Signal Processing Techniques and its Applications
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Lecture - 14
Pole and Zero in Z - Transform

So, today, we talk about Pole Zero, the concept of Pole Zero in Z transform. So, we have already covered what the Z transform is and what the properties of the Z transform we have covered, and we have already said that the Z transform is used to define the systems. So, pole-zero is the concept of the system.

So, in today's lecture, I will talk about that what are the pole-zero concepts in Z transform and what their significance is. How do you locate that pole-zero in the Z plane? All those things we will discuss today we will cover.

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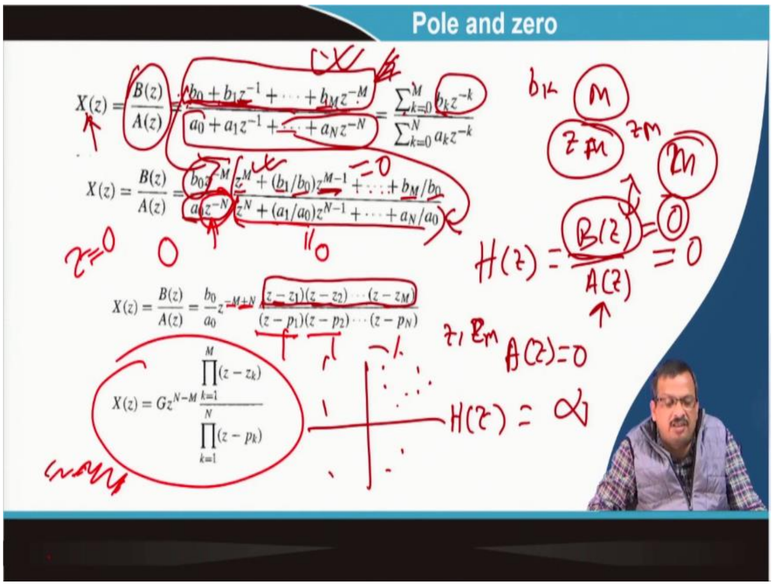

So, let us talk about X (z). X (z) is a rational function that is a ratio of B(z)/A(z).

$$B(z) = \sum_{k=0}^{M} b_k z^{-k}$$

bk is the coefficient, and zk is the z domain transfer function. So, X (z) is the A(z) domain transfer function.

Now, if you see how many solutions of z is available in this equation. So, this is the Mth order equation. So, I can get the M number of solutions of this equation. So, those M number of solutions is the. So, z equal to I can get that M, M number of z value for which this equation becomes 0. So, that is the solution.

So, that solution each of the solutions is called the position of 0. Why is it 0? Why is this solution of this z called 0 and the solution of this z called pole? What is the concept of pole and zero?

Let us say I have a transfer function H(z), which is a function of, let us say, H(z), by which is given B(z)/A(z). Now that I know the M number for the M number of those z values, the B(z) will become 0. Then what will be the value of H(z)? H(z) will become 0. That is why the solution of B(z) is called 0. So, since in the Mth order equation, there is an M number of solutions of z available, which is why I call B(z) an M number of 0.

So, the solution of B(z) is called the 0 position of the transfer function. What is the pole? So, generally, the physical concept of 0 means for those values, the value of the transformation becomes 0. Now, what is the pole? Pole means thus the value of A(z) for the value of z for which A(z) becomes 0; once you said that A(z) becomes 0 then I can say 1 H(z) what will be the value of H(z), H(z) will be infinite.

So, that is why, at the pole, the system will resonate. That is why it is infinite, and the amplitude is infinite. So, the solution of A(z) is called a pole. So, in. So, if I say that z plane. So, you can say the solution of this z. So, if this is my(z) plane, then the solution of B(z) will be located somewhere, and here those are called 0. The solution of A(z) is also located somewhere in the z plane. Those are called poles.

So, this is the concept of pole and zero. Why is it zero? Because for those z values, the H(z) becomes zero. Why is it the pole? For those z value H(z) becomes infinite. So, that is the pole-zero concept of the z-domain transfer function.

Now, if you see this is z^{-1}, I want to make it positive. So, z^M. So, if I take that b0 z^M is outside of this equation, then I can set b0 as 1. So, z^M b1 by b0 into z^{M-1} plus dot dot b M by b0.

Similarly, if I want to express in plus in the z domain, I can take that a0 z outside, and then I can get z to the power N plus this one. Now, if I want to say that the. So, this is the Mth order. This one is the Mth order equation. So, I can say the M number of solution is possible z - z1, z - z2, z^M for which this value becomes 0. So, those z values in this z1 zM are called 0 positions, the position of the 0.

So, if I want to plot those values in the z plane, let us say I just take slides, and I will show you that; suppose this is my entire z plane, this is my entire z plane.

