Signal Processing Techniques and Its Applications
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Lecture - 21
Properties of Discrete Fourier Transform

Okay, so last week, we discussed the Properties of Discrete Fourier Transform. So, we have discussed about the linearity property and periodicity property.

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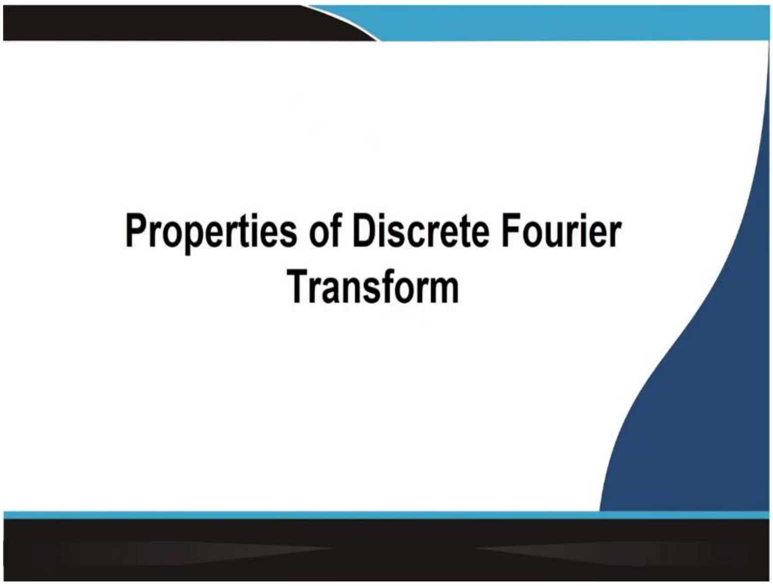

So, this week, we continue with the properties of Fourier transforms; the other properties we will we will be continuing here. Last week, I also covered the circular sequence. How do I sequence a periodic sequence can be expanded, or if x n is there, how can the consequence be extended that I also covered in the last week.

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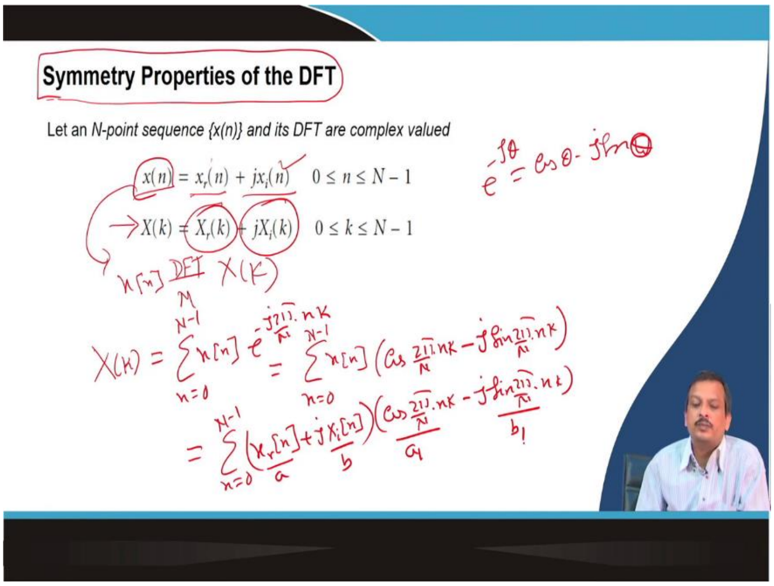

So, this week, we continue with the properties of discrete Fourier transform, and the first property is the symmetry property of discrete Fourier transform. So, last week, I covered the symmetry property of the sequence; here, I will cover the symmetry properties of the Discrete Fourier Transform. So, what is the symmetry property of a discrete Fourier transform? Ok.

So, let us start with the math, and then we go for the physical explanation. So, what is the math? Let us say I have a signal x[n]; x[n] is a signal that is nothing but real and imaginary. So, it is a complex signal that has a real part Xr n and an imaginary part Xi n ok. If I take the DFT of x[n], and if I take the DFT N point DFT, I get X(k). Now, X(k) is also complex. It also has a real part and an imaginary part.

