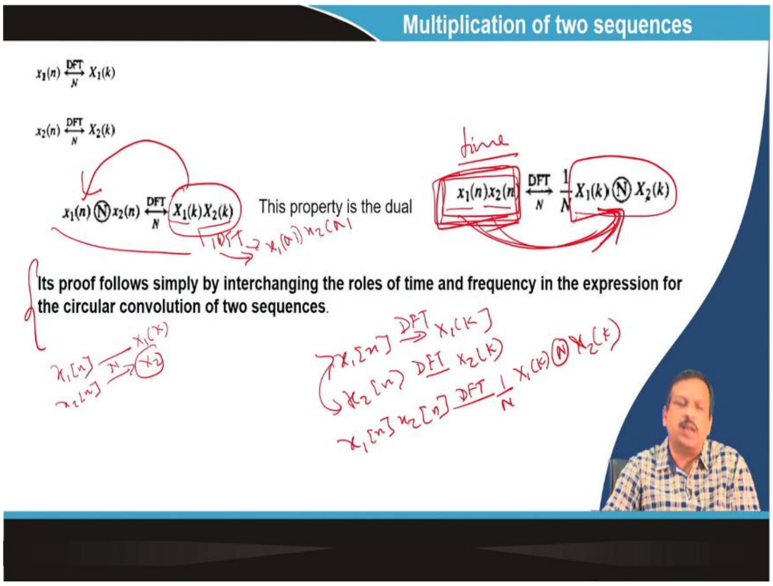
Signal Processing Techniques and Its Applications
Dr. Shyamal Kumar Das Mandal
Advanced Technology Development Centre
Indian Institute of Technology, Kharagpur

Lecture - 23
Properties of Discrete Fourier Transform (Contd.)

Let us continue with that last lecture. We talk about the properties of DFTs, and we also talk about the circular convolution. Let us continue with the Properties of the Discrete Fourier Transform.

(Refer Slide Time: 00:35)



So, today, we discuss the multiplication of two sequences. So, last class, what did we talk about? We talk about if the two sequences, if the x1[n] is 1 sequence and x2[n] is another sequence, if the N point DFT of x1 is X1(k) and x2 is X2(k), then the multiplication of X1(k) and X2(k).

If I take the inverse DFT, then we have proved that it is nothing but a circular convolution of inverse DFT is nothing but a circular convolution of x1[n] and x2[n]. Or I can say if an x1[n] and x2[n] are circular convolutions in the time domain, then the frequency domain is nothing but a simple multiplication X1(k) multiplied by X2(k). Now, if I say the reverse side, let us see that x1[n] and x2[n] are simply multiplied in the time domain. So, in the time domain, x1[n] and x2[n] are simply multiplying.

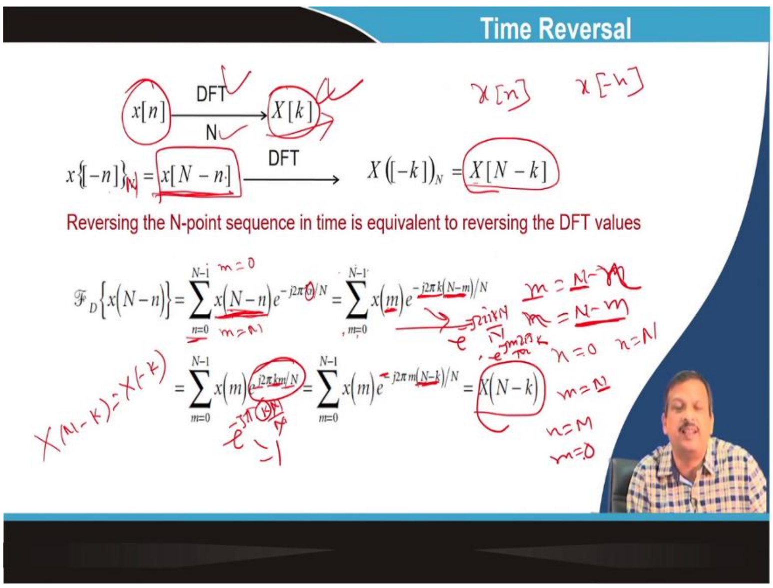
Now, in the DFT domain, if I said x1[n] DFT of x1[n] is X1(k) and DFT of x2[n] is X2(k), then I can say if I simply multiply these two things x1[n] and x2[n] and then if I take the DFT then it is nothing but a circular convolution of X1(k) and X2(k). So, do you understand what I said?