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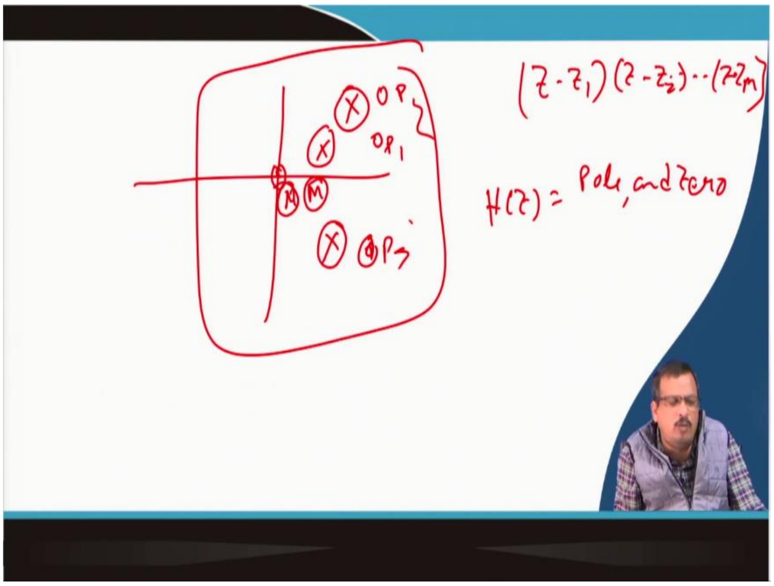

So, this is my entire z plane. So, z- z1 into z- z2 dot dot dot dot dot z-zM those are those for those z values that upper portion becomes 0 transfer function becomes 0 that is why those are called 0; let us locate those 0 by a cross.

So, let us say(z)1 is here. So, this is a 0. Let us say(z)2 is here. So, this is a 0. Let us say(z)3 is here, maybe negative; I do not know. So, those are the zero positions of the transfer function in the entire z plane. Similarly, if I said this one z -p1, z -p2. So, for p1, these functions become 0.

So, I can show that p1, p2, and pN are the poles of the transfer function. So, I can say that, let us say, a pole is represented by a circle. So, this is p1, this is p2, and this is p3. So, all those are poles in the z plane. So, I can analyze the transfer function and determine the pole position of the transfer function in the entire z plane. Then, what will happen here is z^{-M} divided by z^{-N}.

So, if it goes up, then z^{M} plus N. So, I can say that N, the number of poles, are in origin because z equal to 0 is the solution. So, the N number of the pole is at the origin, and the M number is 0 at the origin. So, I can say M number of 0. So, when I go to the plot, I can say that in origin, there is an N number of poles and an M number of 0. So, any transfer function I can say is a combination of H(z), which is a combination of pole and zero pole and zero.

So, what is the meaning of zero? Zero means those values for which H(z) becomes zero; pole means those values for those z values H(z) become infinite. So, these are called the pole and zero of a transfer function. This is called the pole-zero concept. Let us talk about our interest and why we go for the z transform in digital signal processing because the z transform is a well-known mathematical subject. But how do we use z transform in DSP digital signal processing?

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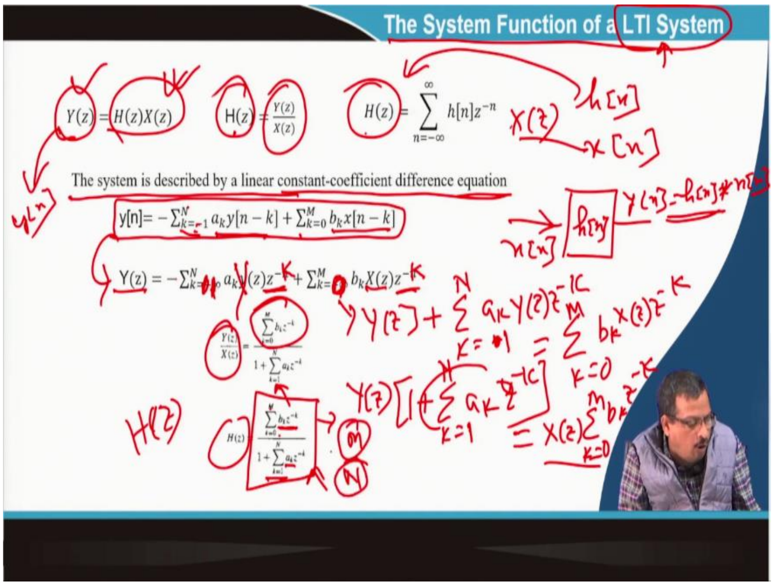

So, our interest is the LTI system. Ok, forget about the written slide. Just listen to it. So, the LTI system is a Linear time-invariant system. Let us say I have an LTI system whose impulse response is h[n]; h[n] is the impulse response of the LTI systems, and x n is my input.

So, I know if I have a system that is h[n], if I apply an input of x[n], I get y[n], which is nothing but a convolution of h[n] convolved with x[n] ok. Now, instead of direct convolution in the time domain, what can I do in the z domain? Suppose I want to find out the transfer function of h[n] in the z domain.