Now, let us derive what the expression of X(k) should be. So, how do I derive this? Now, I know

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi}{N}nk}$$

Now, this can be written as n equal to 0 to N minus 1 x[n]. So,

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

I have written it down now if it is that if I put the value of x[n]; n equal to 0 to N minus 1. What is x[n]?

$$x(n) = x_r(n) + jx_i(n) \quad \text{for} \quad 0 \leq n \leq N-1$$

Now, let us say I consider this is an a, this is a b, and this is, let us say a 1, and this is b1.

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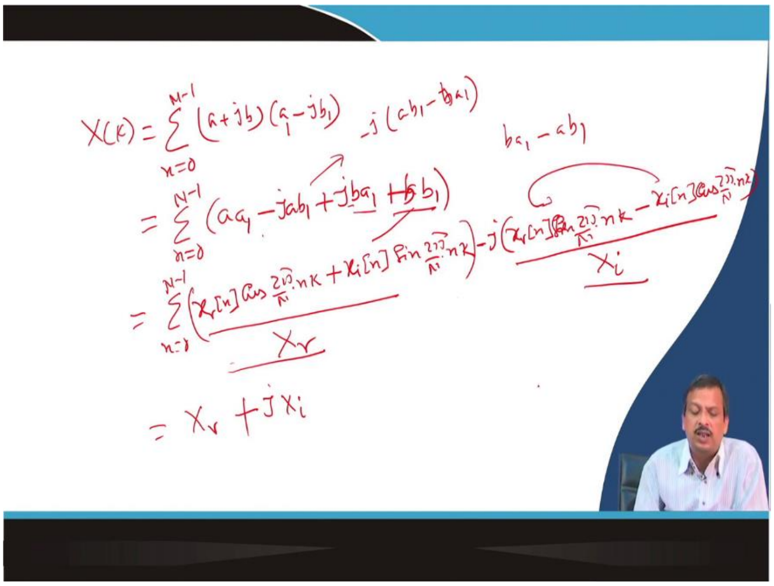

So, I can say that if

$$X(k) = \sum_{n=0}^{N-1}(a + jb)(a_1 - jb_1)$$

So, if I do the product n equal to 0 to N minus 1. So, it will be a a1 minus j a b1 plus j b a1 minus, minus j minus 1, so it will be plus a b1 sorry b b1. So, now, I put the value of a and b n equal to 0 to N minus 1. What is a? a is nothing but a Xr n. What is a 1? a1 is nothing but a cos (2π/N)*nk. So, it is cos (2π/N)*nk, the real part of this one is also the real part. So, plus I write this one first.

So, what is b? b is nothing but an Xi n imaginary part of the input signal. What is b b 1? It is nothing but a sin (2π/N)*nk. So, this is one part. Next part if I take the minus sign common minus j then I can say it is if I take the minus j common. So, it is minus j if I take common a b1 minus b a1 ok. So, minus j a b1 a is nothing but a Xr n into b a b 1; b1 is nothing but a sin (2π/N)*nk minus b a1 b a1 b is nothing but a Xi n into a1 is nothing but a cos (2π/N)*nk.

So, I can say this is an Xr, and this is an Xi. So, this is one transformation this is. So, this is a real part of the transformation and this is the imaginary part of the transformation. So, I can say that X(k) is nothing but an Xr minus j Xi. So, if I want to make it plus if I want to make it plus j Xi, then if I want to make it plus plus means if I so, this will be this side. So, I can say b a1 minus a b1.

So, this is b a1, which will be on this side, and a b1 will be on this side.

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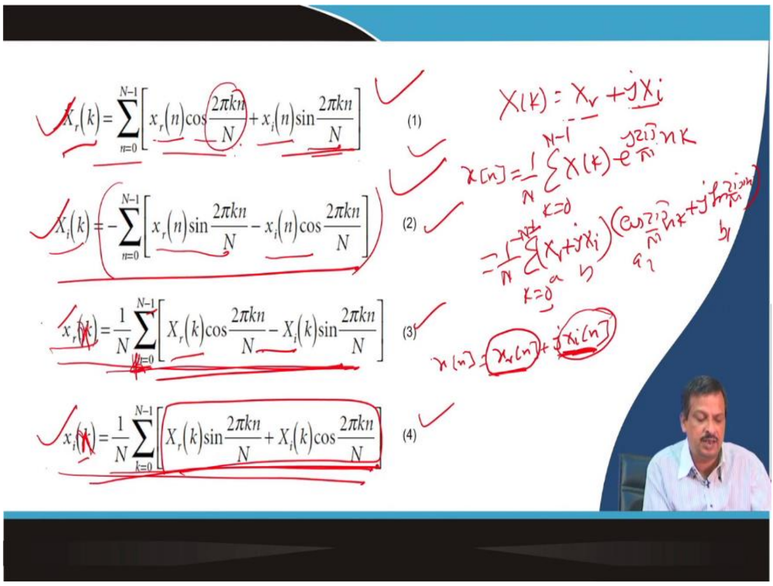

