Signal Processing Techniques and Its Applications
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Lecture - 16
Frequency – Domain Representation of Discrete Signals and L.T.I Systems

So, today, we will talk about the frequency analysis. So, we have already covered the Z transform and discrete signal, discrete system, and implementation of discrete systems. So, we have already covered all those things. So, this week, we will talk about the frequency domain analysis of systems and signals. Then, we talk about the discrete Fourier transform. So, as you know Fourier analysis, the continuous time Fourier analysis.

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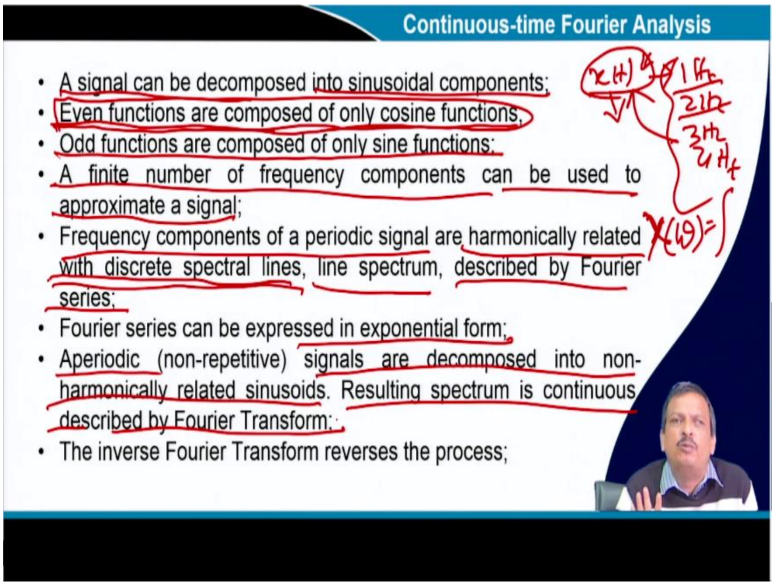

Yeah. You have already studied continuous mathematics time Fourier analysis. So, what is the basis of that Fourier analysis? The Fourier analysis said a signal can be decomposed into a sinusoidal component. So, any given time domain signal that Fourier said can be differentiated or divided into a sinusoidal component.

So, let us say I have a signal time domain signal, continuous time domain signal x(t); this can be decomposed in the sinusoidal component. So, I am saying the signal x(t) consists of different sinusoidal components. So, what is that continuous drawing Fourier analysis you already learned? Then what is the meaning?

What are the physical meanings of those things? This means that if I have a time moment signal, I can consider decomposing the time domain signal in terms of a sinusoidal signal. Let us say I have a signal x(t) in the time domain signal; I can say it consists of 1-hertz frequency and 2-hertz frequency.

So, 1 hertz sinusoidal, 2 hertz sinusoidal, 3 hertz sinusoidal, 4 hertz sinusoidal like that. I can take different sinusoidal components, and I can say that if I take the sum of all sinusoidal components, I will get the back signal x(t). So, that is the basis of Fourier analysis.

So, how is it done? If you see that equation of the continuous domain. So, it is nothing but an x(k) or capital X(ω) is nothing, but you say the integration of minus infinity to infinity x(t) then dt; so, it is $e^{-j\omega t}$ dt. So, why do you do that?

So, as you said, a signal can be decomposed using a different sinusoidal component. So, what Fourier does is create a sinusoidal component and pull it out with the signal. So, as a sinusoidal component actually a filter, so, I pass the signal through a filter and get that component.

So, that is the idea behind that Fourier analysis. So, Fourier analysis why we do the Fourier analysis, the Fourier analysis gives me a pace, or I can say that it is a transformation pace where I can see what the frequency components consist of in the signal; that is the Fourier analysis advantage. Then, what he said? That even functions are composed only of cosine transforms; so, $e^{-j\omega t}$.

So, if I say that $e^{-j\omega t}$ or plus jωt, it is nothing but a cos(ωt) - jsin(ωt) or plus j sin(ωt) . So, e^{j}. So, if you see the Fourier basis of the Fourier. Let us take new slides and explain them to you because the explanation is very important. So, what I said? the, what Fourier said that any signal x(t), x(t) can be x(t) is a continuous signal.

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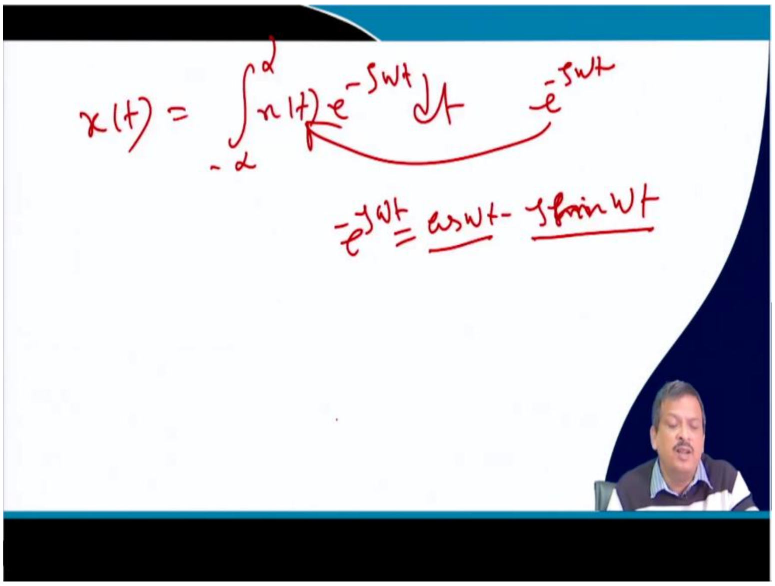

So, x(t) is a time domain continuous signal that can be decomposed in terms of minus infinity to infinity x(t) $e^{-j\omega t}$ dt. So, what is the meaning? This means that I am creating an $e^{-j\omega t}$ signal and convolved with x(t) of that signal exists. So, I get a component.

