Signal Processing Techniques and Its Applications
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Lecture - 25
Two Dimensional Discrete Fourier Transform

So, as we saw last week, we talked about the properties of the Discrete Fourier Transform. So, we have said periodicity, all kinds of things we have discussed, and we have talked about the circular convolution, but all we have applied on a single dimensional signal. So, we said x[n]. So, x[n] is in a one-dimensional signal; if you see n, it varies from 0 to n minus 1, let us say in the x direction. This week let us start with what will happen if my signal is two-dimensional.

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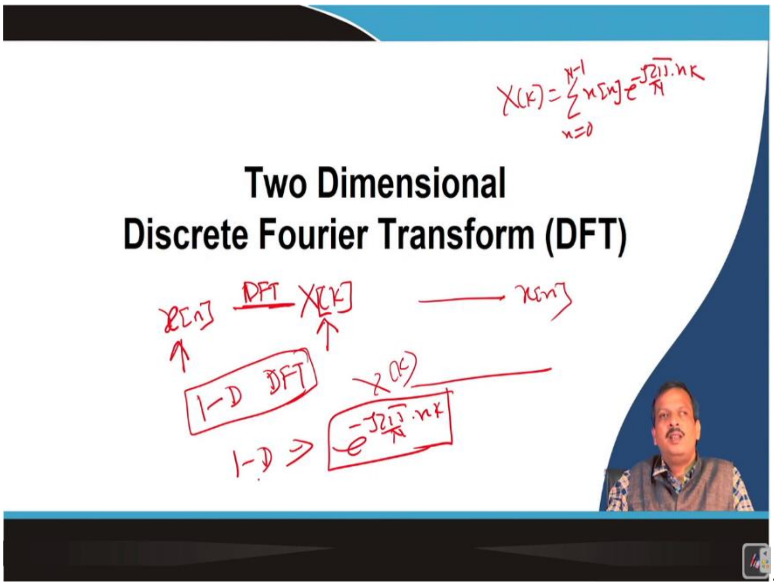

So, what is there? instead of x one-dimension, I have a two-dimension. So, what should the Fourier transform or discrete Fourier transform be for a two-dimensional signal? That is called the two-dimensional discrete Fourier transform; this is a very important issue. So, as far as per last week's number, you can say 5 or week number 4, we talk about x[n], which is only one dimension, and we talk about the DFT of an x[n], discrete Fourier transform of x[n] is X(k).

So, if the signal is one dimension in the time domain and the frequency domain is also X(k), k is varying in one dimension. So, I give a sequence. I get a sequence x n, which is also a sequence capital. X(k) is also a sequence. So, that is called 1D one-dimensional discrete Fourier transform 1 DFT. So, what is one-dimensional discrete Fourier transform? As you said, X(k) is equal to n equal to 0 to N minus 1 x[n] e to the power minus j2π by N is the resolution in one-dimension into n k. So, $e^{-(j2\pi/N)nk}$ is the Fourier basis function, which is called the one-dimensional basis function.

So, that means, how do you do that? We generate different frequency components and convolve them with the original signal to find out whether the energy is there or not; that is my one-dimensional discrete Fourier transform.

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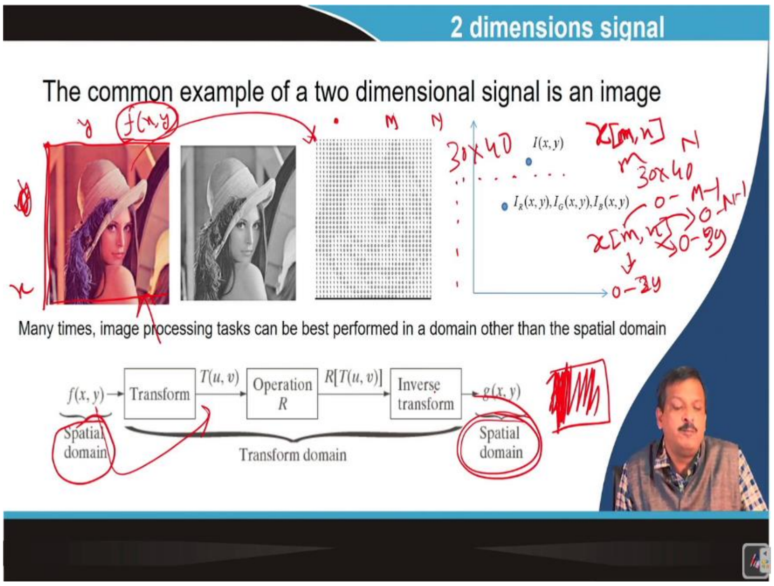