So, what I should do is take the z transform of h[n], and I get the H(z). I take the z transform of x[n], and I get X(z) ok. Now, if I get the X (z) and H(z), I can get the y(z) just by multiplying the H(z) and X (z). Then, suppose I want to get the y[n]. I take the inverse z transform of y(z), and I will get y[n].

Now, suppose I know the output signal and the input signal I want to derive the transfer function. So, if I know the y(z), if I know the X (z), the transfer function is nothing but a y(z) by X (z). Let us say I have an LTI system, which can be expressed in a constant coefficient differential equation; I have an LTI system, which can be expressed using a constant coefficient differential equation.

So, this is the constant coefficient differential equation

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

Now, if I take the z transform of this, this will not be n. This will be k. So, I take the z transform. So, I know. So, this will be capital Y(Z), and this will be capital X (Z). So, I can say Y(Z) if I take the Z transform, it is

$$Y(z) = -\sum_{k=-\infty}^{N} a_k Y(z) z^{-k} + \sum_{k=-\infty}^{M} b_k X(z) z^{-k}$$

Now, if I want to calculate Y(z) by X (z). So, this will be at the top, and this side will go down. So, I can say H(z) which is nothing, but a Y(z) by X (z) is nothing, but a k equal to 0 to M b k z^{-k} divided by 1 plus k equal to 1 to N a k z^{-k}.

So, this is my transfer function of H(z). So, this is my transfer function. Now, how many zeros are there? How number of zeros are there? How many poles are there? N number of poles are there? Now, suppose I said that M has a number of zeros there and N has a number of poles there.

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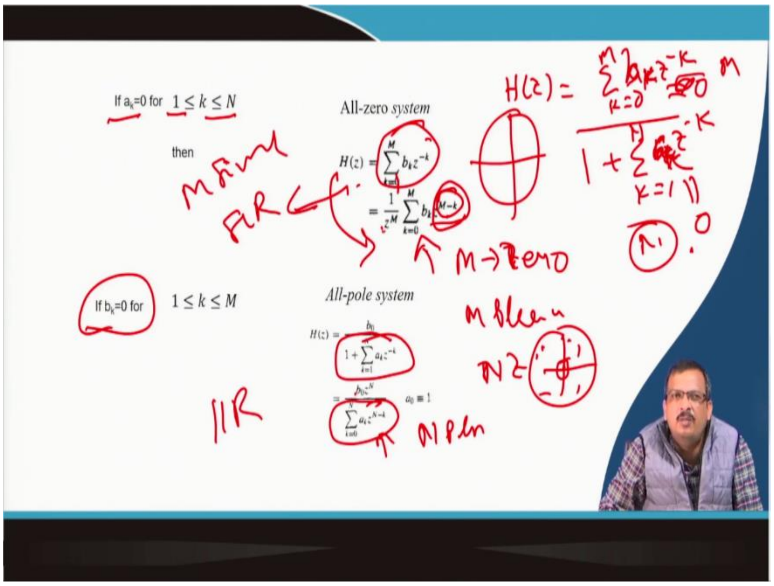

So, I can say the

$$H(z) = \frac{\sum_{k=0}^{M} a_k z^{-k}}{1+\sum_{k=1}^{N} b_k z^{-k}}$$

ok. So, there is an M number of zeros and an N number of poles. Now, I said if a k is equal to 0 for k is equal to 1 to n, this part is equal to 0, and the coefficient a k is equal to 0.

So, if this is 0 sorry a k equal to 0 the, this will be sorry this will be a k this will be b k; b k with X and a k with Y. So, this will be a k. So, if it is, I can say that I am sorry. So, a k is equal to 0. So, a k equal to 0 means not this 1 0 this 1 is equal to 0.

So, if this is equal to 0, then I have only the upper portion z k equal to k equal to 1, not 0 k equal to I can say 1 or 0 k equal to 1 to M b k z^{-k}. If I want to say make it positive, this 1 is positive instead of -k. So, if I take z to power 1 by z^M, it becomes z^{M-k}.

So, if you see this H(z) only contains M number of 0; M number of 0 and M number of the poles at origin and M number of 0 in the entire z plane, then this is called all zero systems all zero systems.

Similarly, if I say B k is equal to 0, then the upper portion becomes 0, then I can get b0 divided by 1 plus this 1, and so, b0 they again I make it positive. So, b0 to the power z to the power N divided by this 1.

So, this has N number of the poles, N number of 0 at the origin, and N number of the poles in the z plane. So that is called a pole system. So, when I want to implement an LTI system while we discuss the implementation of the LTI system, either it will be an all-pole system or an all-zero system.

When it is a zero system, I can say if M is finite, then it is nothing but a FIR system. If M is finite, it is nothing but a FIR system. When I talk about the all-pole system, it is called an IIR system, or the Infinite Impulse Response system. So, all pole systems are zero systems. So, the z transform is used to determine the transfer function of LTI system H(z), and then I can find out the pole-zero systems I can convert to all pole systems or all zero systems using that Z transform manipulation.