So, it is nothing but a convolution. So, as you know, $e^{j\omega t}$ has two parts. If it is minus, then I said cos(ωt) - jsin(ωt). So, cos(ωt) is the real part jsin(ωt) is the imaginary part. So, I know that odd function and even function. So, if it is an even function, it is composed of only a cosine function and an odd function is composed of only a sine function. Then what Fourier said? A finite number of frequency components can be used to approximate a signal.

So, I can say a signal will consist of some finite number of sinusoidal components. The frequency components of a periodic signal are harmonically related to a discrete spectral line, which is called a line spectrum or described by the Fourier series. So, when I say a differencing component of the periodic signal is harmonically related to discrete spectral line $e^{j\omega t}$. Let us say it is a 1-hertz component or a 2-hertz component. Is it a line spectrum, It is a line spectrum.

Because with 1 hertz, I get a line or 2 hertz, I get a line. So, it is a line spectra and described by a Fourier series. Fourier series can be expressed in exponential form also, and a periodic signal is decomposed in a non-harmonically related sinusoidal resulting spectra are continuously described by a Fourier transform, and there is an inverse Fourier transform exists.

So, those are the Fourier transform property and those you have already learned in mathematics that Fourier transform of the continuous signal. So, what is the basis? The basis is very simple. Any signal can be decomposed in terms of a sinusoidal signal. So, that is the basis of Fourier analysis.

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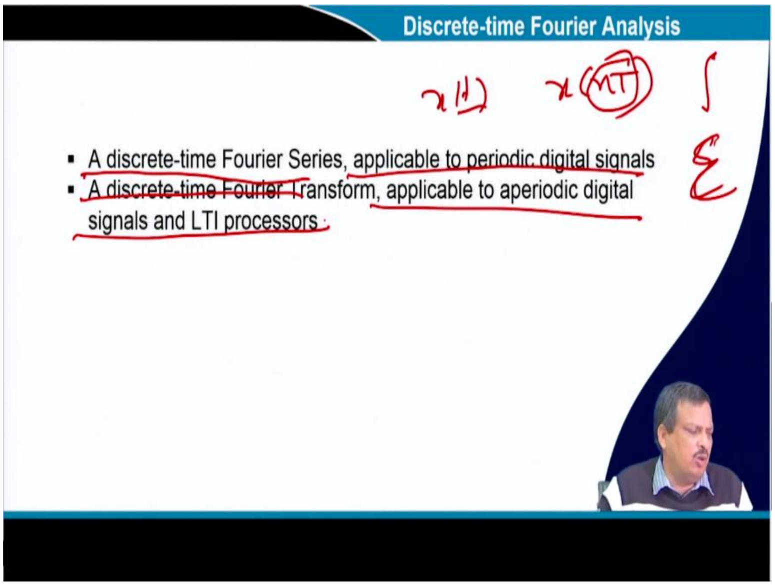

Next, now I have a signal that is discrete; the time domain is not x(t); it is a discrete x(nT), while capital T is the sampling period. So, now, I have a discrete-time Fourier analysis. So, I do not have continuous time. So, you know, the discrete domain, the integration becomes summation, and time becomes time instant, so discrete-time Fourier series.

So, instead of integration, it becomes a summation. So, it is nothing but a series, ok. So, if it is a series, then I can say it applies to a periodic digital signal. Discrete Fourier transform is applicable to a periodic digital signal or LTI process.

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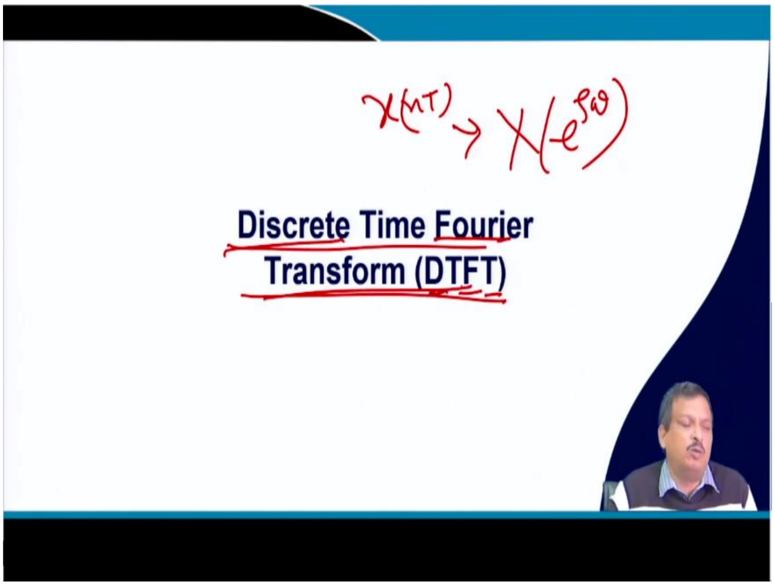

Let us forget about all those things. Let us think about what a discrete Fourier transform is. Discrete-time Fourier transform, discrete-time Fourier transform. So, what does it mean? This is that my signals, whether it is system responses or whether it is signals in the time domain, are discrete, but in the frequency domain, they are nothing but x(t) to the power e j ω is continuous, ω is continuous.

