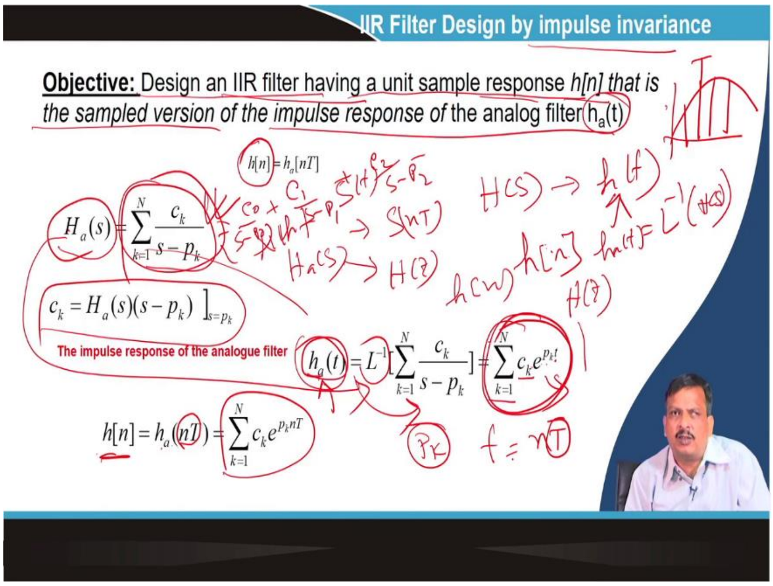
Signal Processing Techniques and its Applications
Dr. Shyamal Kumar Das Mandal
Advanced Technology Development Centre
Indian Institute of Technology, Kharagpur

Lecture - 44
Analogue filter to digital filter transformation

Ok. So, now, what we have done? We have designed that analogue filter H s using the transfer function of H s using Butterworth or Chebyshev, whatever you want. So, you create that filter transfer function H(s). Then, what is required? I have to convert H(s) to H(z). So, how do I do that? Two methods, either bilinear transformation or impulse^{-1} methods.

(Refer Slide Time: 00:48)

(Refer Slide Time: 00:50)



So, now, let us discuss about impulse inversion in the impulse^{-1} method. So, what is objective? Design an IIR filter with a unit response; let us say if the H(s) is my Laplace transform, then if the h(t) is my time domain transfer function of the filter.

Now, if h(t) is the time domain transfer function, if I want to convert the same digital filter, then my transfer of the impulse response is h[n]. So, my objective is to design an IR filter with an impulse response h[n], which is the sampled version of the impulse response of the analogue filter. Is it clear?

So, I am designing the analogue filter, so I know the analogue filter equation ha(t), and then what h[n] is nothing but a sample version that is the impulse response of the digital filter. So, H(z) is nothing but converting that analogue domain to the z domain, which defines my h[n].

So, my objective is not only to get H(z), so I H(z) is not there is a method, but my objective is that if I know h t, I can get h[n] by nothing but a sampling. If I have a signal S t, then I get S(nT), which is nothing, but the digital signal is nothing but a sampling. So, time is sampled and discretized. So, I can say this is nothing but a discrete time response of the filter ok.

So, let us say the analogue filter is Ha(s) is equal to ck by s minus pk where k varies from 1 to N. So, that means I am saying that I am to design an Nth-order analogue filter, which is H and s. Now, what is my objective? I have to find out what H(z) will be. So, ck is the coefficient. What is the ck?

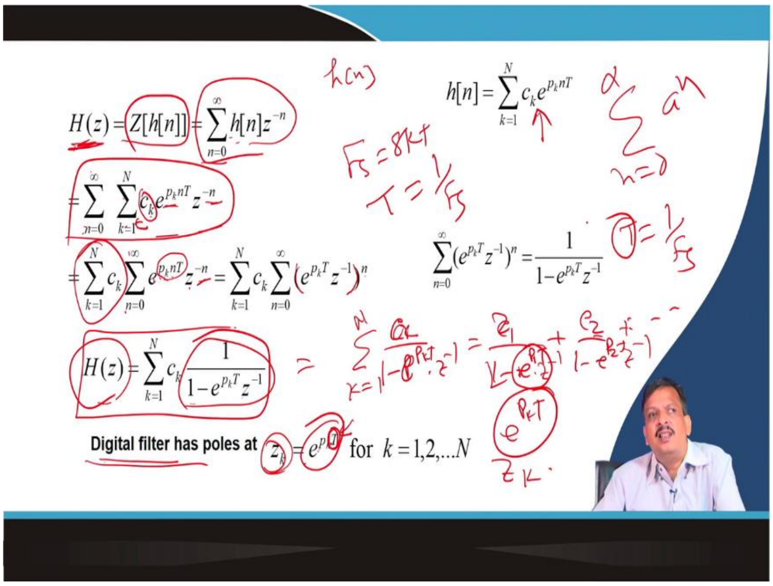
So, I can find out the ck value by an algebraic partial algebra, which is nothing but this one, ok? So, now, if this is my Ha(s), then what is the impulse response of the analogue filter? So, ha(t) so, ha(t) is nothing but an^{-1} Laplace transform of Ha(s). So, I can say ha(t) is nothing but a Laplace^{-1} transform of H(s).

So, I take the^{-1} Laplace transform of H(s), which gives me k equal to 1 to N ck e^{pkT}, pk is the position of the pole. So, pk is the pole position s minus pk, so k is equal to 0 s minus p0 into s minus p1 into s minus summation, so ck is equal to 0; what is this? This is nothing but a c0 by s minus p0 into plus c1 by s minus p1 plus c2 by s minus p2 like that, so that is my H s. So, if the k varies from 1 to N, then the order of the filter is N ok.

So, now, my responsibility is to find out h[n]. So, I know h(t) is nothing but the^{-1} Laplace transform of H(s). So, H(s), I take the^{-1} Laplace transform, and I get this one. Then, what is h[n]? h[n] is nothing but a sampled version of h(t). So, h[n] is equal to h of a sampled version, continuous time is sampled means n into T, T is called the 1 by sampling frequency or sample interval.

So, T is, there is a continuous time I sampled it at every T interval. So, that is why, T this continuous time T is replaced by n into capital T ok, so, which is equal to this one ok.

(Refer Slide Time: 06:30)



Now, what is H(z)? So, suppose h[n] is my impulse response, then H(z) is equal to the Z transform of h[n]. What is the Z transform? $H(z) = \mathcal{Z}\{h[n]\} = \sum_{n=0}^{\infty} h[n] z^{-n}$. Now, I know h[n] is this one because this is the continuous impulse response, the continuous time I sampled the T into n into T, so this is my h[n], so I put this h[n] value is in here ok.

Once I put the h[n] value, I get two summations. So, if you see ck are related to the k, I can say this k summation take out, and here all are related to the n, so I can say

$$H(z) = \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} \left(e^{p_k T} z^{-1}\right)^n$$

Now, this is my H(z). So, H(z), if you see I have a this is my H(z), which is nothing but a

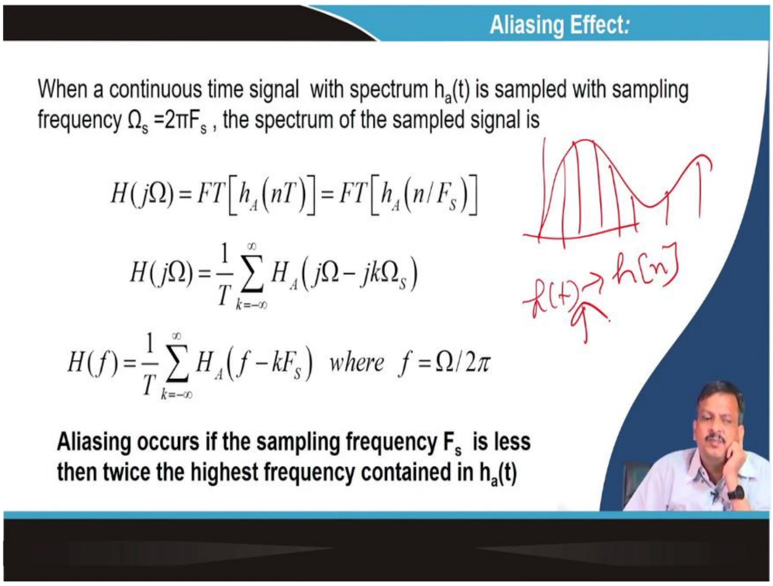
$$H(z) = \sum_{k=1}^{N} c_k \cdot \frac{1}{1 - e^{p_k T} z^{-1}}$$

which is nothing but the c1 by 1 minus e^{p} let us say p1 T plus c2 divided by 1 minus e^{p2T} like that dot dot dot dot ok. So, I can say z^{-1}.