So, I said that if I say if I take the multiplication and frequency domain, we have proved that the time domain is nothing but a circular convolution of point n. Similarly, if I simply take the time domain multiplication and in the frequency domain, it is nothing but a circular convolution, so this proof can just change the role of x1 can X2(k) you can do that proof.

So, your home exercise is to do that proof. Simply interchange the role of time and frequency in the expression for the circular convolution of 2 sequences. You get that. So, what is that? What do I concept I said? If a two sequence X1(k) and X2(k) if I take the N point DFT it is X capital X1(k) and capital X2(k), then X1(k) and X2(k) if I multiply and take the inverse DFT it is nothing but a circular convolution of time domain signal x1[n] circularly convolved with x2[n].

Similarly, if I say that multiplication in time domain x1[n] is multiplied with x2[n] and if I take the DFT, it is nothing but a circular convolution in frequency domain X1(k) circularly convolved with X2(k) ok.

(Refer Slide Time: 04:07)



Now, I come to the time reversal. So, let us say x1[n] is my signal, and the X capital X k is by the frequency domain signal or k domain signal. If I take the DFT N point DFT, I get X k. Now, if I shift it to time reversal, I just reverse the time. So, x[n] was there. So, the reverse of time is x[-n]. So, you know that sequence is periodic at n. So, I can say x[-n] capital N, which is nothing but an x[N-n]. If I take the DFT of x[N-n], it becomes X[N-k] or X[-k].

So, how do you prove it? It is very simple put instead of x n, you put x[N-n] ok. Now I said n equal to 0 to N minus 1 x[N-n]. So, I took this is m. So, m is equal to 0 to N minus 1; instead of n, I put N minus m. So, I said m is equal to N minus m. So, I can say the m is equal to N minus m. So, I put n minus m instead of n, ok. Now, here, if n tends to 0, that means m tends to n, n tends to capital N, and m tends to 0. So, that is why I said n to 0 or 0 to n does not matter.

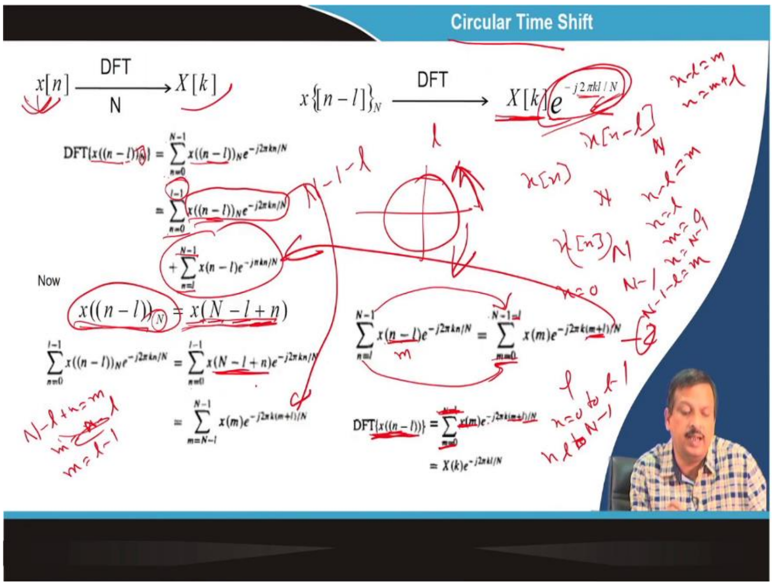
So, I put n minus; do you understand or not? I said I take the signal x[n] minus m. Now, n minus m is equal to m; small m is equal to capital N minus small n. So, n small n equal to N minus m, I put the value of small n here, and instead of n minus m, I put the value of m. When n is equal to 0, n is equal to 0, which means m is equal to capital N, and when n is equal to small n capital N, then m is equal to 0.