Now, if I say my signal has changed from one-dimension to two-dimension, So, I have a signal with, let us say, x[n]. So, x[n] has a two-dimensional; not only n, but I also have an m and n. So, x[n] is a series. Now I have a matrix m by n matrix. What is the example? An image two-dimensional signal is an image. So, if I say an image, the image is called a spatial signal because it is if the image of the f is called a signal. So, the image is a function of x and y; x has an x-axis, it has a y-axis.

Mainly it is started from here. We always start the image from the top corner. This is called the x-axis, this is called the y-axis, and this is called the x-axis ok. So, that is why f and x y is called spatial signals, it has a space that is plane x and y two-dimension because the concept you know that the discrete version of an image, discrete version of image is nothing but a pixel an image is represented by a pixel and every pixel has a coordinate in spatial domain x and y.

Let us say that in the x-axis, I have discretized, I have taken the m number of samples, and in the y-axis, I have taken the n number of samples. So, if I say a pixel is a signal. So, pixel pixels vary along the x and y dimensions; I discretize the x and y dimensions. So, if I take an image, let us say it is a 30 by 40 pixel image. So, in the x dimension, I have a 30 pixel, and in the y dimension, I have a 40 pixel.

So, how do I represent the signal? It is nothing but a 30 cross-40 matrix. So, I can write it down as nothing but an x[m, n] where m varies from 0 to let us 30 minus 1, 29, and n varies from 0 to 39. So, m varies from m; if this is M and this is capital N, this is capital N, M varies from 0 to M minus 1, and n varies from 0 to N minus 1. So, I have a signal in the spatial domain. People say why I should be interested in image transformation in the frequency domain of the image. Why do I do that? Many times in signal processing, it is required to represent the signal in the frequency domain.

So, I have a signal in a spatial domain, which is the original image; I can transport it to a frequency domain and do some operations and again take the inverse transform, and again, I back the spatial domain signal. So, suppose I have an image I want to remove the high-frequency component of that image. So, I think all of you know what is the concept of frequency in a spatial domain signal and the concept of frequency in a one-dimensional signal. You know how the signals vary. So, if more frequency means a higher rate of oscillation.

Similarly, in the spatial domain also more frequent. So, if I take only 1, let us say I take only a white image. What is the frequency? This is nothing but the same single frequency or DC value, or I take a black image, the whole image is black, the same black colour. So, all pixel value is the same value: 0 0 0 0 0 0 0. So, it is nothing, but it is there is no frequency. once the value of the pixels is changing in the x and y direction, then I can say the image has a frequency. So, if the change is very big, then we call the high-frequency image.

So, suppose I want to remove that high frequency. So, I converted the image to a transform domain using a two-dimensional discrete Fourier transform, then did some operations on it and then again inverse transform to get back the spatial domain image. So, how do I do that two-dimensional Fourier transform or discrete Fourier transform?

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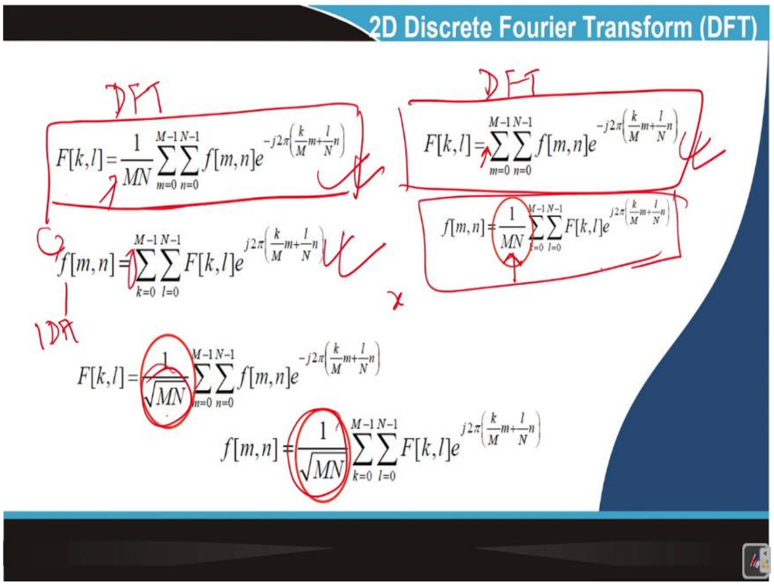