That is why it is called Discrete Time Fourier Transform DTFT. So, the Fourier domain is not a discrete Fourier transform; I said DTFT; Discrete-Time Fourier Transform ok.

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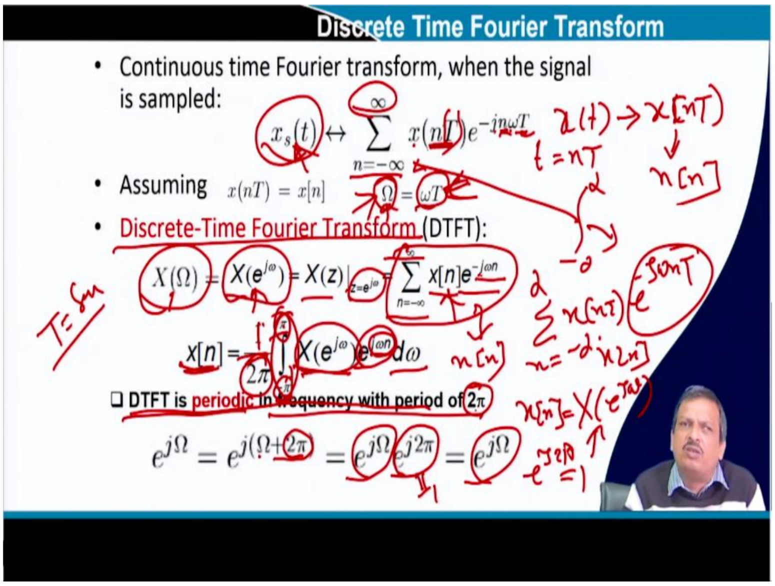

So, what is the meaning? So, I have a continuous signal $x_s(t)$, I have a continuous signal. Now, what I want is a transform in a discrete signal. So, what is the discrete signal? So, the discrete signal is nothing but an x(t) and has to be. I have taken a time instant that is nothing but the x[n] T, which can be said as x[n]. So, t is replaced by nT, where capital T is the sampling period.

So, if you see the Fourier transform, what I said is minus infinity to infinity if it is continuous. If it is discrete, the integration becomes summation n equal to minus infinity to infinity x(nT) $e^{-j\omega t}$. So, ω small t becomes nT.

So, I can say that if it is discrete time Fourier, discrete time signal then, I said that the Fourier transform or I can say the x Fourier transform is nothing but an n equal to minus infinity to infinity x(nT) into $e^{-j\omega nT}$. So, now, this is nothing but a x[n]. So, I can say let us the discrete time Fourier transform of x[n] is represented by capital X $e^{j\omega}$.

So, I said in Fourier frequency domain x is ω domain, it is continuous. That is why I said it is discrete-time, but the frequency domain is not discrete. So, you know the continuous frequency is capital ω, which equals nothing but a ω. So, this is radian per second, and this is radian per sample that I have already covered in my week number 1 lectures.

So, this is radian per second and this is radian per sample. So, discrete-time Fourier transform X capital ω is nothing, but a X $e^{j\omega}$, X(z) also where z equal to $e^{j\omega}$ n equal to minus infinity to infinity x[n] $e^{j\omega n}$, because I do not write the T, T is nothing but a sampling frequency.

So, T is constant everywhere, which is why I said x[n] instead of $e^{j\omega n}$ T, I just omit it j ω n, n is the index the signal.

So, this expression is called discrete-time Fourier transform ok. So, if I want to get the x n back, So, inverse discrete-time Fourier transform x[n], you know that why it is 2π why it is minus π to π you know that.

For any discrete signal, the oscillator's highest rate of oscillation varies from minus π to π. So, it is not infinite, any discrete signal is periodic. So, I can say the sampling period is the period. So, I can say minus π to π the highest rate of oscillation is π or minus π.

So, that is why it is minus π to π. So, the total range of variation is 2π, which is why it is normalized by 1 by 2π minus π to π $X(e^{j\omega})$ $e^{j\omega n}$ $d\omega$. Inverse Fourier transforms - You know the Fourier transform is minus infinity to infinity $x(t)$ $e^{j\omega t}$ dt.

What is the inverse Fourier transform? That is frequency domain integration, which is integration in the frequency domain. So, that is why I asked why it is integration, not summation. Because in the frequency domain, this is continuous; the signal is continuous in the frequency domain, it is discrete in the time domain only ok.

So, I said, then you can see that DTFT is periodic in a frequency period of 2π because of $e^{j\omega\pi n}$. So, let us say I write down

$$e^{j\omega +2\pi} = e^{j\omega} * e^{j2\pi}$$

$$e^{j2\pi} = \cos 2\pi + j \sin 2\pi.$$

So, sin 2π is 0. So, it is 0, and cos 2π is nothing but a 1. So, I can say this is equal to 1. So, this is nothing but $e^{j\omega}$. So, I can say that the DTFT is periodic in the frequency of a period equal to 2π. So, do not confuse continuous Fourier transform, discrete-time Fourier transform, then we go for the discrete Fourier transform.

So, I have not yet gone to discrete Fourier transform; I said continuous Fourier transform. For example, I create a continuous sinusoidal signal convolved with the original signal, and I get back to whether that component exists or not. So, that is the procedure for calculating continuous Fourier transform. Now, I said the signal is in discrete form. So, time instant is not continuous.

So, I got DTFT con Discrete Time Fourier Transform again in the frequency domain; it is continuous, the time domain is discrete, and the frequency domain is continuous. I know that if it is a discrete Fourier transform, the period is 2π because Fs is equal to 2π and it varies from minus π to π. Is it clear?