So, I can say the expression of

$$X_r = \sum_{n=0}^{N-1} \left(x_r(n)\cos\frac{2\pi}{N}nk + x_i(n)\sin\frac{2\pi}{N}nk\right),$$

and

$$X_i = \sum_{n=0}^{N-1} \left(x_i(n)\cos\frac{2\pi}{N}nk - x_r(n)\sin\frac{2\pi}{N}nk\right)$$

So, if I say the plus if I say it is plus. So, I can say minus of this one is nothing but a Xi. So, minus this one is nothing but an Xr sin and x minus Xi cos. So, Xi will be cos and Xr will be sin.

So, I can calculate that Xr expression and Xi expression. So, I know that Xr expression and Xi expression. Similarly, if I take the inverse transform of X(k). So, X(k) is equal to Xr plus j of Xi, and Xr is this one, and Xi is this one. Now, if I say I want to calculate x[n]. So, x[n] is equal to 1 by N k equal to 0 to N minus 1 X(k) multiplied by $e^{+j(2\pi/N)*nk}$.

Now, if you put the value of Xr and Xi j, again if I put the same way if I calculate 1 by N k equal to 0 to N minus 1 Xr plus j of Xi multiplied by cos (2π/N)*nk plus j sin (2π/N)*nk if I do that I get Xr expression and Xi expression. So, I get x[n] is equal to Xr n plus j Xi n. So, the expression of that Xr n will be this one and the expression of the Xi n will be this one. You can do it.

Again, you take this is a b, this is a1, and b1, then take the product and calculate the real part and imaginary part. So, you can get the expression of Xr of n, the real part is Xr of n, and the imaginary part is Xi of n. So, I derive if I have a complex input, this is my frequency domain if DFT differentiation of Xr k and Xi k and then if I take the inverse DFT, I get the Xr k and Xi k if ah or you can say that this is not k this is n.

If it is k, then it will be n n equal to; if it is n, then it will be k equal to ok. So, Xi, I can say n instead of k. I can write n ok and then write this one is k. This one is not n, k equal to 0 to N minus 1 k equal to 0 to N minus 1, this one will be n ok. So, x[n] Xr n and Xi n; Xr n this is the expression, and Xi n this is the expression. So, this is the 4 equation I get. Now, consider the sequence.

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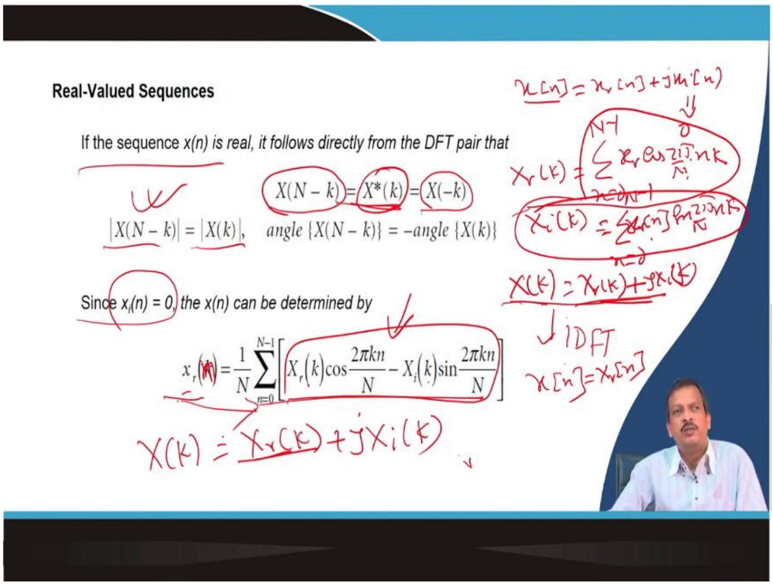

So, what is the sequence? Let us say I have a real-valued sequence, ok.