So, it will be m equal to m equal to n equal to 0, which means m equal to n, m equal to n equal to n, which means m equal to 0. So, I can say 0 to m minus N minus 1, which is the same thing, but I only change it to ok. So, I write down this. Now, if you see that $e^{j2\pi k(N-m)}$. So, $e^{j2\pi km/N}$ into $e^{-(j2\pi kN)/N}$. So, N by N; N N cancel.

So, if we see that 2π k, k is an integer. So, $e^{-j2\pi}$ is nothing but a 1. So, this factor will be 1 because this is nothing but $e^{-(j2\pi kN)/N}$ into $e^{j2\pi m/Nk}$. So, that it will be there. So, if you see it is positive. So, I can make it minus and minus of N minus k. I take it negative.

So, I know X[N-k] is equal to X[-k], and k is negative here. So, I said X[N-k], is it clear? So, time reversal.

(Refer Slide Time: 08:25)



Similarly, if I go to the circular time shift, another property is circular time shift. x[n] is my signal, and DFT is capital X k. Now, if I say circular time shift because I know x[n], if the N point DFT is calculated, then the period is N and x[n] is repeated circularly. So, x[n] is repeated circularly with N. So, I have to make a circular time shift. So, I have to shift within. So, I can say, as you saw when you do the circular time shift, what do you know?

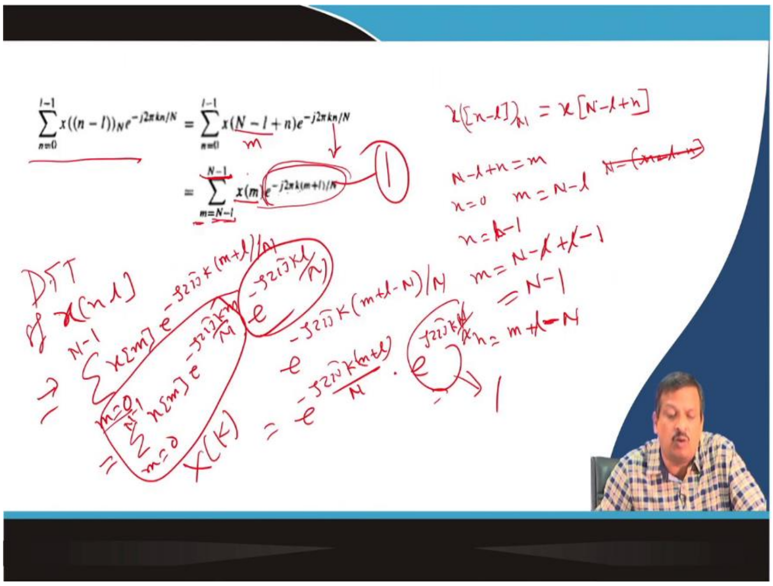
What if I shift this wise? This is a positive index, and if I shift this wise, it is a negative index. So, if I shift this wise, that means x[n] minus l within the period of n, which is nothing but a capital X(k), which is the Fourier transform of x[n] multiplied by $e^{-j2\pi kl/N}$. As if the time shift of l unit, x[n] time shift of l unit as if the frequency response is shifted by $e^{2\pi kl/N}$.

So, X(k) is multiplied by $e^{-j2\pi kl/N}$. The proof is this one. So, DFT of x[n] minus l capital N, so, which is the circular. Now, the circular shift is a point circular shift. So, I put x[n] minus l x[n] minus l is the signal. So, I can say this is nothing but an n equal to 0 to l minus 1; this one is a sum n equal to 0 to N minus 1.

So, I can divide it into two parts: n equal to 0 to l minus 1, and n equal to l to N minus 1. So, because n varies from 0 to N minus 1, the l will be within that. So, I said, let us n varies from 0 to l minus 1 and then varies from l to N minus 1. So, I write down the sum in some form. Now, you know that if it is a circular shift, x[n] minus l plus n is nothing but this one circularly shift because the circle again starts shifting.