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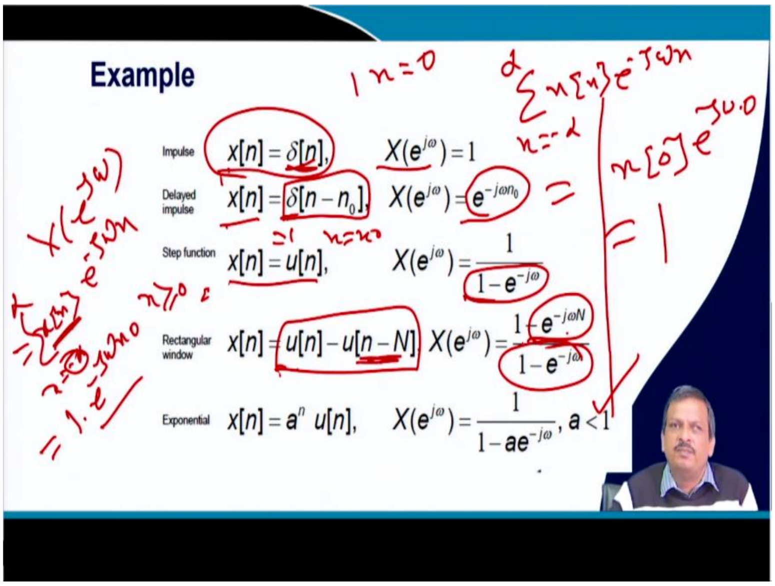

So, now, I go for, let us suppose you know there are some examples given you can calculate x[n] equal to δ[n]. So, $x(e^{j\omega})$, what should be the formula? n equal to minus infinity to infinity x[n] $e^{j\omega n}$. δ

Now, it says n is equal to δ[n]; that means it is 1 when n is equal to 0. Elsewhere, it is 0. So, I can say that this summation only exists x(0), $e^{j\omega}$ into 0. So, it is nothing but a 1. Similarly, if I say x[n] is equal to δ[n-n0]. So, I can say $x(e^{j\omega})$is only 1.

So, this function is only 1 when n is equal to n0. So, I can say it is nothing but a x[n] into n equal to minus infinity to infinity $e^{j\omega n}$; so, at n equal to n0 only x[n] equal to 1; so, 1 $e^{j\omega n0}$. Let us x[n] equal to u[n].

So, what is the formula? u[n] equal n greater than equal to 0 only it is 1; elsewhere it is 0. So, I can say this summation will vary from 0 to infinity, x[n] $e^{j\omega l}$. So, now, x[n] equal to 1, so, $e^{j}\omega$ to the power 0 or j ω 2, so, I can say it is nothing but a 1 by $e^{-j\omega}$.

Again, if you look at this one, it is nothing but a shift. So, this is shifted by n sample. So, that is why $e^{j\omega N}$ will be upside and here will be $e^{j\omega}$. So, those are the examples of DTFT Discrete Time Fourier Transform. The signal is in discrete form, but it is continuous in the frequency domain, clearly ok.

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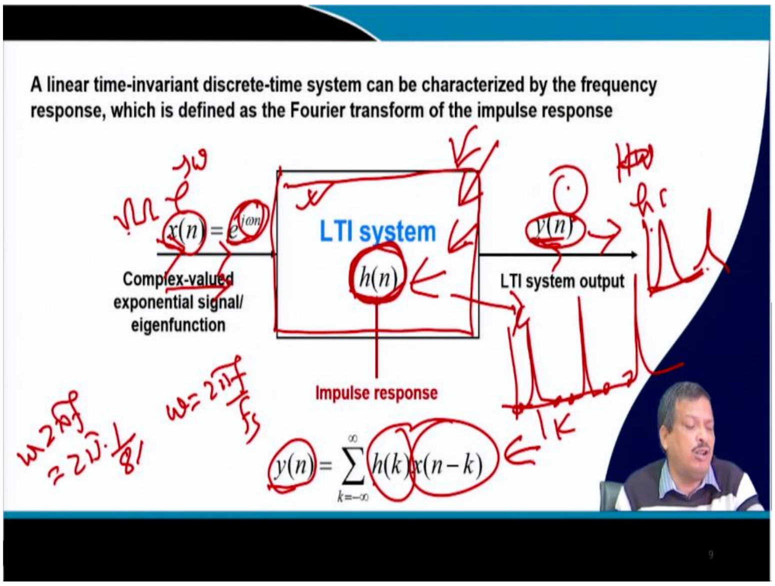

Now, I go for a system. Suppose I have a system. I want to find out the frequency analysis of this system. So, I have already explained the impulse response. So, h(n) is the impulse response. So, what do you how do we get the impulse response? We know if I excited the systems with a unit impulse, then the output is nothing but an impulse response of the system because I know H ω or h(n) is nothing but the output is nothing, but output by input is nothing but a frequency response of the system.

So, the impulse response we get when we excite the system with an impulse, and if I get the output, that output is characterized by the system's impulse response. Now, think that instead of impulse, if I excited the system with $e^{j\omega n}$ let us say ω equal to 1 hertz. So, 2πf, ω is equal to 2πf, 2π into 1 hertz normalized sampling frequency if I said, then if it is 1-hertz, the sampling frequency is 8 kilohertz.

Then, it will be a 1 by 8 kilohertz, which is the normalised discrete frequency component. So, for every normalized discrete frequency, I excited the system, and then I can say the output of the system. So, I excited the system with $e^{j\omega n}$ I get the output y(n), I get the output y(n).

