Signal Processing Techniques and its Applications
Prof. Shyamal Kumar Das Mandal
Advanced Technology Development Centre
Indian Institute of Technology, Kharagpur

Lecture - 13
Z – Transform Properties

Ok, now we will talk about the properties of Z Transform.

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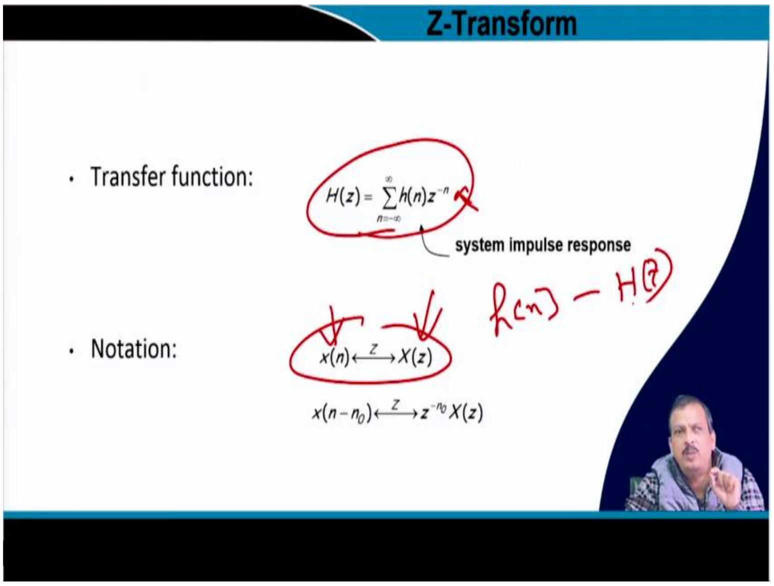

So, before I go to the properties of z transform. So, in the H(z), as I discussed, if H(z) is a system. So, the system impulse response will look like this. So, this is a notation. You have to remember if x(n) is a time domain signal, then X(z) is a Z domain signal. If h(n) is a time domain signal, then the capital H(z) is a Z domain signal. So, when I write Z domain, it will be capital, and it will be Z. Z domain means the complex domain is equivalent to the frequency domain.

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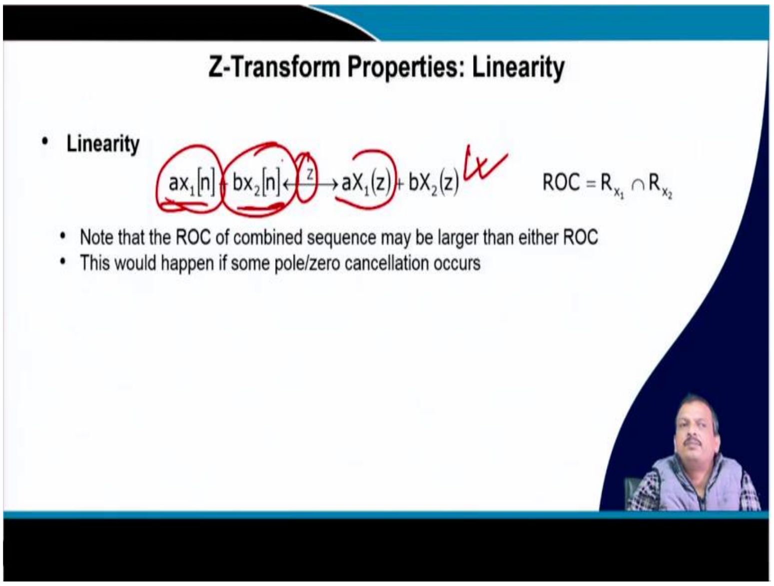

Let us go to the properties of object forms. What is the property? So, the linearity property z transform is linear; that means, if a signal is linear if the system is linear after the z transform, it also becomes linear. ROC may change, but the system will depend on the z transform to be linear. So, how do you do that, how do you prove it? Again, the superposition principle. So, I can say ax1[n] and bx2[n] are two different signals. So, if I took the z transform, it would be aX1(z) plus bX2(z).

Or if I apply the entire signal and then take the z transform, the expression will be the same so which means the z transform supports the superposition principle. So, that is why the z transform is linear. So, the linearity of the z transform is linear.