So, if this is the circular shift, then I can say that instead of writing n minus l mod of capital N, let us write this one, n equal to 0 to l minus 1. So, what is what it will come? So, I can say this one is nothing but a m. So, N capital N minus l plus n is equal to m. So, when n is equal to 0, m is equal to N minus l capital N minus l and when m is equal to when m is equal to l minus 1. So, when m is equal to 0 a, when n is equal to 0, the capital M.

(Refer Slide Time: 12:14)



So, what I said? So, I take the first sum part, and what I am saying is that I am taking the first sum part. Now, I know that x[n] minus l mod N can be written as x[n] minus l plus n or N minus n minus, or I can say N minus n minus l, the l minus n is ok. So, it is circularly shifted after the N point and will be shifted again. So, now I said N minus l plus n is equal to m. Now, I said when n is equal to 0, then m is equal to N minus l. When n is equal to n; n is equal to l minus 1, then m is equal to N minus l plus l minus 1.

So, it is nothing but a capital N minus 1. So, I replace m is equal to N minus l to N minus 1 x[m]; this is nothing but an m $e^{j2\pi}$, now n is equal to m plus l minus N. n small n is equal to m plus l minus capital N. So, if I put it here instead of n, I can get $e^{-j2\pi}$ k m plus l minus N divided by N, which is nothing but an $e^{-j2\pi k(m+l)/N}$ into minus minus a plus j 2π k N by N which is nothing but a 1. So, it is nothing but this one, ok.

The other sum of the other part, n, is equal to l 2 N minus 1 n, equal to n minus l. So, n is now I can shift n minus l n minus l, which is equal to m. So, when n is equal to l, m is equal to 0. When n is equal to N minus 1, it is nothing but a, so m is equal to then l, then N is equal to n minus 1 minus l is equal to m.

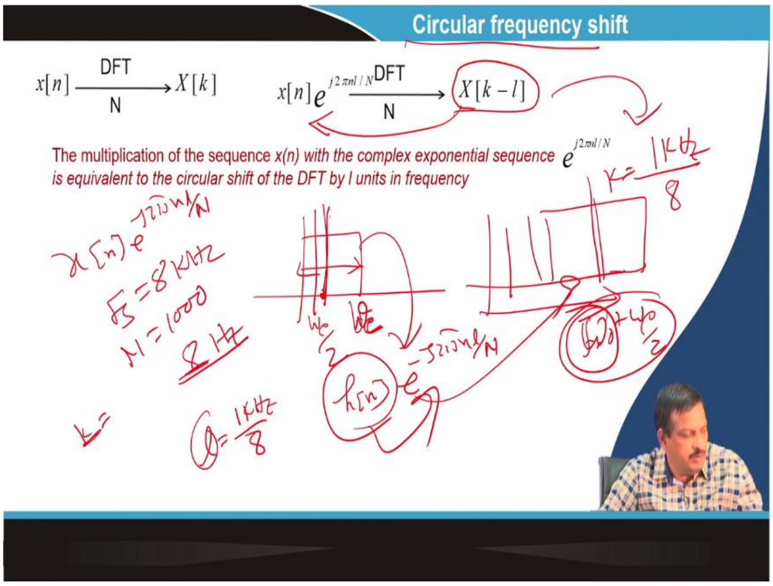
So, it is nothing but a N minus 1 minus l m equal to 0. So, this term will be this one, but n minus l is equal to m. Now, n minus n minus l is equal to m. So, n is equal to m plus l. So, I put m plus l here, ok? So, m plus l divided by N, ok. So, here, I get m plus l divided by N up to N minus 1. So, I can get DFT of x[n] minus l, which is equal to n m, which is equal to 0 to N minus 1.

So, here I get the sum, here I get the sum. m equal to N minus l to N minus 1 and here I got the sum m equal to 0 to N minus 1 minus l N minus 1 minus l and here I get N minus l to N minus 1. So, if I together, it is nothing but a m equal to 0 to N minus 1 x[m] $e^{j2\pi k(m+l)/N}$. So

$$DFT\ of\ x(n) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

This is one part. This is the second part. So, this part is related to this part. So, if I add them, m is equal to 0 to N minus 1. So, this is nothing, but I can say m is equal to 0 to N minus 1 x[m] into $e^{-j2\pi km/N}$ into $e^{-j2\pi kl/N}$. So, this part is nothing but an X(k), and this part is nothing but an $e^{-j2\pi kl/N}$ ok. So, that is called circular time shift. So, a l times shift means $e^{-j2\pi kl/N}$ will be added.