So, I can say this e^{pkT} is nothing but a value of z k because it is nothing but a z minus this one, z upper side z, so the pole position, the pole digital filter pole, is nothing but an e^{pkT}. What is p k? pk you have given s minus p k, the pole position in s plane. So, I know p k, if I know sampling frequency, z k is nothing but a e^{pkT}. So, suppose I have an F s is equal to 8 kilohertz, then I know T, T is equal to 1 by F s, then I if I know the pk value, I know the value of the z k.

So, this is my impulse invariance filter transfer function. So, I know H s. So, if my H s is equal to this one, H s is equal to ck by s minus pk, then my H(z) is equal to ck divided by e^{pkT} into z^{-1}. So, I converted the s domain to the z domain.

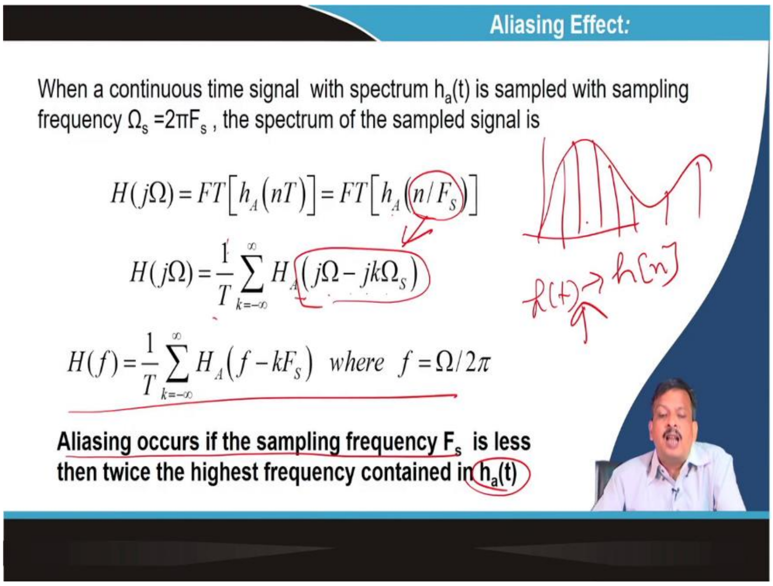
(Refer Slide Time: 09:45)



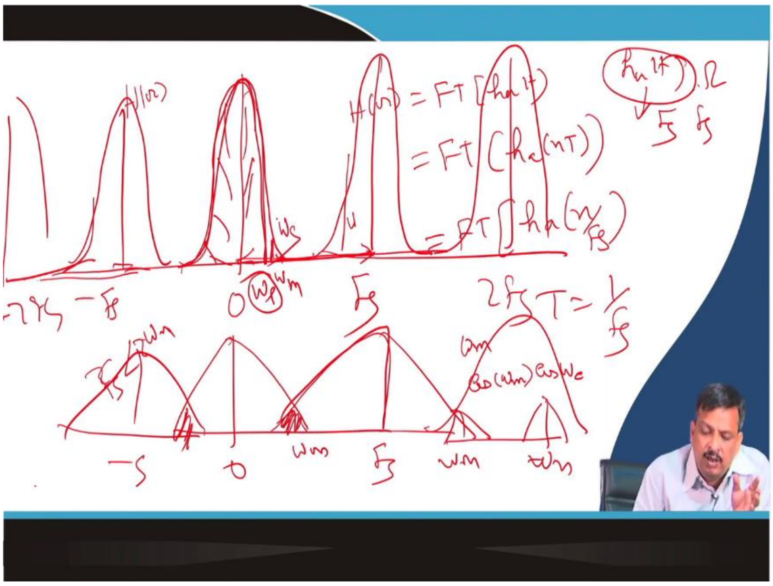
Then, what is the implication? What actually happens if I do that? Mathematically, I can do that very easily, but what is happening inside means there is an aliasing effect; why does this aliasing effect come? Because I have a signal and I have sampled it. So, if you see any signal sampling related to the aliasing, I have to avoid aliasing, which is why I required an anti-aliasing filter.

So, I also only have h t, and I converted to h[n], but what is the anti-aliasing filter? What is that aliasing effect? Any analog-to-digital conversion requires an aliasing effect; why does this aliasing effect come?

(Refer Slide Time: 10:36)



(Refer Slide Time: 10:49)



Now, let us say I drew it, and then it will be better than showing you the slide. Let us say I have a signal whose frequency response; I have a signal whose frequency response is less than I have a signal ha(t) whose frequency response H(ω) looks like this; let us say this one: this is my frequency response.

Now, when I make it sampled, if I sample this ha(t) by a sampling frequency F s, then what is H(ω)? H(ω) is nothing but a Fourier transform of ha(t). Now, ha(t) is periodic, and Fourier transform is periodic now; in the case of the digital domain, the period is s defined by F s 2π sampling frequency is 2π, F s is equal to 2π. So, in mathematics, when I say this one, then what will happen? So, it is nothing but hn by F s.

So, the Fourier transform of ha(t) is nothing but a Fourier transform of h of an nT if T is replaced by sampling, then F of T is equal to h of a T, which means n by F s, and T is equal to 1 by F s. So, I am doing the Fourier transform of n by F s. So, this is nothing but a 1 by T if this is the Fourier transform, which will come into j ω minus jk ωs, k equal to minus infinity to infinity.

So, what is happening physically? So, once I say that, how do I do sampling? basically, I am passing this signal or multiplying this signal with an impulse whose frequency is F s. Once I say I have sampled the signal, that means the signal is periodic after every F s, so the highest frequency component is F s, so if this is 0 and if this is F s, then there will be another repetition of the same signal. If this is minus F s, there will be another repetition of the same signal.

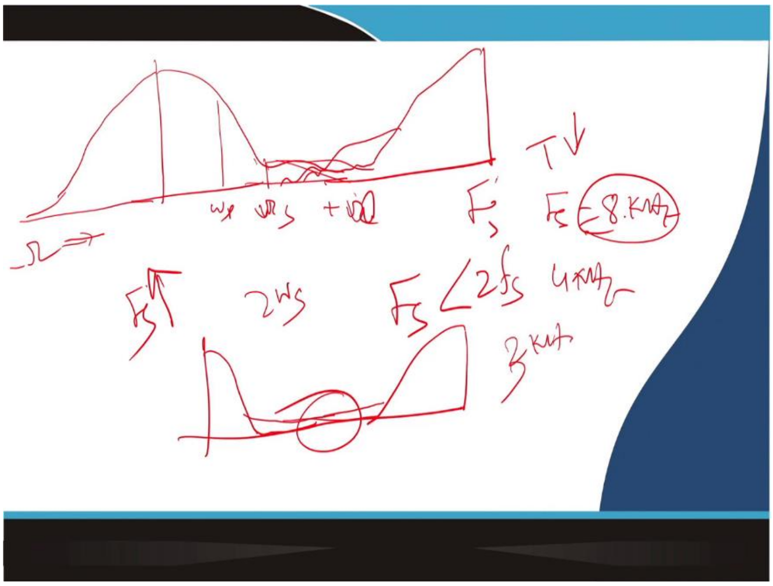
In communication, when you studied that let the ω m is my signal frequency so, if I say cos ω m multiplied by carrier frequency cos ω c, I get two components, ω c plus ω m ω c minus ω m so, at and all when I say ω m has a two-frequency component in plus ω m and minus ω m.