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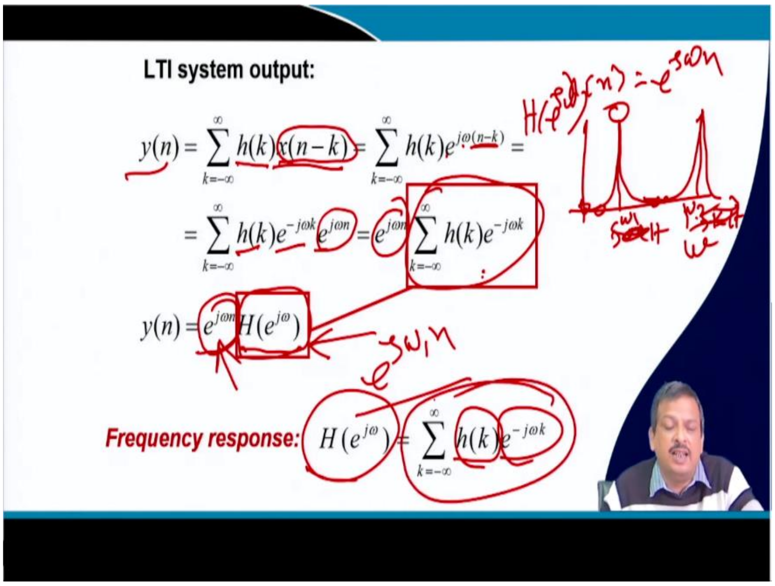

So, that output is nothing but the frequency response of the LTI system. Physically, you understand, forget about mathematics, and think about it. I have a black box system. I exited the system with a 1-hertz signal, 2-hertz signal, 3-hertz signal, and 4-hertz signal, and I calculated the output. So, the output is nothing but a response for a 1 hertz signal. So, $e^{j\omega n}$ is nothing but a pure sinusoidal signal.

So, I exited the system with a sinusoidal signal. So, a response is nothing but the power of that sinusoidal signal. So, basically, I am calculating the system's response to a particular frequency. So, I get the frequency response of the system. So, $e^{j\omega n}$, as you know, the normalized.

So, if the signal is discrete, you normalize discrete frequency. What is normalized discrete frequency? You know that is that this ω is nothing but a 2π f by fs. So, I exited the system for every normalized discrete frequency. I collected the output, and that output is characterized by the frequency response of this LTI system. So, that is the physical.

Now, how do you do that mathematically? Let us say I have a signal x[n], I excited the system with x[n], I get the output y(n), and h(n) is the frequency of the impulse response of the system. Then, as you know, the y(n) output is nothing but a convolution of the impulse response of the system and the signal.

So, it is nothing but a convolution. So, as I think about a system that may consist of d, let us say I have a system that has an amplitude of 1 hertz, it has an amplitude of 500 hertz, it has an amplitude of 560 hertz is less than the amplitude. Otherwise, it is 0. So, when I apply a 1-hertz signal, I get an output; otherwise, it will be 0. So, if I plot the output, I get the frequency response of the system based on the excitation.

So, mathematically, it also has to be true. So, this is a convolution. So, what I said the LTI output y n is nothing but a convolution of system impulse response plus input. So, what is the input? x[n] is equal to $e^{j\omega n}$. I can Instead of x[n-k], I put $e^{j\omega(n-k)}$. So, which is nothing but a h k $e^{j\omega k}$ into $e^{j\omega n}$.

So, if I say $e^{j\omega n}$ outside and take the sum of this one k, what is this? This is nothing but a frequency response of the system, and what is this? This is nothing but a shifting of different frequencies. So, the system's frequency response multiplied by the input frequency gives me that response.

So, suppose in the frequency response of my LTI system, let us know that is why I have an exam by examining. Let us say I have a 500-hertz component and I have a 1.5-kilohertz component. Let us see this is my frequency response, which is nothing but a H of $e^{j\omega}$.

Now, when I am excited about the system for every, let us say for every ω, I am excited. Let us see this if I dropped in ω, let us draw in ω scale, normalize discrete frequency. This is ω 1. This is, let us say, ω 2. So, when my input is $e^{j\omega}$ 1n, then I get output elsewhere it will be 0; because it is a multiplication of H $e^{j\omega}$ to the input. So, I get the frequency response of the system ok.

That way, I can calculate the frequency response of an LTI system. So, what is frequency response? So, how do I interpret it? So, frequency response is nothing but an impulse response $e^{-j\omega k}$. k equal to minus infinity to infinity.

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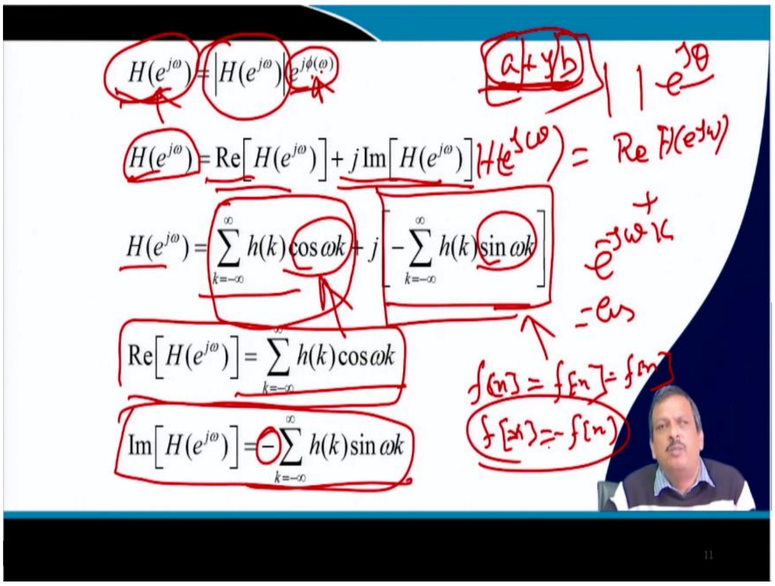

So, I can say this is a complex number. So, H of $e^{j\omega}$ is in complex number. So, let us say a complex number is a plus jb. So, every complex number has two parts: amplitude, which is nothing but a mod part, and θ part, which is nothing but a phase part.