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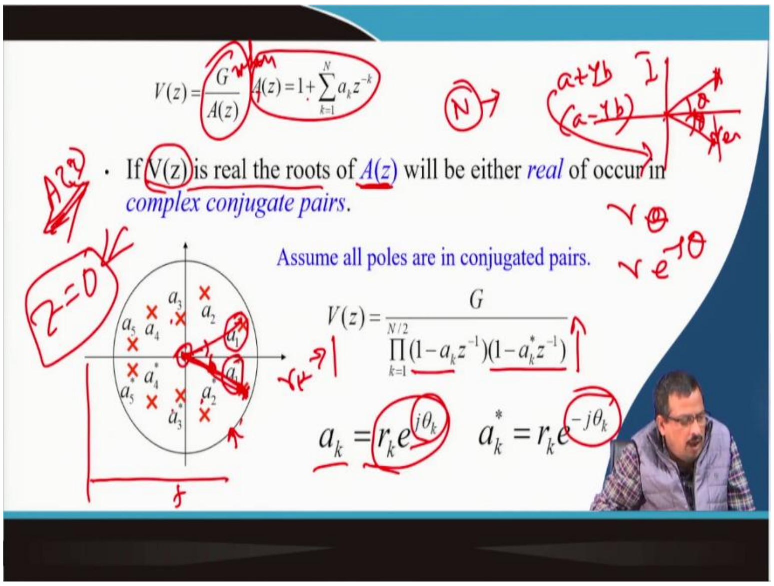

Let us, for example, an application side. Let us say human vocal the human vocal tract can be derived from the mathematical expression of the human vocal track and can be considered as a pole system, which is

$$V(z) = \frac{G}{A(z)}$$ where

$$A(z) = 1 + \sum_{k=1}^{N} a_k z^{-k}$$

So, there is an N number of poles present in the human vocal tract. I want to give an example. So, let us say V(z) is a real V(z) is a real function and is a real root of agent. So, V(z) has a real root of the agent. If it is a real root, N number of poles are there and is the real; then I can say that all poles occur N number of poles are there. So, if it has to be a real pole, then if it is even complex, that occurs in a complex conjugate.

So, I can say that there is an N by 2 number of complex conjugate pairs of the pole. What do you mean by complex conjugate pair? Now, suppose I have a + jb in my 1 pole. Then, if it is a complex conjugate that will be a -jb, then only multiplication becomes real. So, I can say that A(z) is real; A(z) is not an imaginary function. It is a real function.

So, if all roots of the A z's are real roots, then I can say the A(z) consists of N by 2 number of complex conjugate poles. So, if I want to say that, how do I represent a + jb and a -j b in a complex plane, this is the real axis, this is the imaginary axis. So, this is a + jb. is positive when I say a plus a -j b, and the amplitude is the same, but only θ is negative. So, that is why I can say in the A(z) plane if it is inside the unit circle. So, this is a unit circle, then it will be a 1, and there will be a 1.

So, those two are complex conjugate pairs. There will be a 2, there will be a 2, there will be a 3, and there will be a 3. Only the θ is negative, but the amplitude of the poles is the same because any pole is complex. So, a k can be represented r k plus $e^{j\theta k}$ a + jb can be represented in r N θ, which is nothing but r $e^{j\theta}$.

So, if it is a + jb, then r $e^{j\theta}$. If it is a -j b, it is r $e^{-j\theta}$. Now, what is the significance? So, how do you relate pole-zero in the physical system? All of you have already known the Laplace transform bode plot. The bode plot is nothing, but the x-axis is the frequency, and the y-axis is the intensity.

So, it is nothing but the frequency response of the system. Similarly, here also from the pole position, the pole which is close to the unit circle basically gives you a resonance frequency; so, each pole corresponds to the resonance frequency in speech application, which is called formant.

So, if I say θ1 and θ and -θ both have the same frequency $e^{j\theta}$ related to the frequency. So, the same frequency and the formant have a bandwidth if it is not in a unit circle in the origin. So, each pole corresponds to a formant frequency, which is called the resonance frequency of the system.

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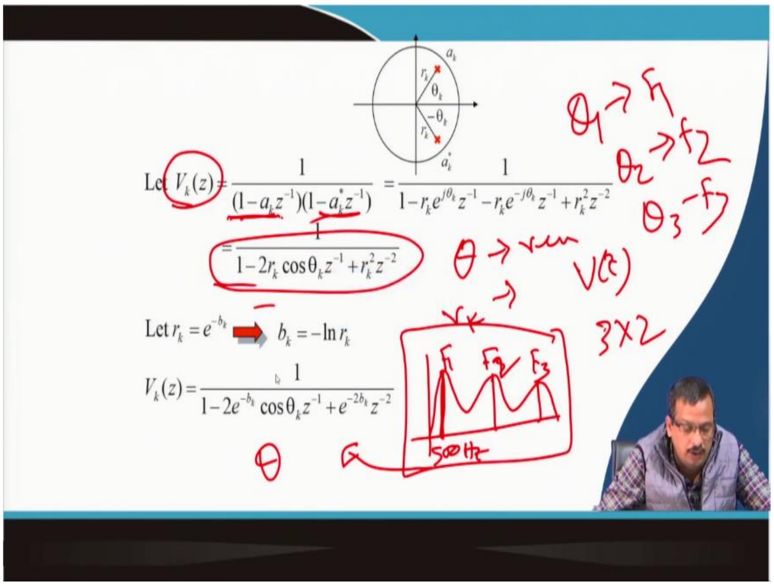