So, the two-dimensional discrete Fourier transform, DFT, is defined by this. This or this both you can write. What is the difference in both cases? Normalization is done in the frequency domain. So, when we converted to the time domain, this is inverse IDFT; the normalization is not there, or I can say that I do not include the normalization here.

So, I include the normalization here when you transport to the spatial domain again. I can use does not matter, or I can do the normalization this way also: multiply by 1 by root over MN and again, time domain also multiply by 1 by root over MN. So, it is nothing but a gain. DFT gain is okay; forget about that. Then what is happening?

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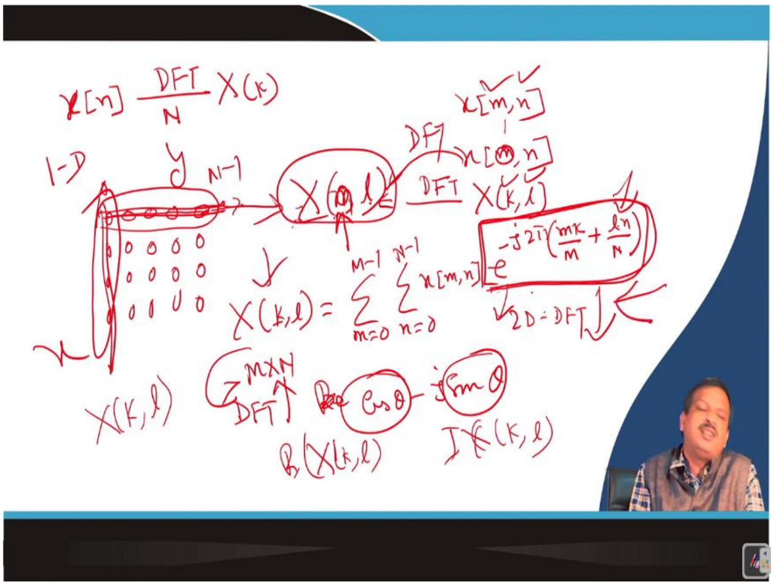

So, if I have a signal in, so, I have a signal in, if I have a signal in x[n] if I take the N-point DFT. So, signal a dimensional signal, I get X(k), which is called 1D, only the one-dimension signal is changing. Now, I have a signal, which is x[m, n], and both dimensions exist. So, if both the dimensions exist means, let us first say that it is a single dimension; let us say x[m], m is 0, and n is varying. So, what is an image?

Image is a value on this axis, and the value on this axis ok or not. So, this is a spatial image. So, I first take those samples, let us say those varying from 0 to N minus 1 and do the DFT. So, let us say I have taken. So, what should my DFT be, that I will get an X(k) for every m, let us m equal to 0, and I get X(l)? So, I transfer to DFT discrete Fourier transform, I get this X(l). Now, what is this signal? This signal is nothing but a again signal that directs this signal to be a frequency transform, but this direction is not a frequency transform.

So, in the first row, I do the frequency transform. So, let us write down the m here; for every m, I get the X[m], l; then I do the m y. For every l, I do the frequency transform. So, I take again one-dimensional DFT for every m, then I get X(k) and l. Since the spatial domain signal has two directions, my frequency domain signal also has two directions, k and l. It is also an n matrix.