So, H of $e^{j\omega}$ has an amplitude, which is the mod part and $e^{j\Phi\omega}$, which is nothing but a phase part. Or I can say H of $e^{j\omega}$ has two components: one is a real part real of with H of $e^{j\omega}$, and another part is imaginary because any complex number has a real part and an imaginary part. So, I can say H of $e^{j\omega}$ has a real part and imaginary part. a plus j b is a complex number it can be has a real part and has an imaginary part.

So, I can say that $e^{j\omega}$ again I can. Instead of writing real and imaginary, I can write in cos and sin terms because $e^{j\omega k}$ has a minus. I think it is minus $e^{-j\omega k}$. So, it is minus j ωk has a two part cos ωk minus j sin ωk . So, this part is the real part, which is the cosine component; this part is the imaginary part, which has a sin component.

So, the real part is the cosine component, and the imaginary part is minus the sine component. If you see that the real part is the even function and the imaginary part is the odd function, see that minus. So, f[n] equals f[-n] equal to f[n], then I say even function and f[-n] is equal to minus f[n] is nothing but an odd function that you have heard. So, even function and odd function.

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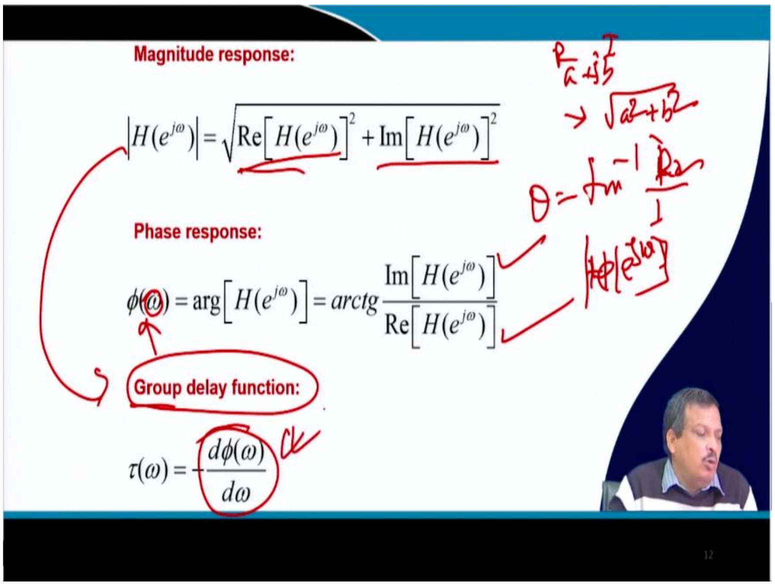

So, I also have a magnitude response. So, what is the magnitude response? For any complex number a plus b real part and imaginary part, magnitude response is nothing but root over of real square plus imaginary square; root over of real spot square plus imaginary square is the magnitude response.

Another phase response, you know the phase is nothing but a tan inverse, θ is nothing but a tan inverse, real part divided by imaginary part divided by real part, arctag tan inverse real part divided by the imaginary part which is also a function of ω. So, I can plot H $e^{j\omega}$ mod of H $e^{j\omega}$ which is called magnitude plot magnitude response and this is called phase response.

So now, if I say the group delay function. So, a difference in the phase derivative of the phase response with the different ω is called the group delay function. We will describe the non-linear phase filter non-linear phase filter. So, linear phase filter means group delay will be 0; for all signals, there will be no phase shift because of what I said.

In the LTI system, I get the frequency response by applying a sinusoidal signal. Now, if you see the system change the phase of the if the different signal the output of the system is different for different inputs is a different phase, then I can say it is an arbitrary phase system, but if it is a linear phase, that means, all the component will come out from the system with the same phase then we call linear phase response. So, that is determined by the group delay function.

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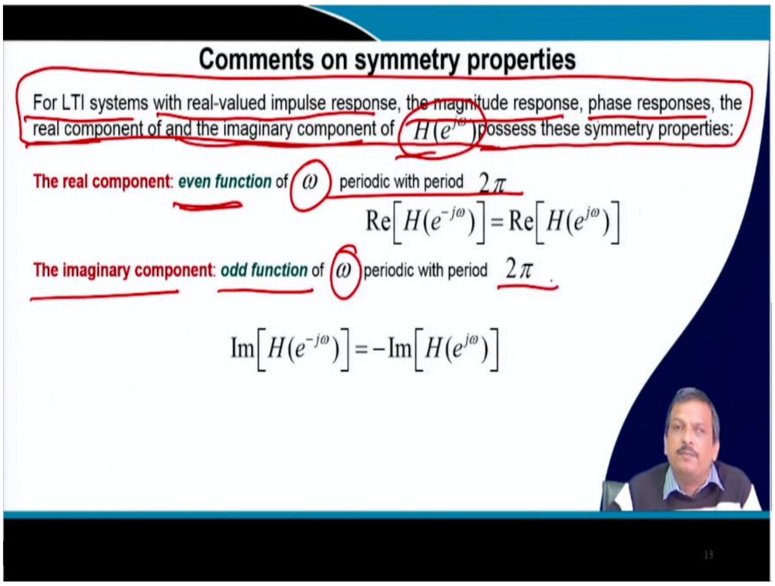

Now, there is a symmetry and asymmetry. So, a mod of H, H $e^{j\omega}$ is a symmetry property. So, for an LTI system with the real-valued impulse response, the magnitude response phase response of the real and imaginary components process the symmetry property.

