is equal to D into ωx or ωy is equal to 2π F 2 by F y, F y is nothing but the F x by D. So, it is nothing but a 2π f 2 into D is equal to 5-kilo hertz into 2π 5 kilohertz. So, 2π 2π cancel. So, if f 2 is nothing but a 5-kilo hertz by 2, F2, it is D.

So, the maximum signal frequency can contain 0.5 kilohertz. So, instead of π, the π is equal to 5 kilohertz, I required π by D, which is equal to 0.5 kilohertz. So, I had to design a low pass filter, which restricts the signal to 5 2.5 kilohertz, and then I chose an alternative sample to get back the downsample by factor 2. Only choosing alternative samples does not give me the downsampling. Maybe aliasing will happen.

Yes, if my x[n] is equal to 1 kilohertz and Fs equal to 8 kilohertz, then I know downsample factor 2 does not matter because it is 1 kilohertz well below that π by D. What is the value of π by D? π by D value is nothing but a π by D. D is equal to 2π by D is nothing but a 4 by 2, 2 kilohertz. If it is π is equal to Fs by 2. So, it is nothing but a 4-kilo hertz divided by 2D. So, 2-kilo hertz is 1-kilo hertz, so it does not matter. So, up to 2 kilohertz, it does not matter. Once it is beyond 2 kilohertz, I must put the low pass filter to clarify it.

So this is called downsampling. When I do a downsampling time scale, it is a reduced frequency scale. It is stretched. So, I required a low pass filter to restrict it within that frequency limit.

Thank you.