Should I derive it? Yes, I derived it because you may have a confusion. All these are greater than Xr and Xi.

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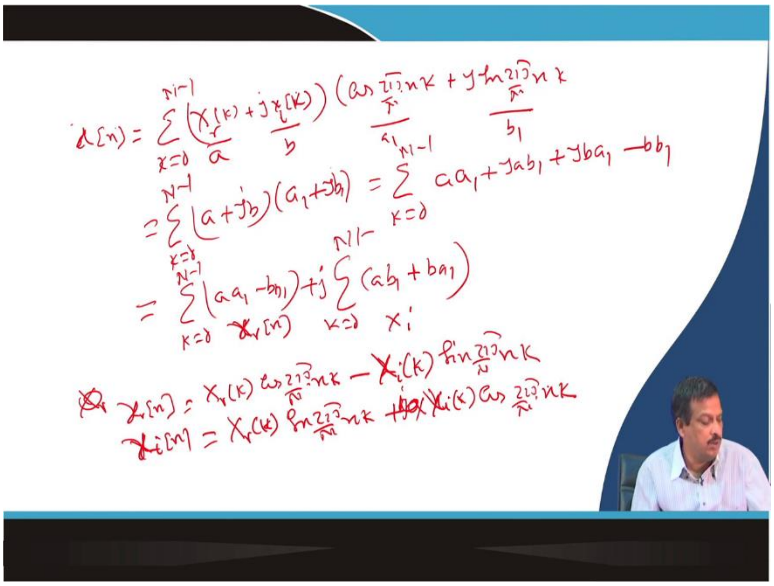

So, let us say what I said: x[n] is equal to k equal to 0 to N minus 1 Xr k plus j Xi k Xi k multiplied by cos (2π/N)*nk plus j sin (2π/N)*nk. So, I consider this as nothing but an a, this is nothing but a b, this is nothing but a 1, and this is nothing but a b1.

So, it is nothing but a k equal to 0 to N minus 1 a+jb multiplied by a1 plus j b1, which is nothing but a k equal to 0 to N minus 1 a a1 plus j a b1 plus j b a1 minus b b1. So, I can say it has two parts: one part is k equal to 0 to N minus 1 an a1 minus b b1, which is nothing but an x of small Xr n and another summation part is k equal to 0 to N minus 1 j if I take common. So, it is nothing but a a b1 plus b a1 which is the Xi part.

So, what is Xr x small Xr n? a a1 a is nothing but a Xr k multiply by a1 cos (2π/N)*nk minus b b is X capital Xi k into b 1; b1 is sin (2π/N)*nk. And Xi i n small Xi n is nothing but a a b1 a is Xr k into b 1; b1 is nothing but a sin (2π/N)*nk plus b a1 j into b means x b means what b means Xi k capital Xi k into a1 cos (2π/N)*nk.

See that expression in here same this is minus and this is plus, this is n, and this will be k, this is n ok. So, now I consider the real, let us just say the x[n] is a real-valued sequence; that means I said x[n] is equal to Xr n plus j Xi n. So, I said Xi n equal to 0; that means a real-valued sequence. So, if Xi n is equal to 0, then what should be my Xr k? Xr k Xr k this part is 0.

So, I can say Xr k is nothing but a n equal to 0 to N minus 1 Xr cos (2π/N)*nk. And what is Xi K? It is nothing but a n equal to 0 to N minus 1. So, this part is 0. So, Xr minus Xr n into sin (2π/N)*nk. So, I can say X(k) is nothing but a Xr k plus j Xi k. So, I can write down X(k) ok. So, X(k) would become this plus j of this minus j of this ok.

So, if the sequence is real, it also can be shown that X[n] minus k is nothing but a complex conjugate of k, which is nothing but an X(-k). If the sequence is real X[n] minus k example.