So, any LTI system frequency response has a symmetry property. So, the real component is a function of the given function ω period is 2π, and the imaginary component odd function of ω period is 2π.

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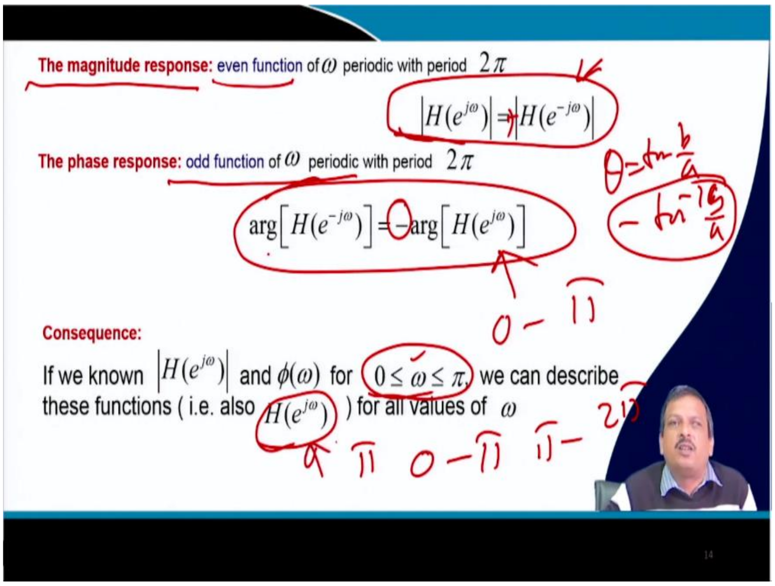

Then the magnitude response is also an even function because my mode of $e^{j\omega}$ minus j ω also the same mod is same for the sign is positive it is positive. So, I can say the magnitude response is an even function whereas the phase response is an odd function because if it is negative, it becomes.

So, tan θ, θ equals tan inverse b by a. If it is negative, it will be minus tan inverse b by a. So, I can say that if it is negative x of minus f[-n], it is equal to minus f[n]. So, that is why it is an odd function. Now, if it is even function then I can say it is a symmetry over the π, 0 to π will be same as π to 2π.

So, if I know $e^{j\omega}$ for 0 to π. So, the frequency here ω is normalized discrete frequency. So, it varies from 0 to π. So, if I know the system for 0 to π, I can draw the system's frequency response. How should we look like this?

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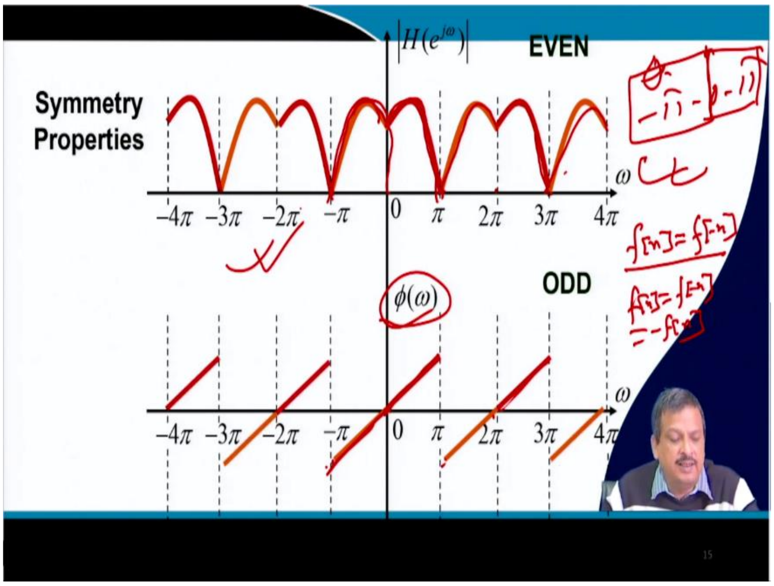

So, this is 0 to π, then what should be the π to 2 π? 0 to minus π will be the same? Because minus π 0 to π is the frequency response of the discrete system. So, 0 to if it is symmetry over here; 0 to π response and 0 to minus π response is same. Similarly, a 2π also will come, then it is coming here, 0 to π, then their negative side.

So, it is a symmetry property, and this is the magnitude response of the LTI frequency response of an LTI system. Now, Φ ω is an odd function. So, 0 to π if it is this side, then I can say 0 to minus π will be negative side f[n] is equal to minus f[-n].

Then, I can say it is an even function. So, here it is the same, but if it is f[n] f of minus f[n] is equal to f[-n] is equal to minus f[n], then I said it is an odd function here if you see it negative. So, it is an odd function, but again, it will be repeated from π to 2π, 2π to 3 π. So, that is the odd function.

So, if you can take any signal and you take the ω and if you plot the ω value concerning ω we will plot the phase value and the magnitude response you get this kind of response. So, this ω is normalized discrete frequency. So, normalized discrete frequency is nothing but a 2πf by Fs .