Let us say I have an N number of poles. So, I can plot it. Let us know that this is the r k. Let us know if the two poles are there. Vk(z) is equal to 1 -a k z^{-k} complex conjugate pole. Those are the complex conjugates. So, if it is a complex conjugate, I can write it down in terms of θ.

So, θ is related to the resonance frequency, and the r k is related to the bandwidth of the resonance frequency. In application, θ is related to the resonance frequency, and r k is related to the bandwidth of the resonance frequency. So, let us say system let us say I have observed the speech signal. I found it has three formants: F 1, F 2, and F 3. F 1, F 2 and F 3; there are three forms in the speech.

This is the frequency transform of the speech signal. So, this is the output frequency representation. So, I want to know how many poles there are in the system in the V z. So, as I said, 1 complex conjugate pair is related to one formant. So, θ1 is related to one formant F 1; θ2 will be related to F 2 and θ3 will be related to F 3.

So, if I say how many complex how many complex poles are there 3 into 2, how many complex conjugate pairs are there 3 pairs? Now, suppose I measure the frequency of this formant frequency resonant frequency. So, let us know if this is around 500 hertz. So, how is this related to the θ?

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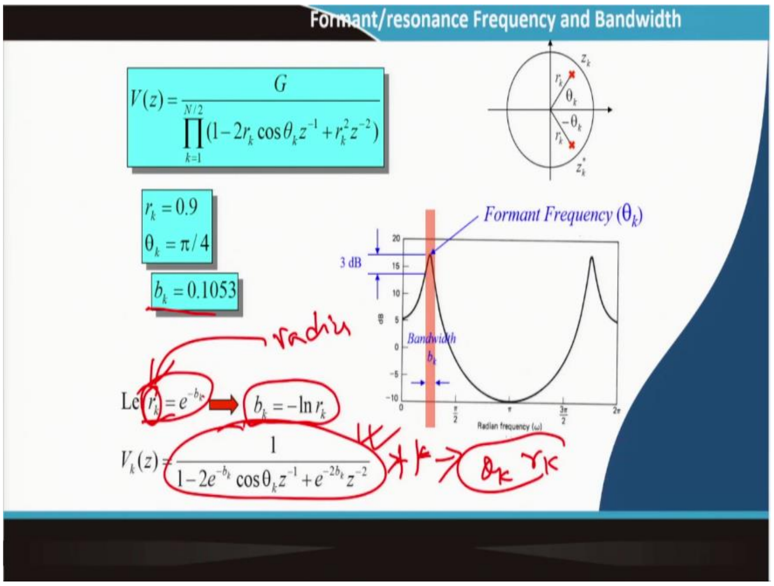

So, I give a complete example. Let us say ah, I will take a slide, I will give you the problem, and then I solve it.

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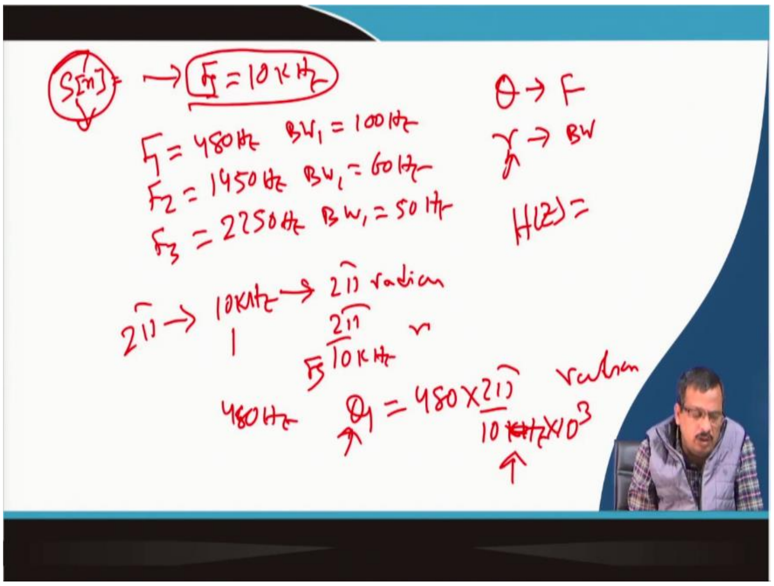

Let us say I said the problem looks like this. Let us say that I have a speech signal. Let us say I have a speech signal and signal time to speech signal is nothing, but let us say S[n] ; S[n] is the speech signal which has a sampling frequency of F s; F s is equal to 10 kilohertz ok.