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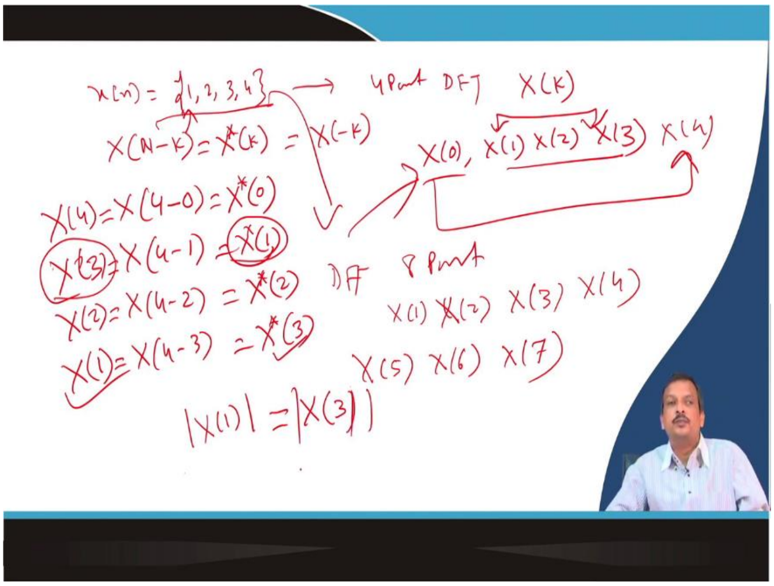

Let us say I have a DFT. I have a sequence x[n]. Let us see whether it is happening; let us say 1, 2, 3, 4. Now, I know what said that if the sequence is real-valued, then X[n] minus k will be the complex conjugate of X(k), which is nothing but a X(-k).

Let us say I computed a 4 point DFT, which is okay. So, 4 point DFT means X(4-k) equal to 0 is equal to star 0. X(4) minus equal to the complex conjugate of 1. X(4-2) is equal to X of star 2. X(4) minus 3 is equal to X of star 3. So; that means, X(4), X(3), X(2), X(1). So, I can say if I take the DFT, I get the sequence X(k). So, X(k) means X(0), X(1) X(2) or X(3).

I will get it after taking the DFT. So, if I take the DFT, I get both sequences. So, what is the relationship between 1 and 3? So, 3 X(3) is the complex conjugate of 1; X(3) and these are complex conjugates if the sequence is real. Similarly, 0, if I said X(4), is there 0 and 4 is a complex conjugate, and 2 is itself a complex conjugate.

You can compute the DFT and see whether this property holds or not. This is a real sequence; yes, this is the 0th sample. So, if I know, let us say I said I have computed 8 point DFT and if I know, let us X 1, X 2, capital X 2, capital X 3, capital X 4. I know I can calculate capital X 5, capital X 6 and capital X 7. Now, what practical sense it is hold? So, if it is in complex conjugate, if I take only the magnitude part, both are the same.

So, I can say the magnitude part of the magnitude part of X of, let us say, X(1) is equal to the magnitude part of X(3) ok.

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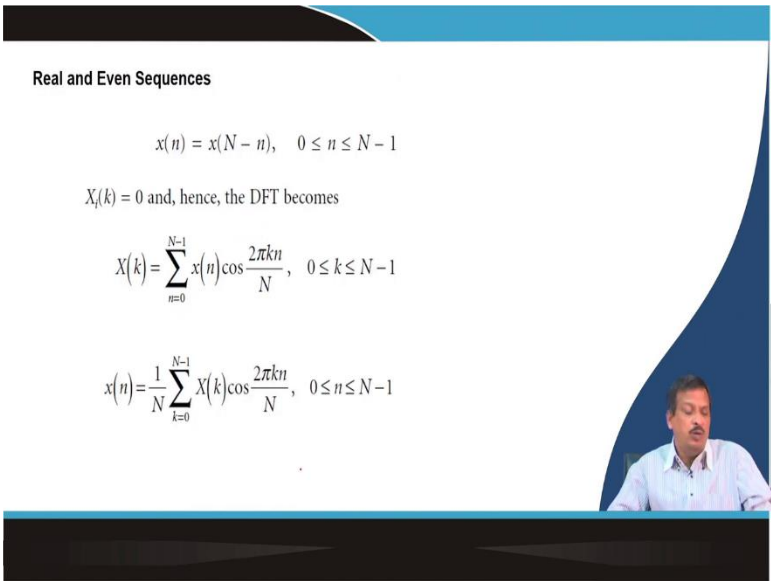