So, if I say what the final representation should be, So, I know X(k) and l should be, you do not have to remember it is nothing but a. So, first, I said the n varies from or m varies from 0 to M minus 1 and then n varies from 0 to N minus 1 x[m, n] $e^{-j2\pi}$. So, the frequency resolution if I say I take m and m cross n point DFT; that means the x-axis is the m order is M and the y-axis order is N. So, this is my y-axis, and this is my x-axis. So, the x-axis order is M, and the y-axis order is N. So, the x-axis I have the value k.

So, I can say m k divided by M plus y-axis. I have a value l, l into n divided by capital N; this is my basis function of 2D DFT, is it clear? So, one-dimensional is m k by n two-dimensional, I have to again add another dimension. So, first column-wise, then first row-wise, then column-wise or first column-wise, then row-wise, whatever I can do. So, this is my two-dimensional basis function. Now, this basis function has two parts. One is the real part, and so you know that $e^{-j\theta} = \cos\theta - j\sin\theta$. So, this part is a real part, and this part is an imaginary part.

So, the basis function has two parts: one is the real part, and the other is the imaginary part. So, you calculate X(k) l if I plot only the real part of X(k) l and the imaginary part of X(k) l or the real part of only the basis function, either this or let us only the basis function real part if I plot it will look like this.

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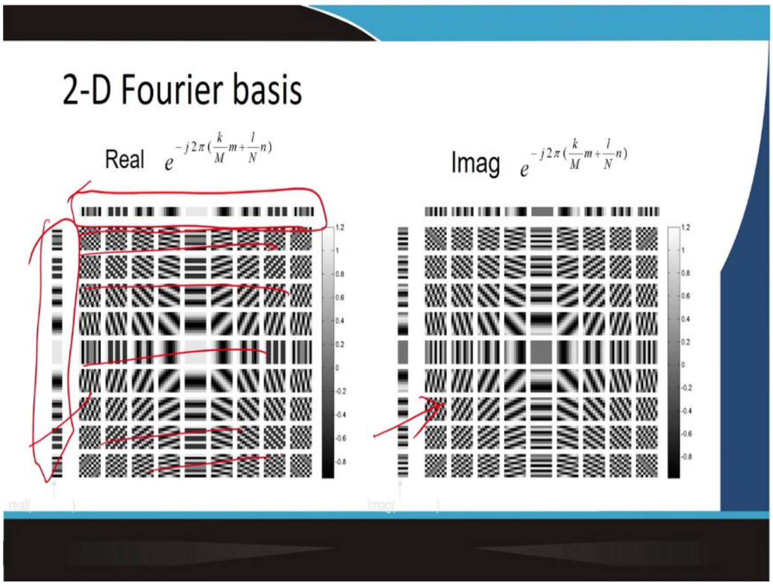

If I plot the imaginary part of the basis function, it will look like this: the real part is also 2D and the imaginary part is also 2D, which is the Fourier basis,? So, this is a one-dimensional basis, and this is another dimensional basis, so there will be just repeated ok. So this is called two-dimensional DFT; let us do an example, and then we go for that other part.

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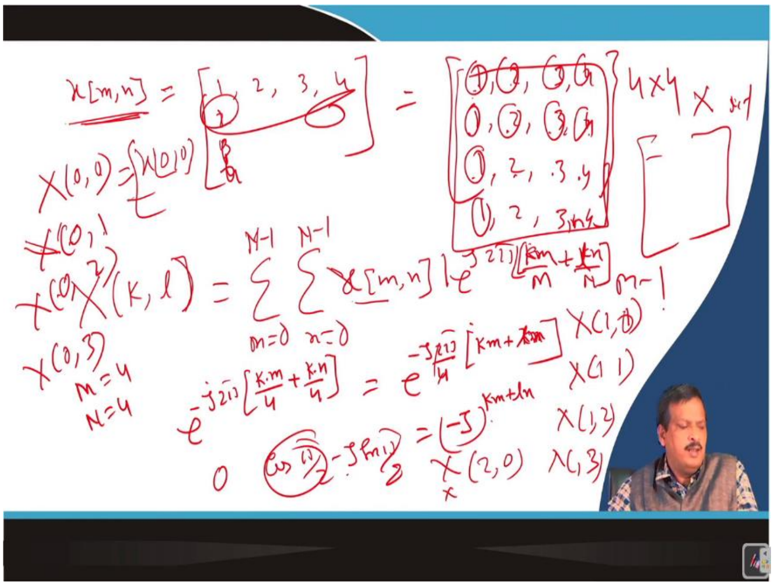