So, I have a digital signal. It may be a speech, let us say, speech, and it has a sampling frequency of 10 kilohertz. Now, after analyzing the signal, I found S[n] has a formant frequency F 1 is related to 48 hertz, and fast formant has a bandwidth this bandwidth is equal to 100 hertz.

Then I have an F 2, which is related to, let us say, for 50 hertz, and bandwidth is equal to, let us say, 60 hertz, and I have an F 3, which is related to, let us say, 2250 hertz and bandwidth is equal to, let us say 50 hertz. Can I derive the transfer function of vocal cords during the production of S[n] ? How do I derive the transfer function of the vocal cords during the production of S[n] ; that is my problem.

So, how do I solve it very easily? So, what I said is that θ is related to the formant frequency, and r is related to the bandwidth. So, as you know the discrete in the first class, I have said the normalized discrete frequency. If the sampling frequency is equal to 10 kilohertz, then 10 kilohertz, then I know that the maximum rate of oscillation is 2π. So, 2π corresponds to 10 kilohertz. On the other hand, I can say 2 10 kilo hertz corresponds to 2π radian.

Then 1 kilo 1 hertz corresponds to 2π by 10 kilohertz radian or nothing, but an F s. Then I have a θ. I have to find out the θ for a 480 hertz. So, I can say the θ1 equals 480 hertz multiplied by 2π by 10 kilo hertz10 into 10 to the power 3 radian.

If I want to multiply, I want to get an angle. So, I can say π is equal to 180 degrees. I get the value of the θ or reverse way. If I know θ, I can calculate the formant frequency. So, if I know the z representation of the S[n] or vocal track, I can find out what is the formant frequency related to pole number 1, pole number 2, pole number 3, and vice versa, then what is bandwidth? What is the bandwidth is related to the r.

So, bandwidth is related to r is nothing, but that b k is r k is e^{-bk} where b k is the bandwidth in radian, or I can say b k is equal to minus l n of r k. So, if I know b k in radians, then I can calculate r.

So, what is given b k is given in hertz. I have to convert it into the radian; once I know the radian in b k, I can calculate r k and vice versa. If I know r k, then I calculate that transfer function. also, once I know θ k and r k, I can find out the transfer function is the z domain of the vocal tract.

So, z domain representation can give me the frequency response of the system analyzing pole and zero. So, this is why it is required, pole-zero concept, as I discussed, instability in the first class of Z transforms that if the poles are at a unit circle, that means it will self an oscillatory system if the poles are within the unit circle; that means, the oscillation will die down or decay down; if the poles are outside the unit circle; that means, oscillation is gradually going up.

So, based on that, I can find out the stability of the Z transform, and I can also find out the stability of the system. So, if the system wants to be stable, all poles must be on the inside of the unit circle. The unit circle means r k is equal to 1 ok.

Now, if you see, then if I say that I have a pole position like this, which resonance frequency will be prominent? The prominent resonance frequency means when the r k value tends to 1 because it is constant oscillation will be there unless it dies down.

So, if the r k value tends to 0, then oscillation will initially die down. So, when I say the poles are at zero position, z equal to 0 is a pole. What is the contribution of this? The contribution of this is that when the poles are at the zeroth position, oscillation is there, but the r k value is equal to zero.

So, if the r k value is equal to 0, it means bandwidth is equal to 0. So, I get an infinite power at a particular frequency at the resonance frequency, but when the r k value is not 0, I get a bandwidth of the formant or resonance ok.

So, this is the concept of pole and zero, and that is why I have to do the Z transform. Let us have a given example; I think not this 1. Let us say, 1 example I may be like this. Let us say I take an another slide.

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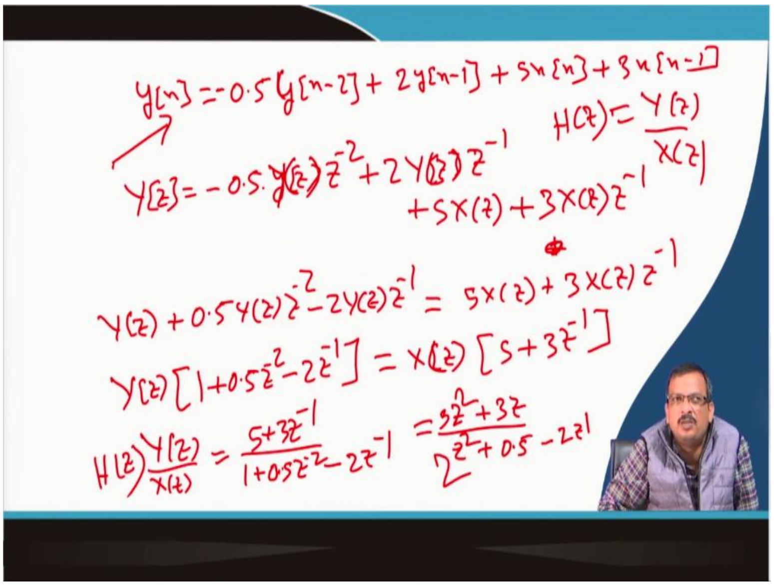