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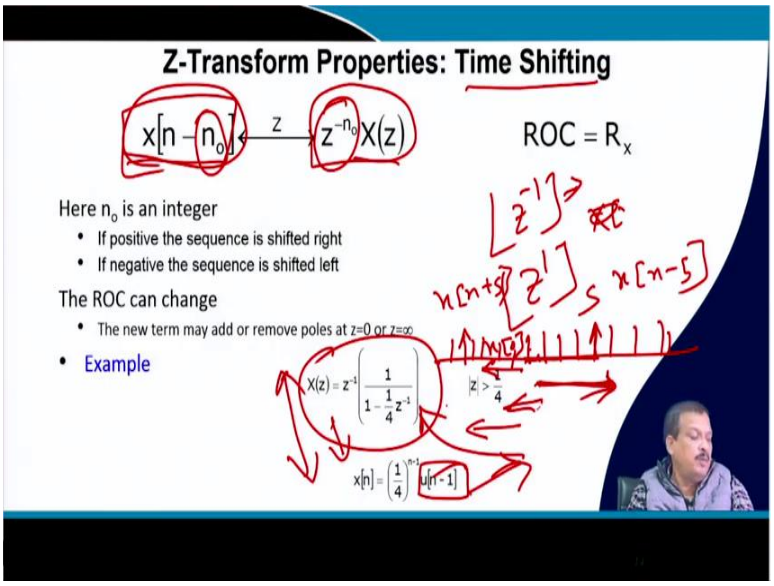

Now, for the time scaling, time-shifting or time scaling time shifting. So, as you are aware, when I draw the system level, that implementation of discrete system structure 1 or structures 2. So, what you say is that z^{-1} means one sample delay and z^{-1} means one sample advance.

So, time shifting means if except x[n] is shifted by time n_0 where n_0 is an integer. So, that means, suppose I have anx(n) if I shifted to this side or this side; if I shifted to this side this is x[0]. So, I want to shift it to here, which is, let us say, 5. So, this is nothing but an x[n-5]; when I go to this side, this is x[0], and this side is negative, it becomes. So, again, here, the 5th sample is x[n-5] because I have shifted on this side. So, n is positive when I shifted to this side, and n is negative when I shifted to this side.

Now, I can prove that $x[n-n]_0$ is nothing but a z^{-n0} into X(z). You can prove it. How do I prove it? Again, I just take slides if I want to do it mathematically. also, I can prove it with no problem.

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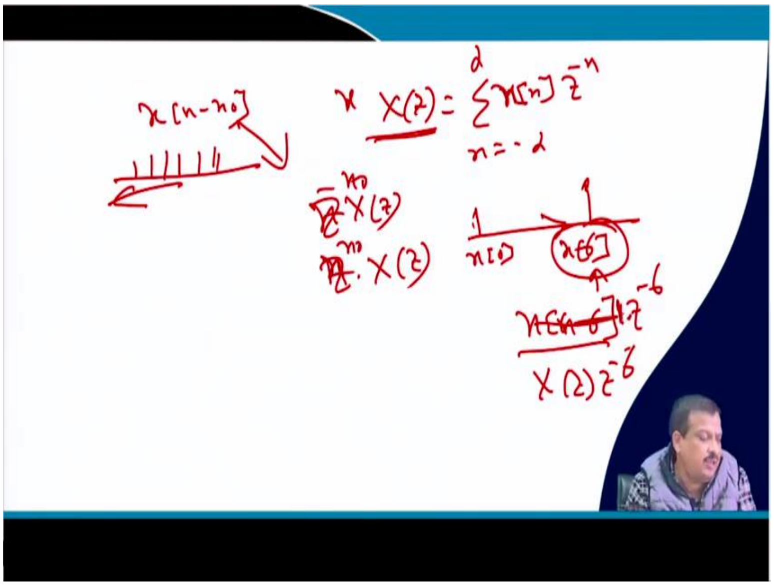

So, now my signal is $x[n-n_0]$. So, I know X(z) is nothing but a n equal to minus infinity to infinity x[n] into z^n. Now I said x[n] is shifted. Change it to n minus n_0, understand.

So, now I put that equation here, and I can change it; I can prove that this is nothing but an X(z). So, if I shifted in this side, n_0 is positive, then I can say X(z); X(z) of this one will be z^{-n0} into X(z). If n is negative on this side, I can say z^{+n0} into X(z). So, when I shifted to this side. So, what I say if I shifted to this side. So, suppose here is my x 0; I want to shift to the 6th position, x[6] here I want to make here.