(Refer Slide Time: 18:07)



Similarly, I can talk about the circular frequency shift. So, instead of shifting in here, I am shifting in here that will be the same as x[n] $e^{j2\pi}$ n l by N, circular frequency shift. Because k also varies from n, after n, it is circularly repeated. Now, what is the reason why I use the circular shift? Circular frequencies: where do I require a frequency shift? So, suppose I have a design and low pass filter. This is f c cut-off frequency, or I can say omega c, not discrete domain omega c. Now, if I want to shift it here,

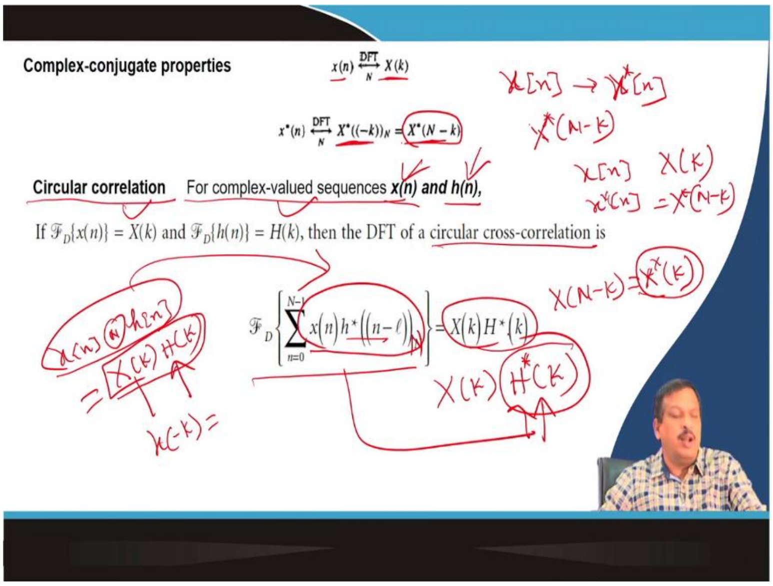
So, I required a circular shift; I required a frequency shift. So, instead of that centre frequency here which is nothing but an omega c by 2, the centre frequency in this part is omega c. So, if I centre frequency 1 2 shifted to here, let us say f 0 omegala 0 plus omega c divided by 2.

So, I required an omega 0 amount of frequency shift. So, if I multiply, let the time impulse response of this filter is h of n. If I multiply by $e^{-j2\pi nl/N}$, how do I define how much omega 0, how much shift is required, and what is the value of l? So, I know the frequency resolution of the transform.

So, if the F s equals 8 kilohertz and N equals 1000, then I know the resolution is 8 hertz. So, 1 k change means 8-hertz change. So, let us say I require I have to shift by 1-kilo hertz; I have to shift it by 1-kilo hertz. So, I, individual k, means 8 hertz. So, I know 1 kilo hertz divided by 8, which is the required amount of the k shift.

So, that much of value l should be 1-kilo hertz divided by the frequency resolution, that much of l if I multiply with the impulse response of the filter, then I get the shifted version frequency response will be the shifted version. So, we will discuss the details when we design the filter. So, this is called circular; why is it called circular? Because after k is equal to n, it will repeat itself. So, that is the circular shift, ok.

(Refer Slide Time: 21:05)



Next is the complex conjugate property. If x[n] DFT is X(k), then the x of, let us say, x[n]; what is the complex conjugate of x[n], x of star of n? Which is nothing but an X star of minus k, which is nothing but an X star of N minus k. So, I know that X star of N minus k.