So, let us say I have a signal x[m] n; let us say I have an image whose pixels are, let us say 4 by 4 matrix. So, let us say 1, 2, 3, 4, and this axis is also 1, 2, 3, 4. So, 1, 2, 3, 4, let us say this axis, then 1, 2, 3, 4 again let us say 1 sorry, 1, 2, 3, 4; again this also 1, 2, 3, 4; again 1, 2, 3, 4; 1, 2, 3, 4. So, this is a 4 by 4 image; those are the value of the pixel value of the pixel.

So, 4 by 4 image each pixel is represented by 1, 2, 3, 4; 1, 2, 3, 4; 1, 2, 3, 4. So, this is nothing but an x[m] n. So, can you come to compute X(k) l? So,

$$X(k,l) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x[m,n]e^{-j2\pi\left(\frac{km}{M}+\frac{ln}{N}\right)}$$

So, what I said is that the value of M is equal to 4, and the value of N is equal to 4. So, what is the value of e to the power minus j 2 π?

$$e^{-j2\pi\left(\frac{km}{4}+\frac{ln}{4}\right)} = e^{-j2\pi\left(\frac{km+ln}{4}\right)}$$

So, cosπ/2 minus j sinπ/2. So, this is equal to 0, this is equal to minus j. I can say minus j to the power k m plus l n. Now I can calculate X 0, k equal to 0, l equal to 0. I put k equal to 0, l equal to 0 and the sum of this part will be 1; this is nothing but an X of 0, 0. So, you can say that all the sample sum. So, it is nothing but the all 1 plus 2 plus 3 plus 4 plus 1 plus 2 plus 3 plus 4 plus 1 plus 2 plus 3 plus 4. So, all the pixel values will be summed, then calculate X 0, 1, calculate X 0, 2, and calculate X 0, 3. Once you do that, then calculate X 1, 1, then calculate X.

So, now this is X 0 0 0, 0 1, 0 2. Similarly, X 1 0, 1 1, 1 2, 1 3; X 2 0, 2 1 like that way. So, all the pixels, so this is also a 4 by 4 matrix. This X is also a 4 by 4 matrix, not a 4 by 4 matrix,. It will be how what should be the length of this matrix. This direction is N minus 1, and this direction is also it is M minus 1, N minus 1, N minus 1 value and M minus 1 value. So, you can calculate that.

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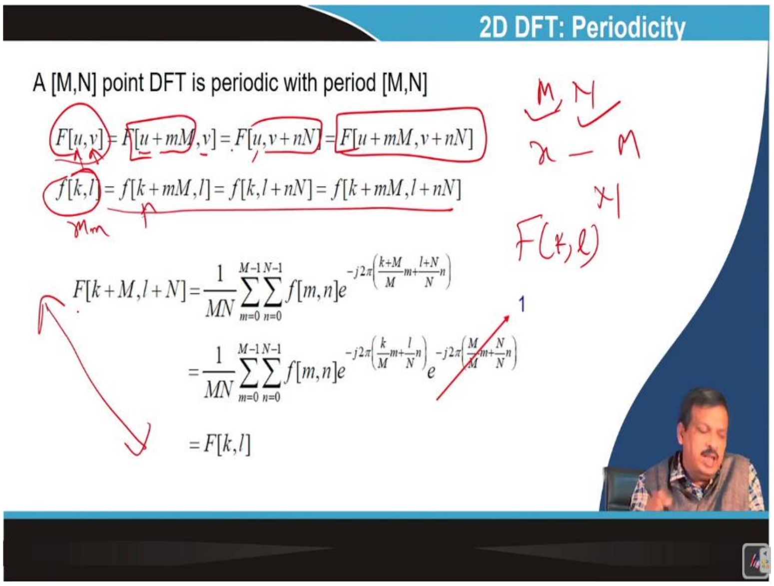

Let us say the properties. So, whatever the properties exist in 1D DFT that is whole for 2D DFT also. So, in the case of periodicity, if M and N are the DFT of two different directions, x direction DFT is M length and y direction is N length. So, now, if I say the frequency domain also, the period is M and N. So, F[u], v, or I can say F[k], l whatever. So, u is the one-dimension, v is the one-dimension, u plus m into v is equal to F[u] into v, plus n is equal to this one.