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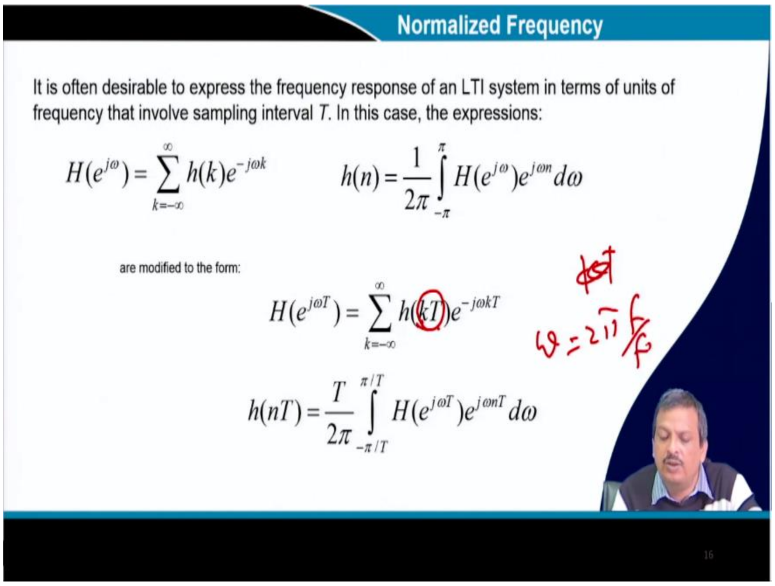

So, again, I am summarizing the normalized discrete frequency. Let us say this one. So, normalized discrete frequency is nothing but an e^{kT}. So, n or n T, whatever you can say, the T is divided into k and multiplied by T. So, I can say ω is normalized distributed to 2πf by Fs , which I have already covered in my first lecture.

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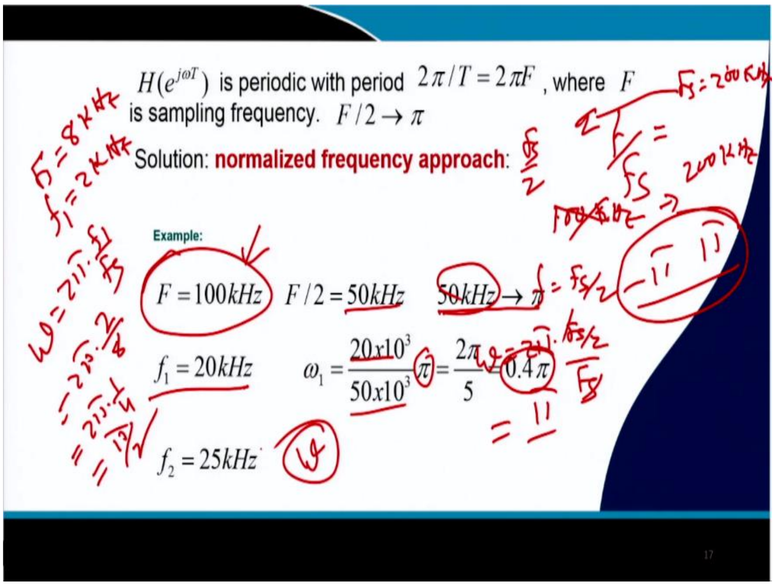

So, you know that f by Fs. So, f by Fs, what is the maximum? So, what is the maximum baseband signal? Ok, forget about that part. So, suppose I have a signal. Let us say, given an example, forget about I have a signal of, let us say, 100 kilohertz signal, ok? Let us say I have sampled it as 200 kilohertz. So, you know that if Fs equals 200, forget about 100 kilohertz. Let us say Fs is equal to 200 kilohertz, and then you know that the baseband signal's highest frequency component of the signal is Fs by 2.

So, f is equal to Fs by 2. So, I can say ω equals twice π Fs by 2 divided by Fs. So, I can say it is nothing but a π. So, the maximum normalized discrete frequency is π; if it is minus π, also, if it is minus f, it is minus π. So, minus π to π is the maximum frequency content of the signal. What is the period? It is nothing but a 2π after every 2π signal will repeat itself ok; so, normalized discrete frequency.

So, F by 2 is equal to 50 kilohertz. So, f 1 is equal to 50 kilo hertz equal to π. So, if f 1 is equal to 20-kilo hertz and the sampling frequency is 100-kilo hertz, I can say it is nothing but a 20; let us say π is 50-kilo hertz; so, I can say how many π will be there. Or I can do it directly f by m. Let us say Fs is equal to 8 kilohertz, let us say 8 kilohertz, and my baseband signal f 1 is equal to 2 kilohertz.

Then I know normalized discrete frequency ω is equal to 2πf 1 by Fs, which is nothing but a 2πf 1 is 2 divided by 8, kilo cancel. So, I can say 2π into 1 by 4 is equal to π by 2, π by 2 or yes π by 2 is the normalized discrete frequency.

So, you can explore a lot when we design a filter; when you calculate the frequency response, we can do this kind of thing when we plot. So, you convert the normalized discrete frequency to the again normal frequency or frequency to normalize discrete frequency.

So, those are the things we will do. So, ω is in normalized discrete frequency. So, this is the frequency response of a system; again, I am saying that this is called discrete-time Fourier transform. The time domain signal is discrete, but the frequency domain signal is continuous. So, that is why I said H of $e^{j\omega}$ where the ω is normalized discrete normalized frequency ok, ok. So, in the next class, I will talk about DFT, Discrete Fourier Transform, ok?

Thank you.