So, suppose I told you to derive the transfer function of an LTI system which is expressed in a differential equation

$$y[n] = -0.5y[n-2] + 2y[n-1] + 5x[n+3]x[n-1]$$

This is my differential equation representation of the y n to derive the transfer function of this system. The system that produces y n finds out H(z). So, if I want to find out H(z). H(z) is nothing but a y(z) by X (z). How do I do that? I take the Z transform. So, Y(z) z^{-2} plus 2 into Y(z) z to the power first bracket z^{-1}.

Then, I can say plus five into X (z) because there is no delay. So, z to the power 0 means once plus three into X (z) into z^{-1} plus ok done. Now, I put the Y on this side. So, Y(z) plus 0.5 Y(z) into z^{-2} -2 Y(z) into z^{-1} is equal to 5 X (z) plus 3 X (z) into z^{-1}.

Now, I take Y(z). So, that is nothing, but a 1 plus 0.5 z^{-2} -2 z^{-1} is equal to X (z) 5 plus 3 z^{-1}. Now, if I say Y(z) by X (z) is equal to 5 plus 3 to the power z^{-1} divided by 1 plus 0.5 z^{-2} -2 into z^{-1} ok. So, this is the transfer function H(z).

Now, if I say how many poles are there, 2. How many zeros are there I can make it positive also. So, if I want to make it positive, so, z to the power. So, if I multiply by(z) z to the power 2, both sides are upper and lower.

So, I can say(z) to the z^2 multiplied by$(z)^2$ both. So, 5 z^2 plus 3 z divided by z^2 plus 0.5 -2 z^1, I can make it positive z ok. So, this is the z transform representation of the transfer function ok.

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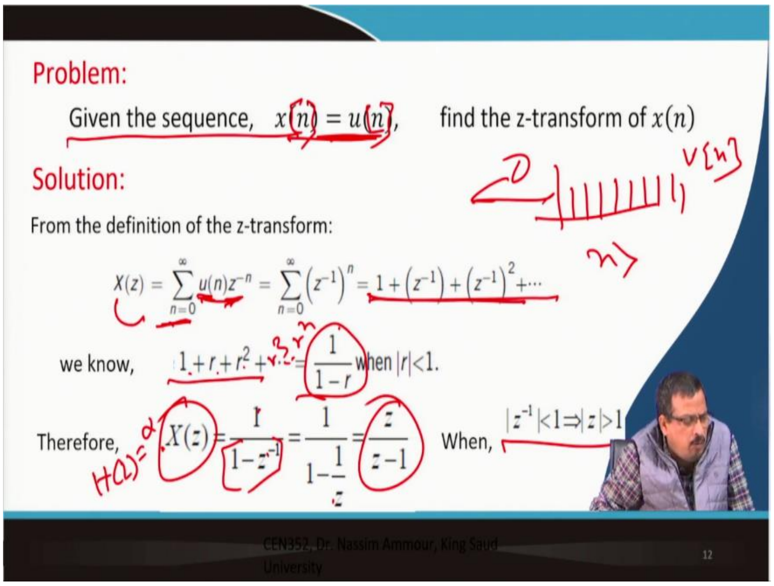

Now, I have some problems which is related to the z transform I will discuss, and then I will end this class because this class is related to the. So, I have already covered the z transform. So, that is why some problem I will discuss let us say that given in sequence x n equal to u[n], you know what is u[n]; u[n] is in the unit step function; that means, at any N greater than equal to 0 then all r 1 so, this is u[n].

Now, if I want to take the z transform of x n X (z), I know this N is equal to 0 to infinity because on this side, it is 0. So, I do not take N equal to -infinity to infinity; I take N equal to 0 to u[n] z^{-N}.

So, I can say$(z)^{-1}$ into n. So, as a series, if it is series, you know that r^2 1 plus r plus r^2 plus r cube dot dot dot r to the power n is equal to 1 by r. So, I can say X (z) is equal to 1 by 1 $-z^{-1}$. So, I can say -1 means 1 by (z). So, z by (z-1), then what is the ROC region of convergent for which X (z) is not infinite? Here is another point for every pole X (z) is infinite.

So, that is why the poles are not within the convergent of the z transforms poles are outside of the region, or I can say the poles are not included inside the ROC poles are not included inside the ROC understand. So, I can find out which X (z) is finite, which is the region of convergence, and that is ok.