Similarly, the time domain is also the same. Here, in the time domain, I have taken k, l. If it is I have taken m n, then instead of k, it will be m and n, ok. So, you can prove it. This is the proof. So, periodicity is held in 2D also.

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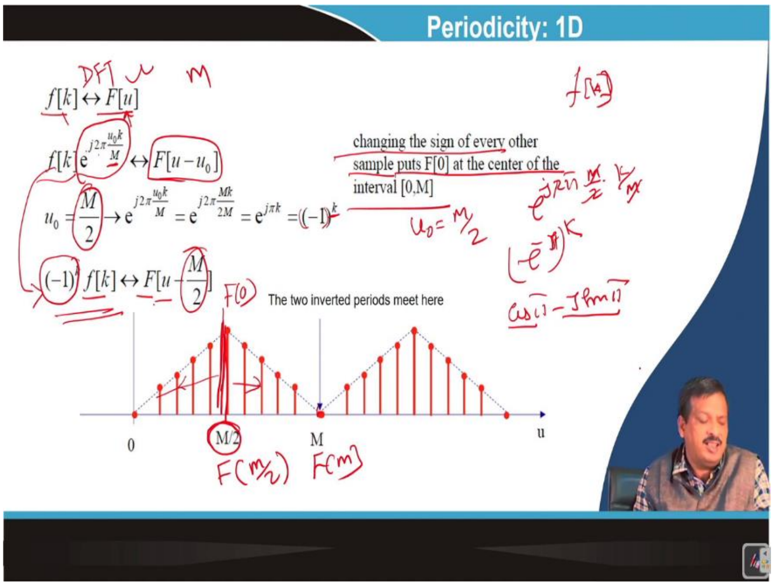

Similarly, all periodicity, let us say another interesting thing, is that periodicity is 1D; let us say f[k] is equal to f[k], which is a one-dimensional signal in k. So, instead of k and F[u], it is the frequency domain signal. So, this is the DFT, a one-dimensional DFT of length. Let us say it is ok. Now, if I say f[k] multiplied by $e^{j2\pi \frac{u_0 k}{M}}$, so it is nothing but shifting in the frequency domain that we have already proved in 1D. So, now

$$M_0 = \frac{M}{2} \rightarrow e^{j2\pi \frac{M_0 k}{M}}$$.

So, it is nothing but e to the power j to the power j π to the power k. So, what is $e^{j\pi}$

$$e^{j\pi} = \cos(\pi) - j\sin(\pi) = -1 + 0j = -1$$

So, I can put minus 1 to the power k. So, I can say this is nothing but a minus 1 to the power k f[k] is equal to F[u] minus M by 2. So, if this is my F M by 2, if this is F of M by 2, and this is F of M.

Now, if I say if I put this M by 2 is u0 at the centre, f of 0 let us say f of 0 at the centre if I put, then if you see both the side negative or positive is nothing but a sign change changing the sign of every other sample put f 0 at the centre of the interval 0 to M ok. You know that it is foldable. So, after every M, it can be folded together.

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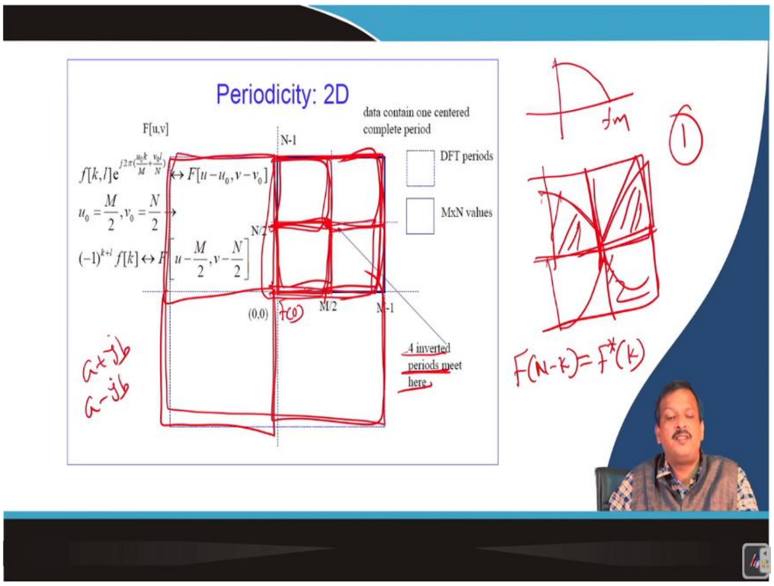