What is the meaning of how I represent x[6]? x[6] is nothing but a n minus 6 into z^{-6}. So, when I transform the Z domain, it becomes X(z) into z^{-6}. So, amplitude into z^{-6} you know that. So, that is called shifting time shifting. You can do this example. Also, let us know if this is my X(z) and see if I converted to the x n, it will be shifted by one sample inverse transform, or if I do the z transform of these things, it will come here ok.

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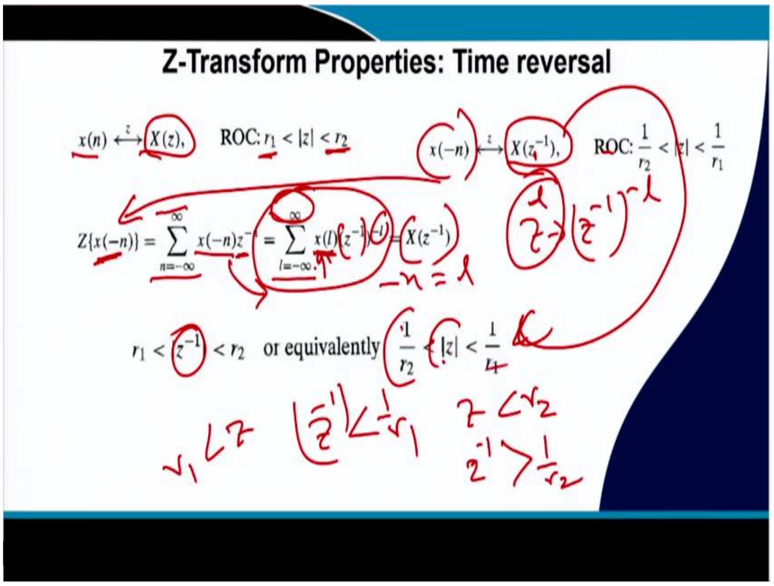

Then I say time reversal, which means if x[n] is z transform if X(z), then x[-n] z transform will be X(z^{-1}), time reversal. So, how do you prove it? Let us say I want to z transform x[-n]. It is n equal to minus infinity to plus infinity x(-n) z^n. This is z transform. So, now let us say minus n is equal to l. So, I replace this minus n with l. So, when it is, I can say the l equal to minus infinity to plus infinity x[l-n] represented by l z^n is nothing but a l.

So, if I want to make it minus l, how do I do z^{-1} into the minus l? So, minus 1 into minus l is a z to the power l. Now, if you look at this form, it is a z transform of x l. So, it is nothing but an $X(z)^{-1}$. Now if the region of convergence is r1 and r2, let us say r1 so both sides it is there. So, I said the region of convergence is z is greater than r1, and z is less than r2. Now, let us replace z with z^{-1}.

So, if it is when the z is greater than r1 when it is z^{-1}, that means z should be less than 1 by r1, understand. If z is replaced by z^{-1} when the z is less than r2, when it is replaced by z^{-1}, it will be greater than 1 by r2. So, I can say the region of convergence of this transform this X(z) is nothing but a 1 by r2 greater than z less than equal to 1 by r1 ok.

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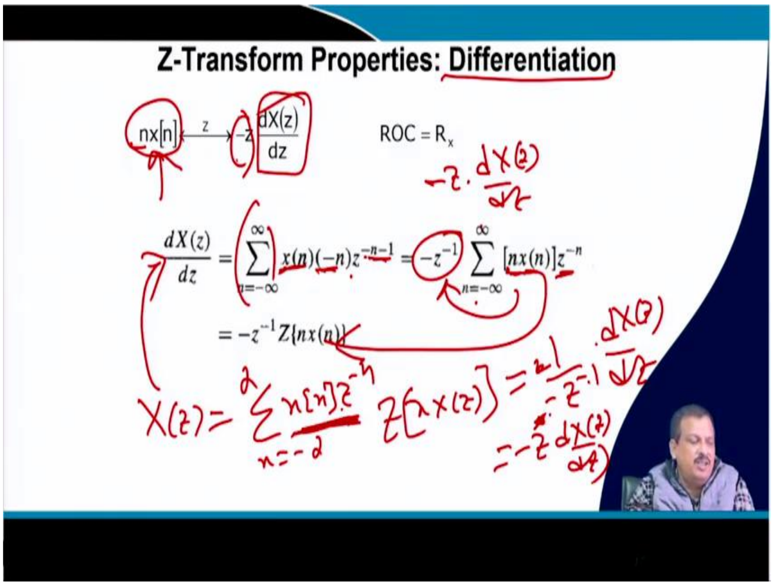

Then next is the differentiation. Many times, we will use it during the z transform.