So, in the case of a single frequency, I said ω m, so this will look like this if the ω 0 to ω m is my signal frequency. So, if this is my signal frequency response, then once I sample using sampling frequency F s, I get this kind of series, there will be another again repeat at twice F s, here also another repeat at minus twice f s, so this will be infinite in both side because it is analogue infinite filter, IIR response.

Now, if you see if this length is 0 to F s, this distance unless this is my ω m so, if this ω m is twice ω F s is less than twice ω m, what will happen? This kind of thing we will get this is ω m, this is F s, this is 0, so there will be an aliasing effect on both sides another, again will be this side and again will be this side minus F s, so there will be the aliasing effect.

So, what is the requirement when I design an ω p? Let us say this is my ω p; if the ωp and here also there will be a ωp so, ωp is my cut-off frequency. Let us say this is my passband edge frequency is ωs.

(Refer Slide Time: 15:52)



So, suppose I want to let us take another slide. Suppose I want to design this filter; let us say my filter frequency response looks like this. So, this is my ωs, let us say this is my ω p. Now if this is so, this side is also the same thing as the negative side: this is minus ω, this side is minus ω, this side is plus ω. Now, once I multiply with F s once I sample it, then this ω becomes sampling small ω.

So, in that case, at F s, the same thing will occur. So, there may be an aliasing. So, if twice ωs is F s is the F s is so, F s is less than; if F s is less than twice the small cut-off pass band or stop band edge frequency, then I can say the both the filters stop band will be overlapped in natures so, there will be in overlapping. So, aliasing will happen. So, I cannot design a filter where there will be aliasing is there. So, to avoid aliasing, my F s should be such that so high that stop band attenuation is almost 0.

So, I required a filter like this, and my Fs are like this. So, I can guarantee the stop band attenuation is close to 0, so there is no overlap.

So, what is the limitation? The F s should be very high; that means T should be very low 1 by F s is equal to T. So, if I want to design a low pass filter, I have to think that what suppose I want to say that let us say I said F s is equal to 8 kilohertz, then maximum allowable stop band frequency is equal to 4 kilohertz, but the problem is stop band attenuation is 40 db so, if that attenuation is not 0, then also there will be a aliasing.

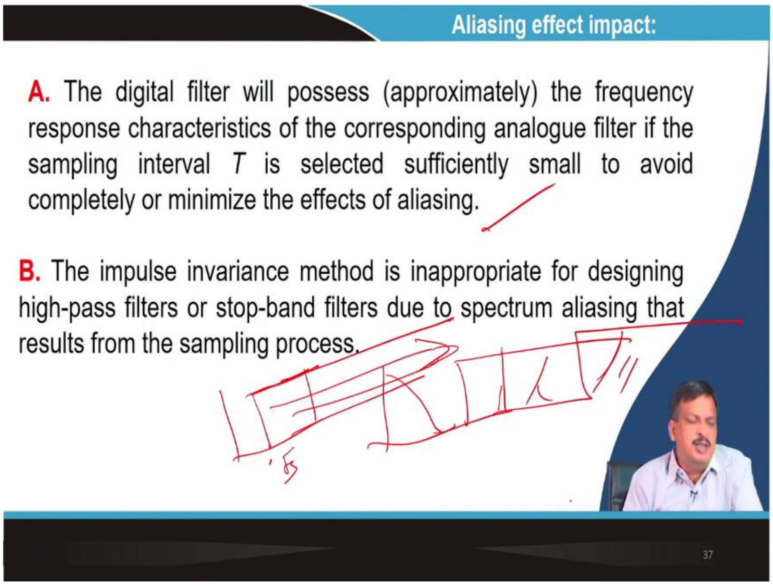
So, let us say if I want to allow it to sufficiently, I can only design up to 2-kilohertz loss of 2 to 3-kilohertz low pass filter if the sampling frequency is 8 kilohertz. So, for the impulse in variance methods, the main problem is designing a low pass filter, and I have to choose a sampling frequency well above or well high; that is the problem, okay? So, that is explained in mathematics like this.

So, aliasing occurs when the sampling frequency is less than twice the highest frequency component of ha(t). Is it clear? So, this will look like this: this is the picture. So, this is my H f. So, this is repeated at F s, again repeated at twice F s. So, if you see if this is my decaying like this and this is decaying like this, there is an aliasing portion there. I understand.

So, what is the mathematical representation? This is f, F s is equal to 0, one is it is a plus F s that means, f minus F s and it is this is minus F s so, it is a plus F s, this side negative side and this side positive side ok. So, the impulse in variance methods I can easily design the Z transform, but this T, T determines the aliasing effect because T is equal to 1 by F s.

So, if T is not sufficiently small, then aliasing can happen. To avoid aliasing, T should be sufficiently smaller, which means the sampling frequency should be sufficiently high ok.

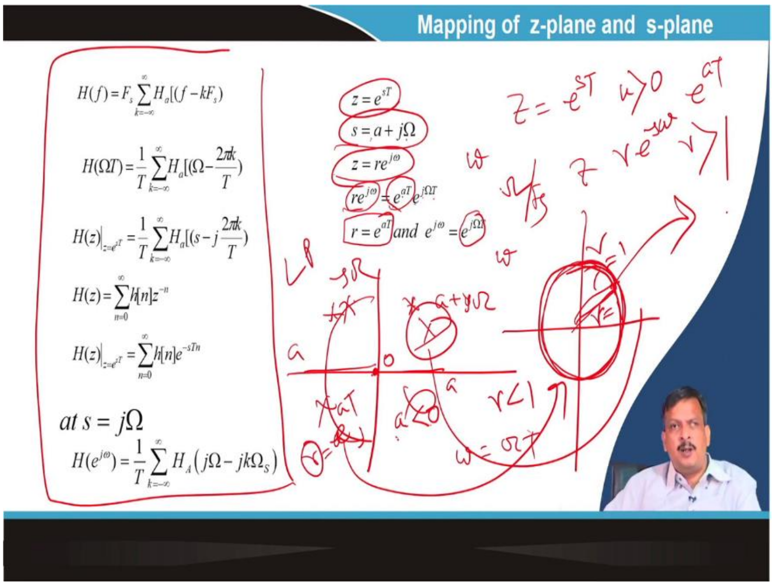
(Refer Slide Time: 20:28)



So, that is written here, and the same thing. But I see that I have an aliasing effect in the case of a low pass filter. However, in the case of a high-pass filter, what is the high-pass filter response? A high pass filter frequency response is this kind: This is your high pass filter. So, a high pass means all frequencies after a certain frequency.

Now, if I multiply by f s, what will happen? I will get the same Fs at the high pass, so here I also get, here I get; it does not matter if aliasing happens. So, I can easily design high-pass filters using impulse invariance methods.

(Refer Slide Time: 21:24)



Now, when mapping the z-plane to the s-plane, whether this means that the transform is stable or not, what is the mapping of the z-plane to the s-plane? I am not going into details on this side, but I have already explained. Is that okay? So, what is z? Z is equal to e in the power sT. If you see that z is equal to e^{pkT}, the position of the pole so, s is the pole position ok.

So, if now my s is equal to so, let us see this is my s plane, this is my s plane, so this is a real axis, and this is the j ω axis, and this is my z plane. So, the z plane is defined by r so, unit circle; unit circle means r equal to 1, so what are the stability criteria? All my poles must be inside the unit circle, ok?