So, now, if I say the same concept if I apply for 2D, this is my F 0, ok? So, 4 inverted point periods meet here. So, this is my M cross N value, and this is my DFT period. So, why is it coming? Think about. You know, if you have a baseband signal like this, this is my fm baseband signal. So, when you make the DFT, you get this one, ok or not, DFT is symmetry at N by 2 points.

Now, if you look at it, I have two dimensions. So, I have this portion for one dimension. So, if I apply for two dimensions on one page, it will be 4 parts. So, that is why if you see there is a 4 part, is the periodicity of the DFT ok or not? M by 2 point, M by 2 point symmetry, so this is one positive, this is one part, and this is one part, and this is one part.

So, what you know, the F of N minus k is equal to nothing but an F star k, which, if you know, is a complex conjugate. What do you mean by complex conjugate? e a plus j b is my number, and the complex conjugate is a minus j b. So, I know that from the DFT symmetry property, if it is one-dimensional, then I know symmetry here. So, if it is two-dimensional, I get 4 coordinate M with a 2 4 value. So, calculate it for an image see what you are getting and then try to explain it.

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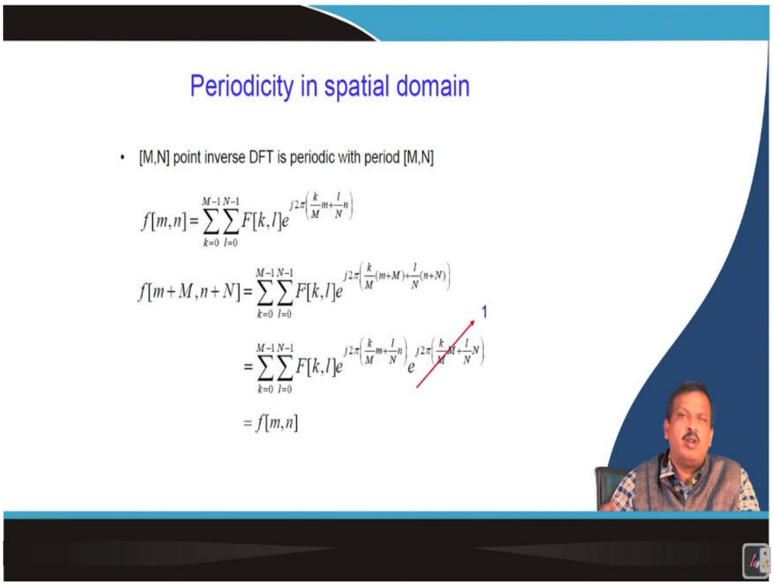

Then periodicity and spatial domain you can also prove it.

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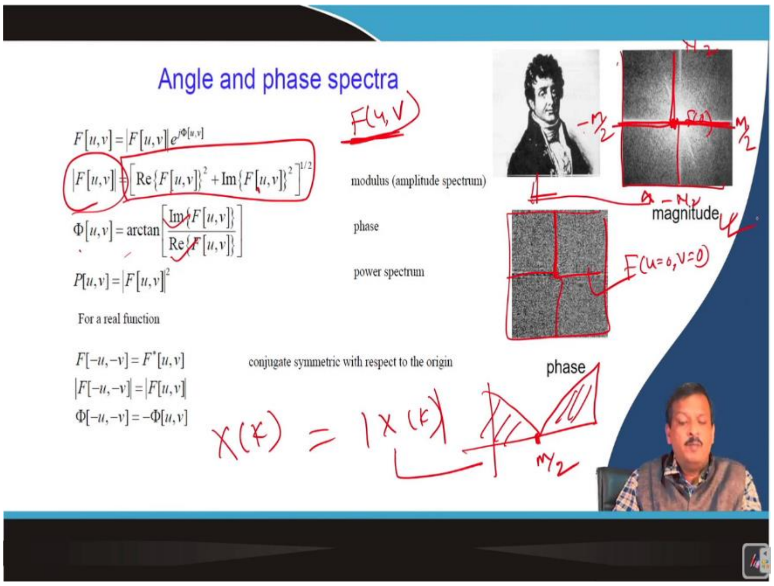