That n is the z transform of

$$nx[n] = (-z) * \frac{dX(z)}{dz}$$

So, if I do that, what is X(z)? X(z) equals n equal to minus infinity to infinity x[n] z^n. Now, if I want to find out what dX(z)/dz Is, I have to take the differentiation of this thing. So, it is ok; this is x[n]. So, this is my z first-order difference is z^n minus n into z^n minus 1. Now let us say I said minus and minus z^{-1}. I take outside the summation.

Then I said n equal to infinity x n in so, n will be there, and x[n] will be there, and z^n will be there. So, it is nothing but a z transform of nx(n). So, I can say the nx(n) the z transform of nx(n) is equal to minus. I can say the 1 equal to this will be 1 by minus z^{-1} dX(z)/dz, which is nothing but a minus z to the power minus z minus 1 will go up to plus 1 to minus z dX(z)/dz

So, the z transform of nx(n) is nothing but a minus z and dX(z)/dz ok. So, many times, we will use this formula for inverse transform or z transform when you take a log 2 z, let us say to log of x n plus 1. So, how do you do that z transform? So, use this differentiation method.

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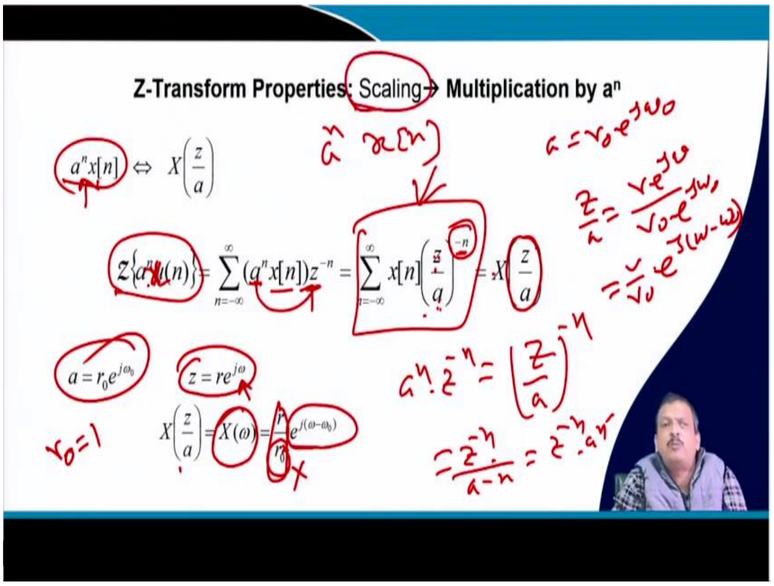

Then scaling is a very important issue: frequency scaling and time scaling. So I can scale its amplitude and frequency both, so scaling. So, let us say $a^n x[n]$. So, x n is my signal and scaling is done by a^n.

So, $a^n x[n]$. So, a^n is a scaling factor. So, what is the z transform of $a^n x(n)$? So, I want to calculate the z transform of a^n x(n) let us say x[n] is u n let us say or let us say x n do not take u n let us say x[n]. So, I can say $a^n x(n)$ z^n.

So, I can say a can be divided by z. So, z by a^n because a^n into z^n can be written as z by a whole power n, which is nothing but a z^n into divided by a^n which is nothing but a z^n into a^n.

So, that is why I wrote down a^n and z^n in the form of z by a, a^n. Now you can see this one. This one looks like a z transform where instead of z, I have a z by a. So; I can say it is nothing but an X(z) by an ok. Now a is a value; a can be a real, or a can be a complex also, let us see a is represented by $r0e^{j\omega 0}$ where I have written down and what is z, z is nothing but a z is $re^{j\omega}$.

Now, if I say the value of z by a. So, z by a is $re^{j\omega}$ divided by $r0e^{j\omega 0}$. So, that is nothing but a r by r_0e into $e^{j\omega}$ minus ω_0, ok or not. So, I can say X(z) by a represented by xω is nothing but r by $r_0 e^{j\omega 0}$. Now, let us say r_0 is equal to 1. So, if r_0 is equal to 1, I can say that this scaling amplitude scaling is not there. So, it is only the rotating part that is there in the z domain; it is rotating only. So, shifting ω minus ω_0 is okay; that is called scaling z transform scaling.