So, the radius is r equal to 1. All poles must be inside the unit circle. How do I define a pole as nothing but $re^{j\omega}$ that is digit discrete, normalized discrete frequency, and when I define a pole in here so, this is nothing but a less, this is my pole. So, this is nothing but a plus j ω x, y coordinate.

So, s is equal to a plus j ω, and z is equal to $re^{j\omega}$. So, I can say since z is equal to e^{sT} so, r $e^{j\omega}$ is equal to e^{aT} so, s is nothing but a into j ω so, e^{aT} into $e^{j\omega}$ T. So, the real part, the magnitude part r is related to e^{aT} and j ω is related to $e^{j\omega}$.

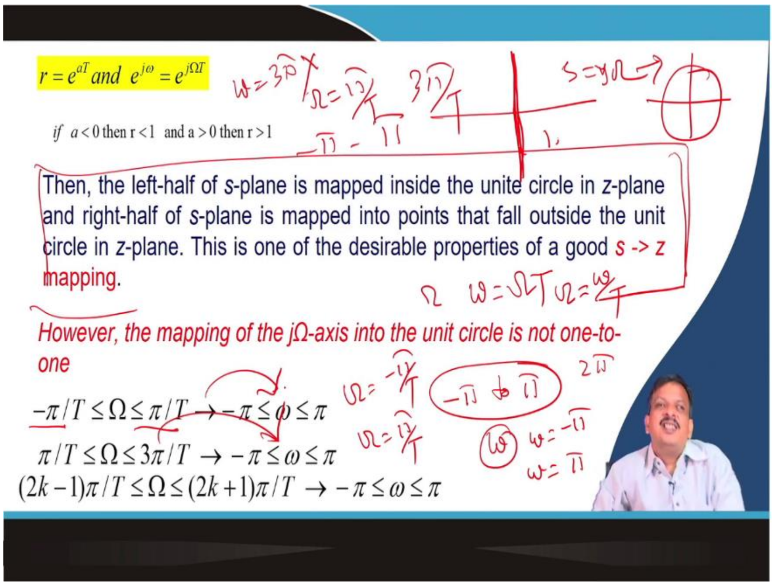
That is why if you see this ω is normalized discrete frequency, this ω is analogue frequency; so, that is why the analogue frequency the once I say that this is divided per sample F s T means 1 by F s, then I can say it is not radian per second, radian per sample which is nothing but a small ω.

So, I know a small ω is nothing but a ω T, and r is equal to a to it, and r is equal to e^{aT}. Now, why did I say when I designed the Laplace domain filter that all poles must be on the left-hand side, and all poles must be on the left-hand side? What is the meaning? This means that the value of a is less than 0 because this is the 0 point, so this is a negative. So, if the value is less than 0, the value of a is less than 0, then I can say r can be within 1, so r is less than 1.

At an equal to 0, the r is equal to 1. So, an equal to 0 means at origin that r equal to 1, s plane origin is nothing but a z plane unit circle. So, if all my poles are on the left-hand side of the Laplace domain, I can easily say that all poles are guaranteed within the unit circle. Now, if any pole in the right-hand side of the Laplace domain is mapped to outside the unit circle because if a is positive, a is greater than 0, then e^{aT} is always r is always greater than 1. Is it clear?

So, that is the mapping of the z plane and s plane that you have already studied in z transform and s transform Laplace transform; I just repeat those things.

(Refer Slide Time: 25:57)



So, we see again that s z is mapping, so that is the mapping which I explain is written in here. So, the left-hand side s-plane plane poles are within the unit circle of the z-plane, and the right-hand side pole is outside the unit circle, and that pole is ok. Now, how the mapping of j ω axis, how do I map the j ω axis?

So, what is the meaning of this? That suppose this is my ok if my all r is in the j ω axis; let us say my all poles are in the j ω axis, so that means s is equal to j ω; then how will this map to the z plane? So, let us say the real part is r equal to 1, then how is it mapped to the z plane or the z plane to the s plane? How is it mapped?

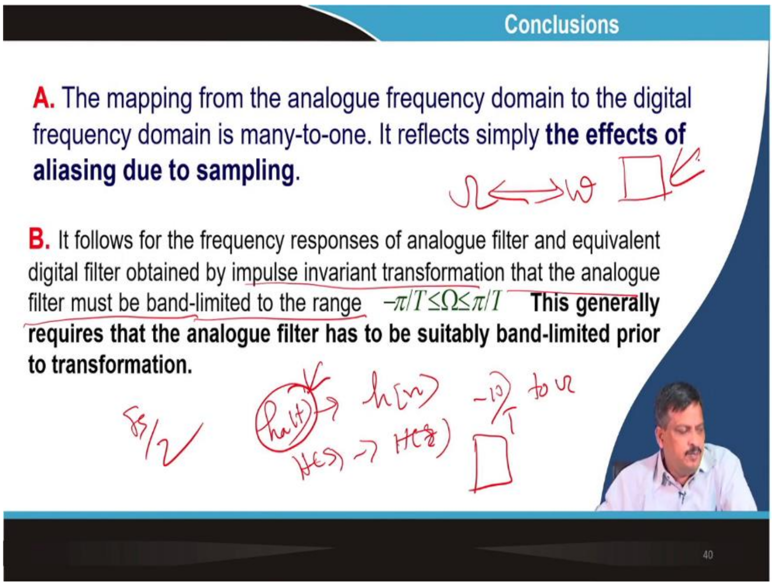
I know in discrete frequency, the frequency range is minus π to π, which is the normalized discrete frequency range of the sampling. So, 2π is the maximum frequency, half 2π half is π so, maximum rate of oscillation is possible so, the value of ω is minus π to π ok or not.

Now, if you see if it is π by T if the ω is equal to what is ω? What is small ω is equal to capital ω into T. So, I can say that capital ω is equal to analogue ω by T. So, if this ω is equal to minus π by T, then I can say small ω is equal to minus π. If it is equal to π by T, then I can some ω is equal to π. So, this minus π by 2π by t is all mapped with minus π to π.

Now, let us say small ω is π by T and also ω 3 π by T. So, π by T, 3 π by T, which is more than π, let us say. So, if it is 3 π by T, then the so, what is the value of the small ω is equal to ω 3 π, but 3 π cannot be because ω is varied from minus π to π only.

So, this is also mapped with these things. So, there is a many-to-one mapping. Is it clear? So, there is a many-to-one mapping; this is called aliasing because all are compressed, so they are overlapping in nature. Due to this, this aliasing is happening ok.

(Refer Slide Time: 29:24)

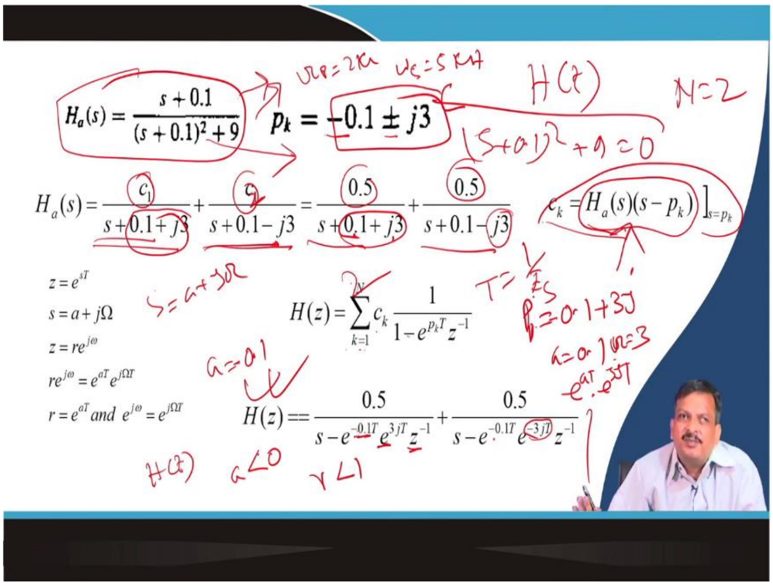


So, in conclusion, we can say the mapping from the analogue frequency domain to the digital frequency domain is many-to-one. It reflects simply the effect of aliasing because analogue frequency ω capital ω, discrete frequency small ω, this mapping is many-to-one. It follows the frequency response of the analogue filter and the equivalent digital filter obtained by impulse invariant transformation that the analogue filter must be band limited.