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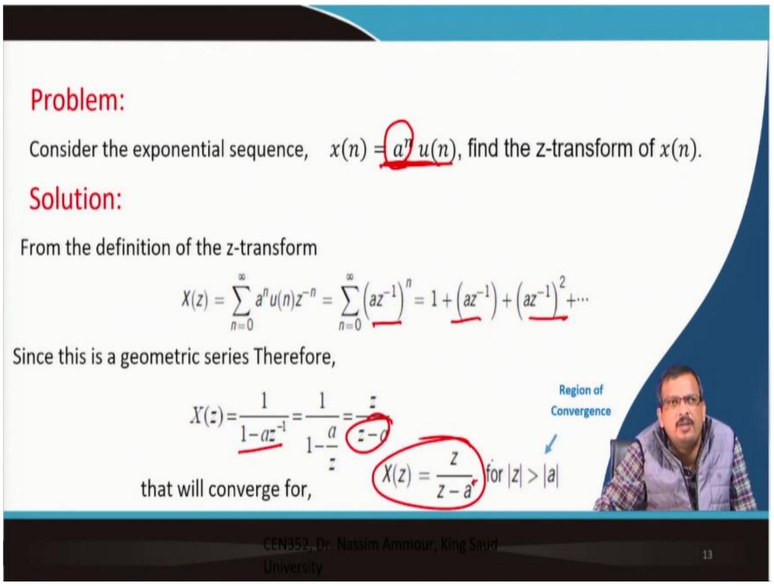

So, that way, we can solve another problem. Let us say consider X n is equal to a to the. So, if there is a constant multiplication a to the power a nu[n] same problem, I multiply by a constant. So, it is nothing, but a so; a will be multiplied region of convergent I can calculate ok. So, it is a to the a/(1 -a) ok.

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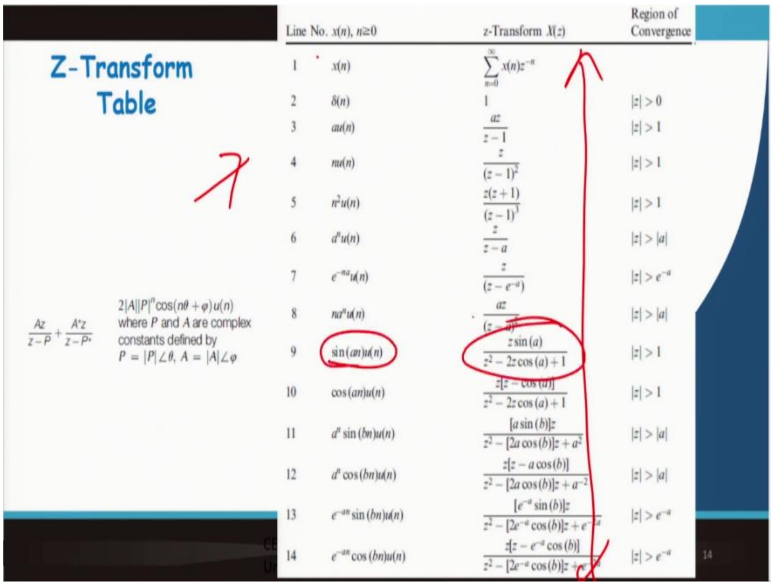

Similarly, those are the tables where you can get the table; table of z transform, which is the table for which z transform is already given using that table when we go for the z transform, we can use this table, or when we go for the inverse z transform we can use this table.

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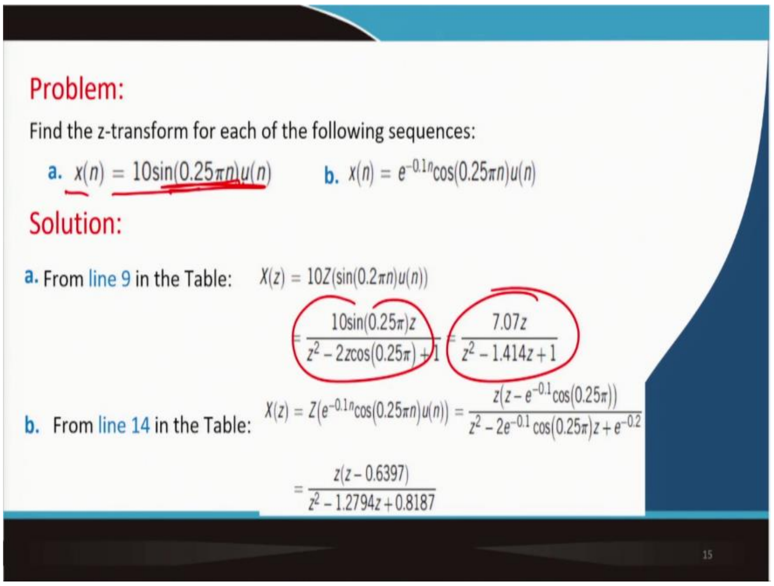

Let us say another 1 find the z transform for the X n is equal to you can do it 10 sin 0.5π n u[n] line number 9, what is line number 9; sin of a n u[n] is this 1 is the z transform.

So, you just put the value, you can get this z transform, or you can do it you can do it z to the n equal to -n infinity, put that value, and find out the z transform ok. So, that is the complete z transform we have covered. Now, we go for the inverse z transform in the next class.

Thank you.