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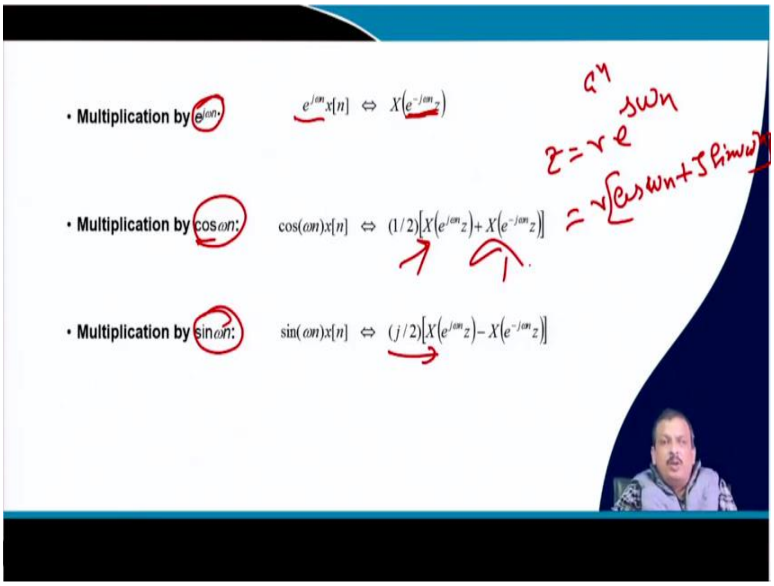

Similarly, instead of a^n, I can scale it by $e^{j\omega n}$ where r_0 is, so it is nothing but $ae^{j\omega n}$ into z. I can multiply by cos(ωn); I can multiply by sin(ωn). So, those are the z values because here, when I go for the cos(ωn) I can say z is nothing but $re^{j\omega}$ n. So, in that case,

$$Z = r^*(\cos(\omega n) + j\sin(\omega n))$$

Then I have an infinite series, and then I do the z transform ok.

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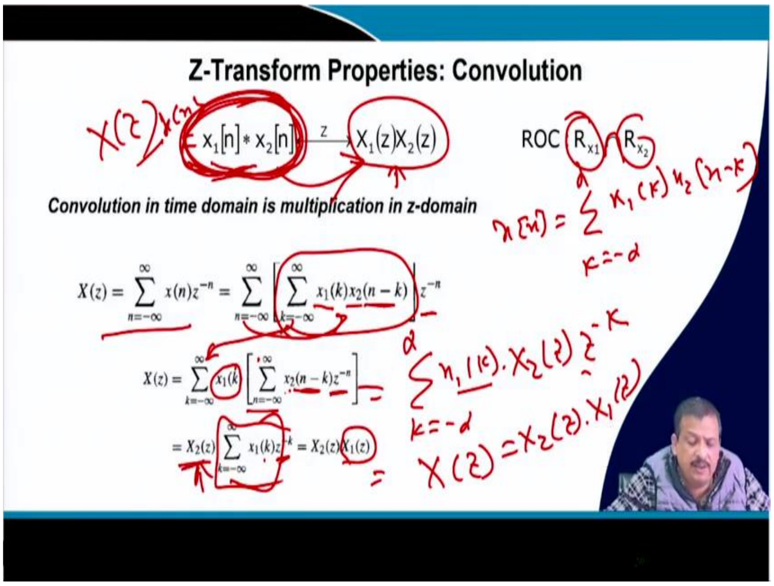

Similarly, another property is convolution. You have heard about time domain convolution frequency domain multiplication z, which is like a frequency, but it is not frequency domain z domain in z domain. Also, that is valid; that means time domain convolution in the z domain is simple multiplication. Time domain convolution is represented by a z-domain simple multiplication. You can prove it also. So, let us see that x1 and x2, two signals, are convolved with each other. I want to find out what is X(z).

So, let us say this is x[n]. This x1 convolved with x2[n] is my new x[n]. So, this is the z transform of x[n]. Now, I will represent the convolution. So, x1[k] into x2[n-k] z to the power minus k. So, this is my convolution term x[n] is a convolution which is nothing but a k equal to infinite to plus infinity x1[k] convolved with x two; that means n minus k is the convolution sum. So, I represent the convolution sum here.

Now, let us say this sum I wrote first and x1[k] and then this sum was written here n minus n equal to minus infinity x2[n-k] z^n, what is there it is nothing but the time shifting. X[n-n_0], z^{-n0} X(z). So, here, n minus k is nothing but an X2(z), X2(z) into z to the power minus k. So, here I can say k is equal to minus infinity to infinity x1[k] into X2(z) into z to the power minus k.