So, that means, ha(t) when I say ha(t), I converted to h[n] or H s I converted to H(z) that ha(t) must be a band limited to avoid aliasing, which is nothing but a minus π by T to π by T that means, F s by 2, highest frequency component that is the property of analogue to digital conversion, highest frequency component must be F s by 2.

So, what is required? If you see any ADC, the first step is an anti-aliasing filter. So, if I use the impulse invariance method, I have to guarantee that the design filter is band limited, which means it is guaranteed that, like an anti-aliasing filter that means, F s is sufficiently high so that there is no aliasing that happens ok.

(Refer Slide Time: 31:07)



Let us give an example and then, I stop, and then, I can. Ok, I can continue with the other methods also, and then I will stop. Let us say I have a transfer function in this one. I want to use impulse invariance methods to convert H(z). So, what is the root? The root is nothing but a s plus 0.1 whole square plus 9 equal to 0. If I solve it, I get this is the root, so this is the root. How many roots are there?

The order of the filter N is equal to 2 ok. So, I can say summation, this summation k equal to 1 to 2, so I can say it is nothing but a c1, this will be c2 sorry c1 by p1 let us say the p1 is equal to plus and p2 is equal to complex z minus so, c1 by this one plus c2 by this one. Now, how can I solve it by an algebraic expression? Is the expansion at ck equal to H(s) into s minus pk at s equal to p k? I explained this method earlier, and it is okay.

From there, I can find the values of c1 and c2, 0.5 and 0.5, and this is my Laplace transform. Then, what is I known? I know z is equal to e^{sT}, or I can say H(z) is equal to k equal to 2 ck divided by 1 plus e^{pkT} z^{-1}.

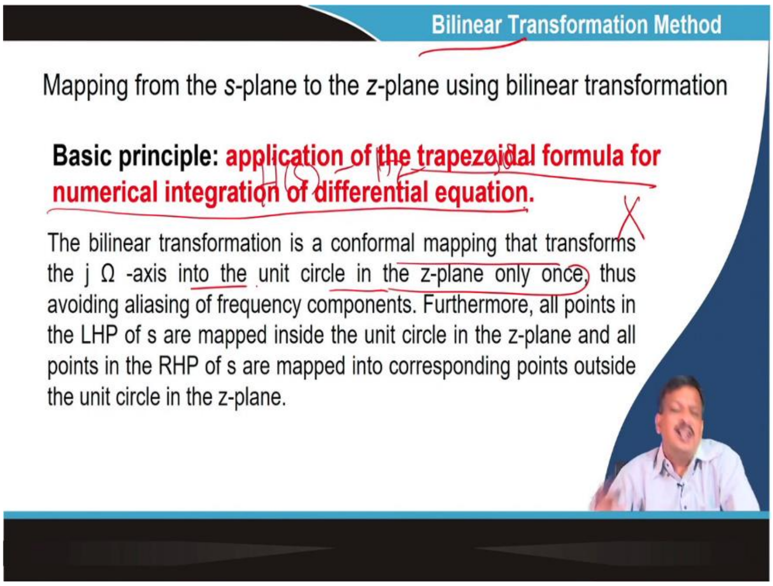
So, I can say that from this equation, if you see that p k, the value of pk p1 is equal to 0.1 plus 3j. So, I can say a is equal to 0.1 and ω is equal to 3 because s is equal to a plus j ω kind of thing, so I write down s equal to a plus j ω ok.

So, I can say e to the power; e^{aT} into e^{3jT}, so it is nothing but an e^{aT}. Why is minus because this is minus, so this is minus 0.1, a is equal to minus 1 minus 0.1 into T, and e^{3jT} into z^{-1} and another pole is this one, only this one is minus so, this one is minus now, I get the H(z).

Now, if you see the value of a is less than; a is less than 0 because a is equal to 0 points; minus 0.1, so that is why I can say r is less than 1, so I can say H(z) is within that unit circle ok. So, I have designed H(z). So, once I know H s, I can get the H(z) using impulse in variance methods.

Now, if I say my cut-off frequency is this design, This cut-off frequency will be defined by the position of the pole, so if I say my cut when I design this transfer function, I take the cut-off frequency ωp is equal to, let us say 2-kilo hertz and ωs is equal to 5-kilo hertz, then if I say if you to avoid aliasing so, around above 5-kilo hertz so, I can say the t will be 1 by F s, F s should be more than 10-kilo hertz ok. So, only the stop bands are aliasing, but at least the transition bandwidth is not aliasing ok.

(Refer Slide Time: 35:31)

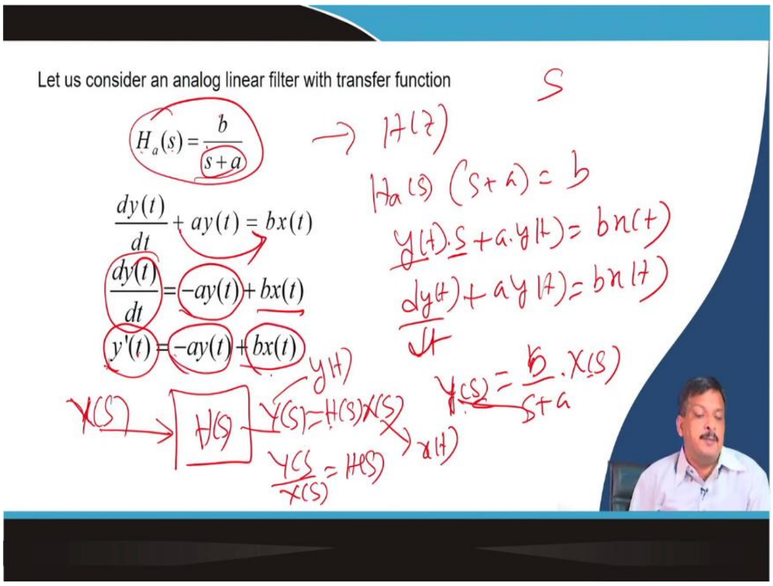


Now, the bilinear transformation method. Mapping from the s-plane to the z-plane, I have already said that ok. So this is called you can say the this is called I said impulse invariance method. Then, there is another method, which you can say is called the bilinear transform method.

So, what are the bilinear transform methods? Bilinear transform is a mapping of the j ω axis to the unit circle in the z-plane only once. So, what is the problem with impulse invariance methods? There is a many-to-one mapping of the j ω axis. In bilinear transform, this problem is solved, so the aliasing effect is not there if I use the conversion of H s to H(z) using bilinear transform.

So, how do we do that? We apply the trapezoidal formula for numerical integration of the differential equation; let us see it, and then you can understand.