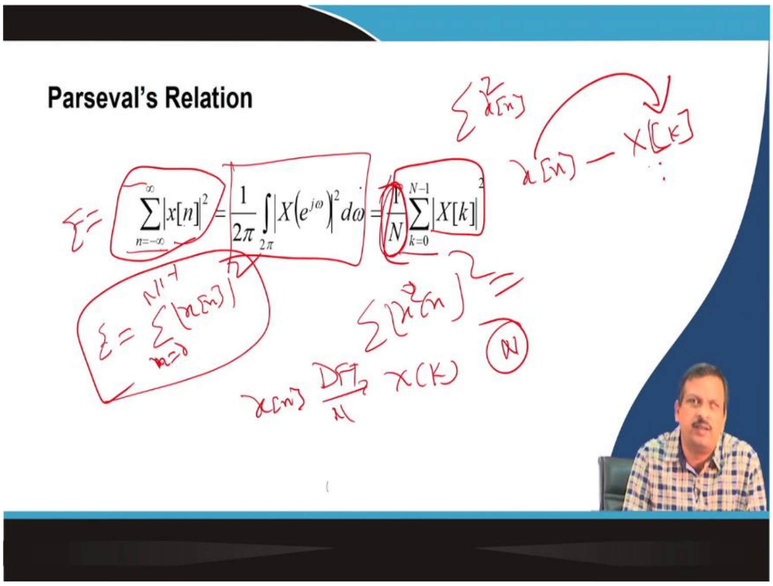
So, if x[n] has a frequency response of X capital X k, then I know that x star n has a frequency response that is nothing but an X star of N minus k. So, I also know complex conjugate and other properties that X[N-k] is nothing but an X star k that property. X[N-k] is equal to X star k ok.

Now, F D let us say circular correlation another property; circular for a complex value sequence x[n] and h n; x[n] is the signal, h n in the impulse response and let the frequency response is x k and h k then the circular correlation is the correlation circularly shifted. So, correlation and convolution circular convolution, a circular correlation which is equal to nothing but an X(k) multiplied by h of k h of the star.

You know that if this is the sequence, then you know x[n] if the circularly convolved with h of n, which is nothing but an X(k) multiplied by H of k. Now, instead of circularly convolving, I calculate circular correlation instead of circular convolution. So, that means holding does not happen. So, without holding, I get just the multiplication of the frequency response of X k and H k.

Now, in the case of correlation, it will be a complex conjugate of H k. Here, since in correlation, H k is not folded, that is why I get the complex conjugate. So, you know that X[-k], or I can say X[-k], is equal to we have proved that X[-k] minus n is equal to X[N-n], which is nothing but an X star. What did I say? Which is nothing but an X-star k complex conjugate of k. So, you know that this folding is not happening, that is why it is star ok. So, that is circular correlation.

(Refer Slide Time: 24:35)



Then, Parseval's relation, that if you see that if x[n] is my signal, then you know what is the energy of the signal summation over the time square and sum over the time. So, if it is an infinite duration sequence, then it is n equal to minus infinity to infinity x[n]. The whole square is the energy of the signal. If it is a finite duration signal, then energy is equal to n equal to 0 N minus 1 x[n] mod x[n] whole square that is the energy of the signal.

So, whether I calculate the energy of the signal in the time domain and I can also calculate the same energy of the signal in the frequency domain. So, x[n] is the frequency domain representation of X(k). Now I know all the energy is transferring here. So, the if I calculate the energy here, x[n] whole square must be equal to the energy in the frequency domain, which is nothing but a 1 by N k equal to 0 to N minus 1 X(k) whole square.

You may say that, sir, we have an x[n] DFT is X(k), then where do I get the 1 by N? You know the DFT is nothing but an N point N multiplication. So, its gain is increased by N times. That is why when you take the reverse DFT, we decrease it by 1 by N. Since it is energy, energy will be increased by N times, which is why 1 by N has to be multiplied.

So, that is what you can say: Parseval relations are ok. So, the energy I can compute the energy in the time domain, and I can also compute the energy in the frequency domain. In the tutorial, I will show you how this can be done.

(Refer Slide Time: 26:45)

So, these are the properties of discrete Fourier transform. So, these are the properties of discrete Fourier transform ok. So, all the properties somehow will be used during the solve, during solving the problem or when you go for the frequency domain analysis, all kinds of things. So, we may use one of these properties. So, you have to remember those properties of discrete Fourier transform.

Thank you.