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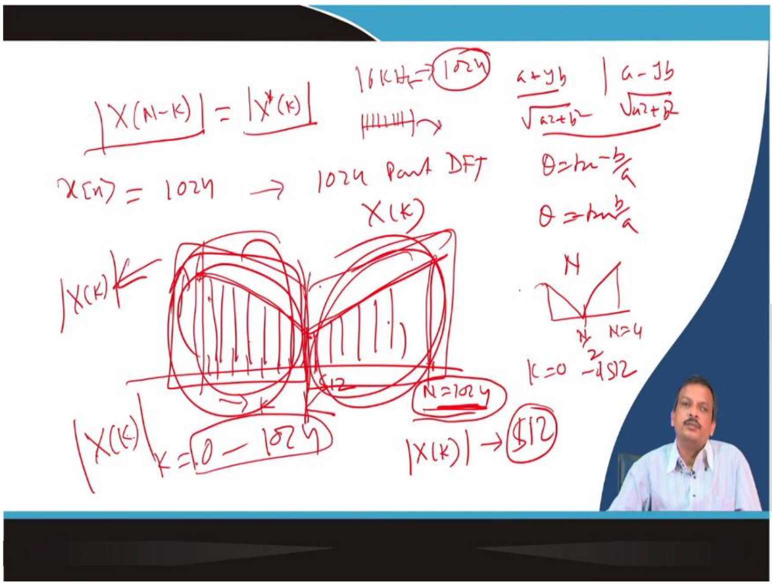

Now, suppose if I want to plot, I will take one extra slide here. What I know? I know the magnitude of X[n] minus k is equal to the magnitude of X complex conjugate k. Both are equal because a+jb, the complex conjugate, is a-jb.

So, the magnitude is root over of a square plus b square magnitude is root over of a square plus b square, so both are the same. And what is the relations between the θ? Θ is tan inverse b by a; now, here θ is tan inverse minus. So, it is minus tan inverse b by a θ is same one is positive, and the other one is negative. Now, suppose I told you I have a real sequence x[n] of length. Let us say 1 0 2 4, and I compute 1 0 2 4 point DFT, I get X(k).

Now, if I plot the magnitude part of the X(k), what should I get? This axis is k, and this axis is the magnitude part of the X(k). If it is 4 points, you know the mod. If it is 4 points, the magnitude of 4 is equal to the magnitude of 0, the the magnitude of index 3 is equal to the magnitude of index 1, the magnitude of 2 is equal to the magnitude of 2 and magnitude of 1 is equal to the magnitude of 3.

So, I can say it is 4 points. So, all if it is 4 points N equal to 4. So, I can say N by 2, I get a symmetry. So, if I plot the magnitude part, let us say 1 0 2 4 is equal to N, and then N by 2 is 512. Let us, if I plot the magnitude like this, let us say this magnitude is like this those are the k value k value is changing.

So, this part will be repeated here because the magnitude of X[n] minus k is equal to the complex conjugate of X(k). If it is complex, conjugate magnitude is the same. So, I get a symmetry over here. So, when I compute DFT of 1 0 2 4 points, if I am only interested in the magnitude spectra, then N by 2 points is sufficient for me. This part I is not required also to consider. So, I compute 1 0 2 4 point DFT; I get a DFT symmetry at N by 2 for a real-valued sequence.

If the sequence is real-valued, then the magnitude response has a symmetry property. So, that is called DFT symmetry. Is it clear to all of you? So, when I calculate, suppose I told you in a real problem record a voice. Let us say record a sound with 16-kilo hertz, ok? Now, compute, let us say, 1 0 2 4 point DFT for a signal length of 1 0 2 4 point. Let us say then I get an X(k) value, which varies from k to 0 to 1 0 2 4.

But I can say that that all values are not distinct all magnitude values are not distinct. So, mod of k will mod of X(k) when we have a value of 0 to 1 0 2 4. So, if I only take mod of

X(k) of length 512 points k, it varies from 0 to 512. I will get the magnitude response of the signal because this portion will be equal to the portion that is called DFT symmetry. Is that clear?