(Refer Slide Time: 36:51)

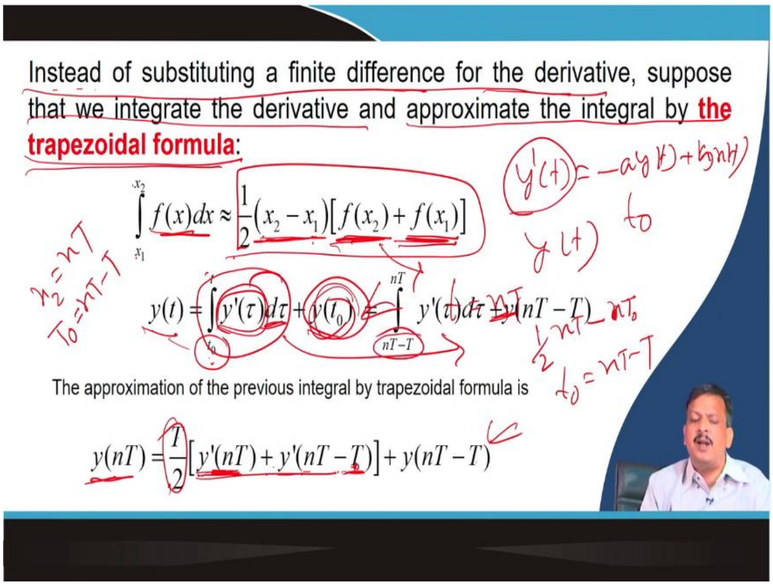


Let us say I have an H s a b by s plus a, so I have to convert H(z) using bilinear transform, a trapezoidal formula I will come to that. So, if I say that H of a s is equal to so, I can say H of a s into s plus a is equal to b. So, now, if you see, if I have a system which has an H, s is my transfer function, if I apply X(s), I get Y(s), so Y(s) divided by X(s) is equal to H(s). So, I can say $Y(s) = H(s) \cdot X(s)$,

So, if the Y(s) in the time domain is y(t), X(s) in the time domain is x(t) so H(s) is defined by b by s a. So, I can easily write down this y(t) into s plus a into y(t) is equal to b into x(t) because this is nothing, but I can say y(s) capital Y s is equal to b by s plus a into X(s). So, this will be here in X(s).

So, Y(s) gives me the y(t), which is multiplied by s. What is the meaning of s? s is called Laplace transform, s is nothing but a differentiator. So, I can say this is nothing but a dy t by dt plus a y(t), which is equal to b into x t. So, I can say that the day t by dt takes this one this side minus a t into bx t so that I can see the differentiation of t is nothing but a minus a t plus bx t.

(Refer Slide Time: 39:11)



So, what is the meaning? Instead of subtracting a finite difference from the derivative, suppose that we integrated the derivative and approximated the integral using a trapezoidal format. So, the derivative is nothing but a here you can see; it is a difference differentiate. Instead of that, if I do the integrated so, I can say y(t) is equal to t 0 to t y prime t plus d t, so this is ok. Think about it.

Trapezoidal, what is the trapezoidal formula that formula said that

$$\int_{x_1}^{x_2} f(x)\,dx \approx \frac{1}{2}\left[(x_2 - x_1)\left(f(x_2) + f(x_1)\right)\right]$$

is called trapezoidal formula. Now, what I said is instead of differentiation of y by t is equal to minus ay t plus a t a b into x(t) I said let us I integrate this one to compute y(t); instead of taking the difference, I take integrate differentiation of t to compute y(t) directly.

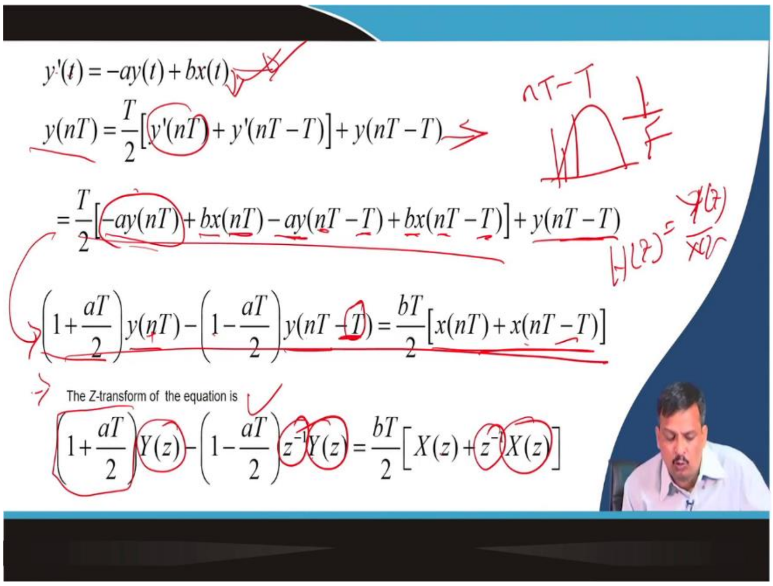
So, what do I do? y(t) is equal to compute your t 0 to t first derivative into dT d τ into y plus y 0 t so, this is the initial value y 0 is the initial t 0 is the initial value. Now, I said that t is nothing but a n of, so T is digitized, so I can say y of nT is nothing but a T by 2 y prime nT. Just put it this formula.

So, instead of T, I put n into T, so it is t 0 to t. So, I can say this integration is nothing but a half of x2, x2 is nothing but a T, n of T minus x1 is nothing but a t 0, so t 0 is nothing but a n of T 0, so, there will be a T 0. So, it is T 0 is nothing but a 0th instant so, let us say T 0.

So, these differences, so, x2 minus x1, give me the half of T and is nothing but a y prime nT plus f of x2 and in plus f of x1. So, f of x2 is y prime n 2; nT, this is f of x, f of x dT plus this is x2, so I can say y prime nT plus y prime nT minus T plus t 0 is nothing but a nT minus T, T minus T, t 0 is the initiation time ok. Is it clear?

So, if it is so, in this case, my x2 is equal to nT, and my T 0 is equal to nT minus T ok. So, x2 minus x1 will give me the T, so that is why T by 2 is into x f of x2 plus f of x1, and this will be here. Now, if it is that, then instead of y prime nT, I can say that so, this is t 0 is nT minus T ok.

(Refer Slide Time: 43:36)

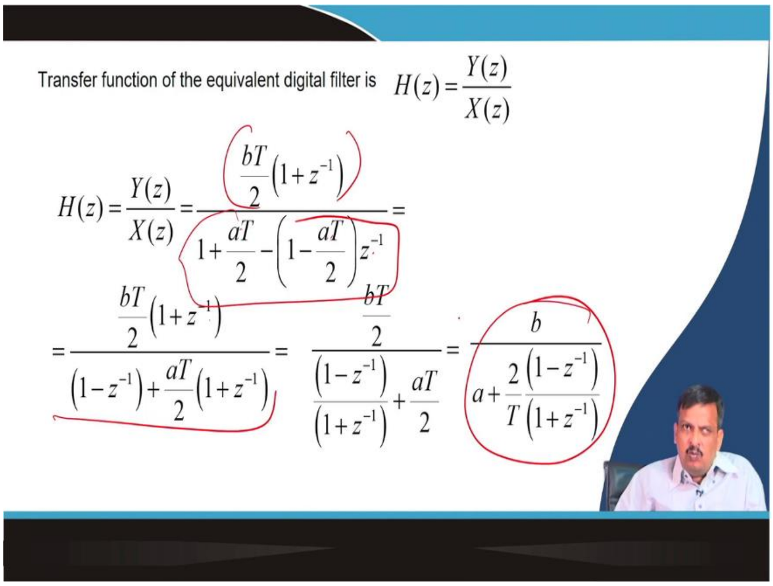


So, I can say the prime t is equal to this. I only know this one; we already know this one, and we have already derived it. So, I can say that I know this is ok. Then this I derived right now. Now, instead of this one, I replace it with this one so I can say y prime t, y prime nT is nothing but a minus a nT plus bx nT plus minus ay prime n; nT minus T plus bx nT minus T plus y nT minus T. Then, I can say y of nT, just simplify this one, this will come to simplify, just simplify this one.