This is an example. So, if I say F[u], v is my frequency domain representation of the image. So, this is the image. So, F[u], v has a two-part; again, I can say that the X(k) has a two-part mod of X(k) and the angle. So, phase and magnitude, here also F[u], v has two parts. One is a magnitude real square plus an imaginary square, which has an imaginary phase by real. So, if I compute the DFT of an image, then if I play if I plot only the magnitude part, I will get this kind of image. So, this is the F of 0, F[u] equal to 0 and v equal to 0. This is the centre.

So, this side it is n, let us say, M by 2, this side it is M by minus M by 2, let us say, and this side is N by 2, and this side is minus N by 2. So, it is 4 4-point symmetry, and this centre is F 0 because if you plot a magnitude plot of a signal, you get this kind of plot of this portion, and this portion is the same. Now, this portion is the centre portion M by 2. So, I made this M by 2, which is my 0 point. So, this side is only a sign of change. Similarly, if I plot only the phase part, then I can also get this kind of image. So, this is the phase part, and this is the magnitude ok.

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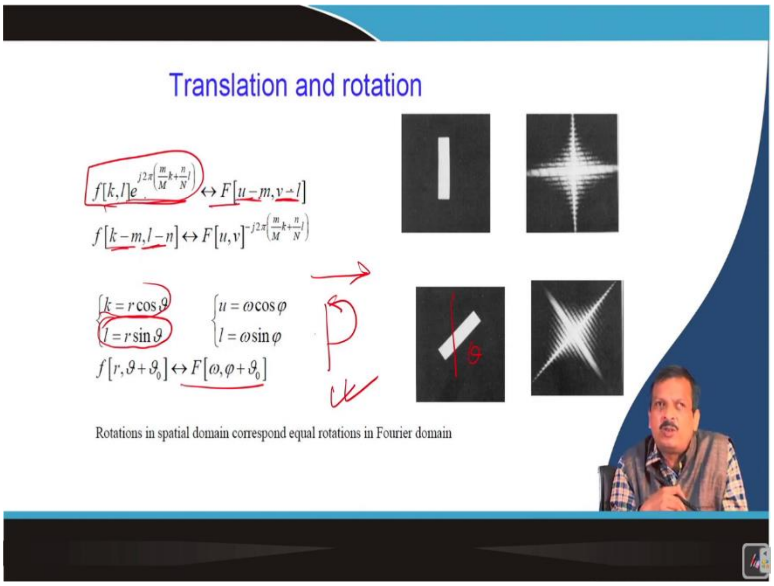

Now, if I go into translation. So, instead of, let us say here k minus m or l minus n. So, this is nothing but a f[k] e to the power j this part. So, it is nothing but a so I take a shift in the frequency domain u minus m, v minus l ok. Now, instead of k, let us k equal to r cos theta and l equal to r sin theta. So, instead of spatial shift, if I say the translatory shift rotation shift, I make. So, this is a spatial rotation shift. So, if it is a rotation shift, then you can see that in that image in the translate domain, it is in the frequency domain, and it is also rotated by theta. So, that is called translation and rotation.

So, if I shift time, this is translation, and I can rotate it also. So, if it is shift, it is theta because it has a spatial domain. So, rotation is also possible. So, I can rotate, or I can translate both if possible ok.

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So this is called the two-dimensional discrete Fourier transform, two-dimensional discrete Fourier transform. I applied the concept of single dimension into two-dimension m and l ok. So, if this is my forward transform, this is my inverse transform; if this is my forward transform, this is my inverse transform, ok.

Thank you.