So, this is DFT symmetry. Now, suppose I want to calculate that. So, this is my X(k). Now, if I take the IDFT, I should get the x[n] back. So, X if we see Xr, not Xr k. So, since Xi n is equal to 0. So, this is not k; this is n. So, Xi n is nothing but an Xr n because my signal is a real-valued signal. So, Xr n, if I take the IDFT of this one, I will get this one. How do I get this one?

Now, I know that IDFT of x[n]. So, X(k) is equal to Xr k plus j of Xi k. Now, you know Xr k is only this expression, and Xi k is only this expression. So, these two expressions only consist of X(k). So, if I say that here, come to here Xr k Xr n. So, this part only exists. This part will not exist. So, I get Xi n Xr n is equal to this one. Ok,

to the power minus $e^{(2\pi/N)*nk}$. I can convert sin cos 2 e to the power so, which is nothing but an inverse transform, and I get the value of the real sequence ok.

So, this is real-valued sequence symmetry. This is an important issue. So, when you calculate DFT, if the sequence is real-valued, then magnitude spectra are not required to be calculated for another part after k equal to N by 2.

Also, if I calculate X(k) up to N by 2, I can assume what should be after N by 2 and what should be the value. What is what is the meaning? Let us take another slide here. I will give an example and explain it clearly to you. If I had told you that I had a signal, ok?

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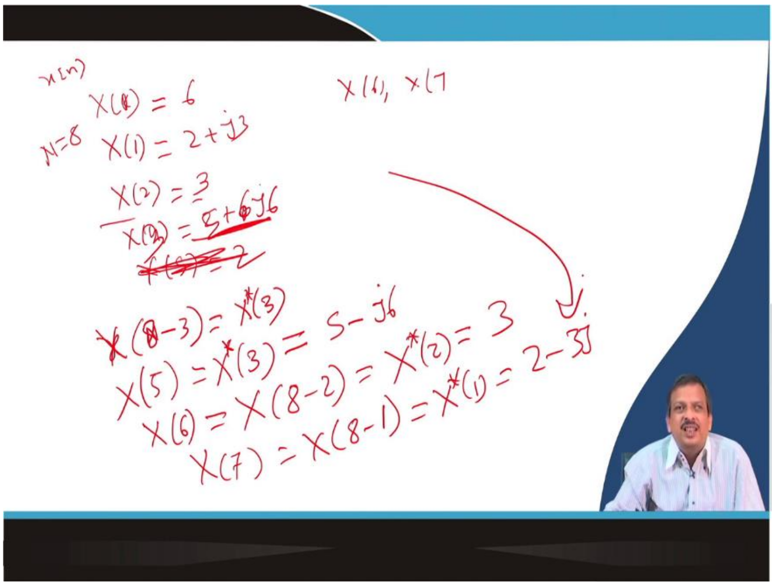

Let us say I have an X(k) I have taken an 8-point DFT of a signal x[n] the N is equal to 8, the value of X 0 is equal to let 6, and values of X 1 are equal to let 2 plus 3 j, X(2) is equal to let us say 3, X(4) is equal to let us say 2 5 plus 6 j.

Then, if I told you what the value should be, let us say X(5) is equal to, let us say, 2. Then what should be the value of X(6) and X(7)? Sorry, 5. So, what should be the value of X(5)? So, I can say 8 point DFT X[n] minus 3. So, N is 8 minus 3, which is equal to an X star of 3; k is equal to 3, I put. So, it is nothing but a capital X 5 is equal to X star of 3. So, what is the 3? Sorry, 3 is this one. So, 3 is 5 plus 4 j. What is the complex conjugate? 5 minus 6 j.

Similarly, he says X(6). I want to say that X(6) is nothing but a capital X(8-2), which is nothing but a complex conjugate of 2. So, which is nothing but an X(2) is 3. So, a complex with 3 is also a complex conjugate of 3. X(7), which is nothing but an X(8) minus 1, which is nothing but a complex conjugate of 1, which is nothing but a 2 minus 3 j.

So, this is a complex conjugate this one. So, I do not have to calculate all those values if the sequence is real. In most cases, when you calculate frequency, the transform sequence is real. So this is called DFT symmetry properties.

Thank you.