Now, if I take the Z-transform on both sides, why do I simplify this one? I get this one, so I take the Z-transform for all the y on one side and all the x on one side now. So, y n so, this is an a plus aT divided by 2 into Y z. Now, if you see one, this is ok; this is minus T, what is the minus T? nT minus T. What is T? T is nothing but a 1 by F s, one sample duration, this is T distance between the two samples.

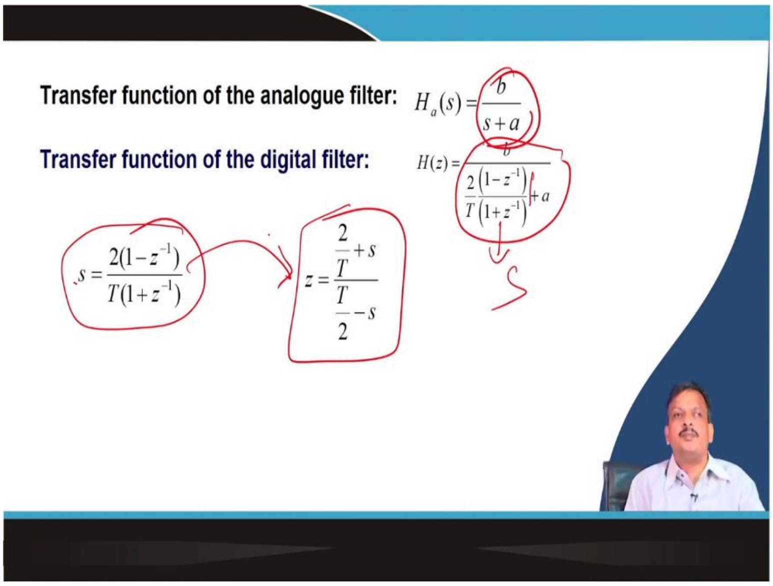
So, what does minus T mean? It is delayed by one sample. So, I can say it is delayed by one sample, so it is nothing but a z^{-1} into Y z. Y z is delayed by one sample. Similarly, here, this is also minus 1 X z. Now, I know H(z) is equal to Y z divided by X z. Do that, do that Y z by X z, so it will be bT by 2 into 1 plus z^{-1} divided by this one.

(Refer Slide Time: 45:43)



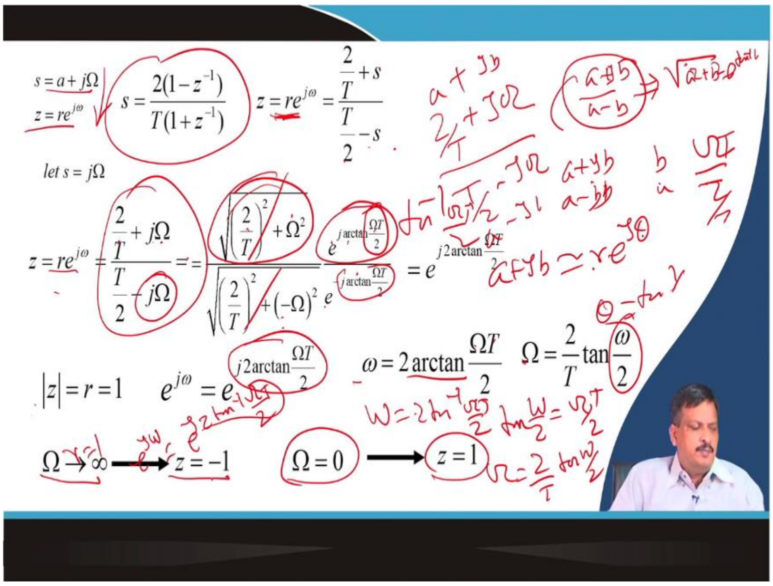
So, I get H(z), just then, the just simplification bT by 2 a 1 plus z^{-1} into aT by 2 so, all aT will be z^{-1} will be there and so, all if I make that simplify it, it will come in this form b by a plus 2 into 1 minus z^{-1} divided by 1 plus z^{-1}.

(Refer Slide Time: 46:22)



So, what I have taken as an H s? So, H s is equal to b by s plus a. H(z) is equal to b by this plus a. So, I can say this is nothing but a s, if this is s, then I can from this equation, I can calculate what is z ok.

(Refer Slide Time: 46:44)



And then mapping, then else is what? The mapping between s-plane and z-plane. So, s is equal to a plus j ω, z is equal to $re^{j\omega}$ so, s is equal to this one I know. So, I can say z is equal to $re^{j\omega}$. So, what is z? z is nothing but a 2 by T plus s divided by 2 by T minus s. So, r $e^{j\omega}$, which is nothing but a T by 2 plus s T by 2 minus s.

So, I know this one is nothing, but how much is it? So, 2 by T plus j ω divided by T by 2 minus j ω. So, this is, let us say, a plus j b, this is a minus j b. So, a plus j b divided by a minus j b. If I want to return in polar form so, this will be amplitude is nothing but a square plus b square e to the power; e to the power j $\tan^{-1}$ b by a.

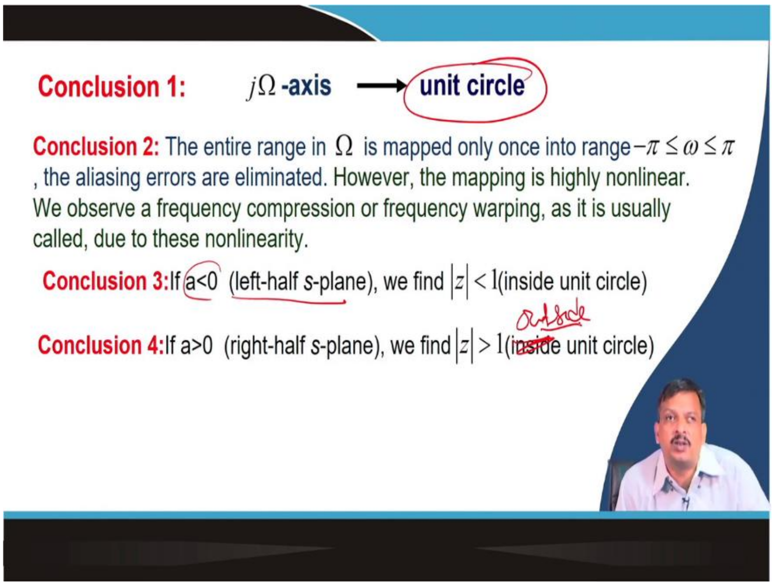
So, this is a plus j b. How can I write? a plus j b I can write in term of r into j θ so, where θ is equal to $\tan^{-1}$ b by a and r is equal to magnitude root over of a square plus b square, this is a this is b so, this is the magnitude and this is the $\tan^{-1}$ j $\tan^{-1}$ ω T by 2 because b is ω so, b $\tan^{-1}$ b by a, b is ω and a is 2 by T so, ω T by 2. So, j arctan means $\tan^{-1}$, $\tan^{-1}$ ω T by 2, so, arctan ω T by 2.

This is only the minus part there, so only the θ will be negative, and the amplitude will be the same; a plus j b and a minus j b both have the same amplitude; only the one is θ is positive, another one is θ is negative so, $e^{j\theta}$ divided by $e^{-j\theta}$. So, when I do it, $e^{2j\theta}$ is nothing but a.

So, which is equal to so, r is equal to 1 because this one is cancelled. So, I can say that r is equal to 1 and $e^{j\omega}$ is equal to e to the power j 2 $\tan^{-1}$ ω T by 2 understand. So, I can say ω is equal to nothing but a 2-$\tan^{-1}$; ω is equal to 2 $\tan^{-1}$ or arctan ω T by 2, $\tan^{-1}$ 2 $\tan^{-1}$ ω T by 2.

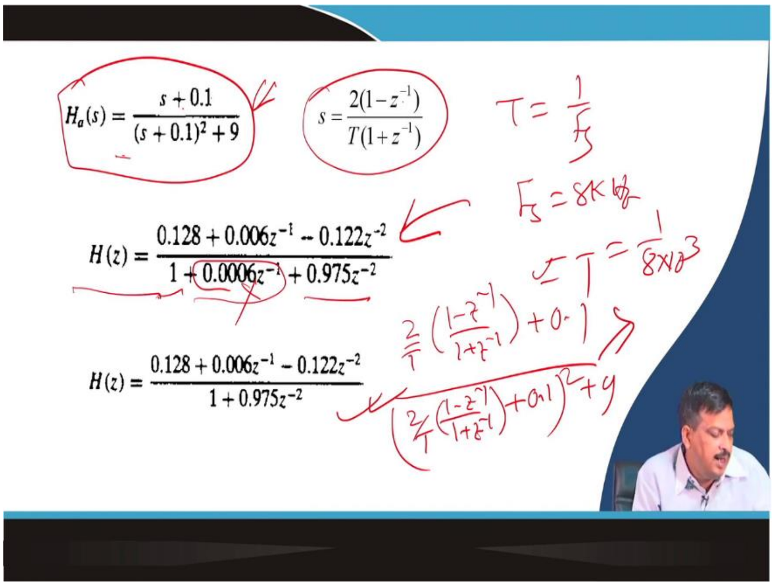
So, what is if I say this capital ω so, it is nothing but a ω by 2 tan is equal to ω by T ω T by 2. So, I can say ω is equal to 2 by T into tan ω by 2. Now, if you see ω is equal to 0, then z is equal to how much? 1. If ω is equal to infinity, z is equal to minus 1. So, z can vary from 1 to minus 1. So, whatever the value, I can say so, z always z will be within that unit circle; infinite frequency can also map within that unit circle, so the problem of impulse in variance methods has gone away.

(Refer Slide Time: 50:42)



So, this is the conclusion. So, the j ω axis is within that unit circle, I have proved. So, the entire range of ω is mapped into minus ω to plus ω, so there is no error, and there is no aliasing is there. Now, if a is less than 0, left-half plane, then we can say the mod of z is equal to less than 1 inside the unit circle. If a is greater than 0, then the mod of j is equal to greater than 1, so it is outside that unit circle. So, this will not be inside; this is outside. So, this is a bilinear transformation.

(Refer Slide Time: 51:19)



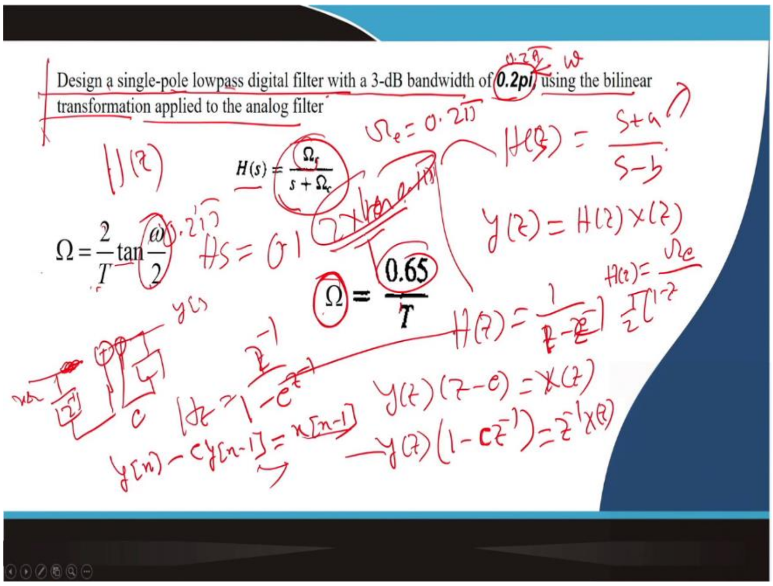
Now, take an example to help you better understand. Let us say H s is equal to s plus 0.1, the same example as that impulse in the variance method and here, you take the same example, only you have to put s equal to this one where T is equal to 1 by F s. Now, let us say I say the sampling frequency F s is equal to 8 kilohertz, then you know the value of T, T is equal to 1 by 8 into 10 to the power 3.

So, I get the value of the T; then I know 2 by T value I know so, that value is constant, then 1 plus z^{-1} divided by so, I can say this is nothing but a let us say 2 by T into 1 minus z^{-1} divided by 1 plus z^{-1} plus 0.1 divided by 2 by T; 2 by T 1 minus z^{-1} divided by 1 plus z^{-1} plus 0.1 whole square plus 9. Now, just simplify, and you will get this one. Simplify it, and you will get this one. I think T is equal to the 8-kilo hertz I have taken; you can simplify it.

If you see it, then you can also simplify that 0.00 z^{-1}, so this is not a significant thing. So, I can write 1 plus this one, so that is the answer. So, that is called bilinear transformation. Simply, s is equal to 2 by T 1 minus z^{-1} 1 plus z^{-1}.

So, in summary, what can I say? So, when I design an IIR filter using the analogue filter approach, I will first my job is to form the given specification of the filter and determine the Laplace transform of our Laplace transform function of the filter so, H s I have to derive using Butterworth methods, Chebyshev methods or elliptical methods, anyone methods I can use, I can design that H s. Once I get H s, either I use impulse in variance methods or bilinear transformation methods to find out the H(z).

(Refer Slide Time: 53:45)



Once you get the H(z), let us say you get the H(z) like this; this is another example: let's say H(z) is equal to get s plus a divided by s minus b; then how do I implement it? Very simply, y z is equal to H(z) into x z. So, let us say once s, this is H s now, let us say I after H s, I take the bilinear transformation I can compute H(z), H(z) is equal to, let us say 1 by; let us say 1 by z minus c. So, I know y z is equal to H(z) into x z, so y z multiplied by z minus c is equal to X z.

So, y z multi, so this will be 1. This will be in the form of 1 minus z^{-1} unless I have to take it this way: H(z) is equal to take z common, so it is equal to 1 by 1 minus c z^{-1} and z^{-1}. So, I multiply both sides by z^{-1} so, it will be 1 minus c into z^{-1} is equal to z^{-1} into X z. So, I know that it is nothing but a y of n minus c into y of n minus 1 is equal to x[n] minus 1.

Implement in structure 1 or structure 2; this is x[n], delayed by one sample, added with this one so, this is not x n is not there, only n minus 1 and then, goes to this is y of n delayed by one sample multiply by c, added with this one, structure 2 or structure 1 whatever you can do that.

Here is a real-life example. Design and single-pole low pass digital filter with a 3-dB bandwidth equal to 0.2π using bilinear transformation applied to the analogue filter H s are equal to this one. So, what is ω c cut-off frequency? 0.2π. So, what is this one? H(s) is equal to ω c. So, what is ω? Ω is nothing but a 2 by T tan ω by 2 so I can say ω.

So, this cut-off frequency is normalized discrete frequency. So, this is small ω so I can capitalise ω as analogue frequency is nothing but a 2 by T tan ω by 2. So, ω is nothing but a 2; 0.2π so it is nothing but a 0.2π by 2, so it is 0.1 π. So, tan 0.1 π 2 by T. So, 2 into tan 0.1 π or 0.1 π divide; 0.1 π divided by 0.1 π divided by T. So, this value is 0.65 and T.

Once I get this ω value, then what can I do? What do I require? I said that to design a single pole low pass digital filter, I have to find out H(z), so what is H(z)? H(z) is nothing but a H(z) is nothing but a ω c divided by T by 2 into 1 minus so, s value of s you can put there and you can calculate that H(z), you do it in yourself ok.

So, I will stop here because the length of the video is increasing because this is the concept, which is there; you have just to read it and go through this video, understand what bilinear transformation is, how you convert it, and you do it ok.

Thank you very much.