Now, if I say X2(z) here and I check the sum to here x1[k] into z^{-k}, this is nothing but X1(z). So, it is nothing but X2(z) multiplied by X1(z). So, I can say time domain multiplication is nothing,. Time domain convolution is nothing but a multiplication in the z domain, and the region of convergence is nothing but an Rx1 and Rx2 intersection; the intersection of both signal convergence is ok.

So, suppose instead of doing convolution in the time domain, I have an x(n) signal, I have a y(n) signal. Suppose I have given a 2 signal let us say instead of writing convolution, what can we do?

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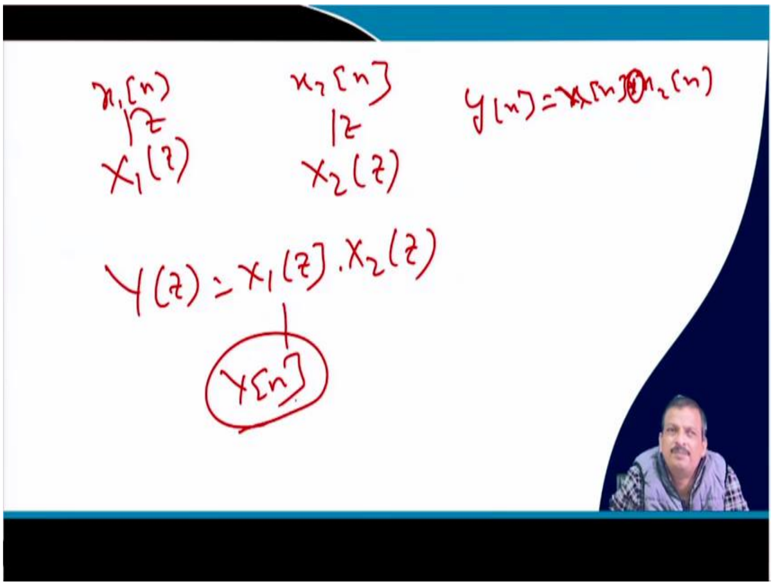

What we can do is suppose I have a signal x1[n] is given, and I have another signal x2[n] is given. So, I can convert to x of X1(z). I can take a z transform; z transform I can convert to X2(z) then I can create Y z is equal to X1(z) multiplied by X2(z), then I get the inverse z transform I get Y[n]. So, instead of computing y[n] is equal to x1[n] convolved with x2[n]. I can take individual signals to the z domain, multiply them and then take the inverse transform. I get Y[n], and I can also do it instead of computing convolution, ok?

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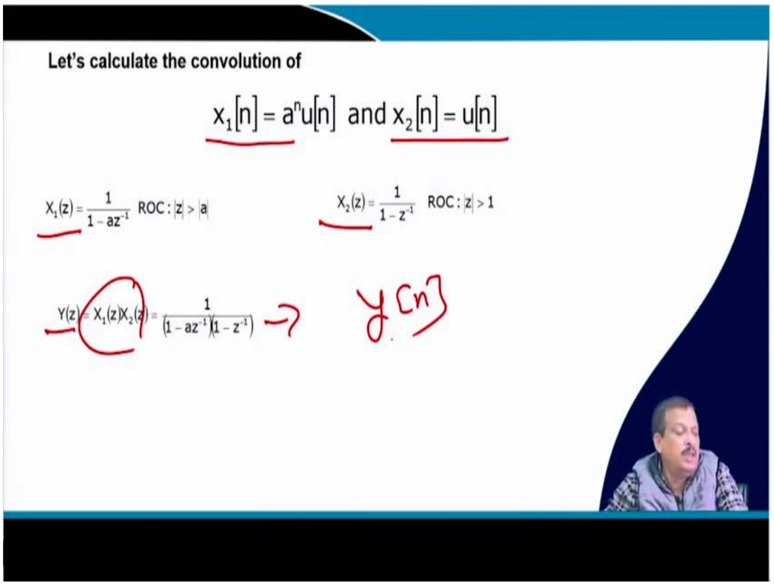

The next property is ok. Let us convolution. I have given an example you can do. For example, I will say that you will try to do it x1[n] is given, x2[n] is given. Now you calculate X1(z), X2(z), and then Y(z) is this. Now you can calculate, you take the inverse transform, and you can get y(n) ok.

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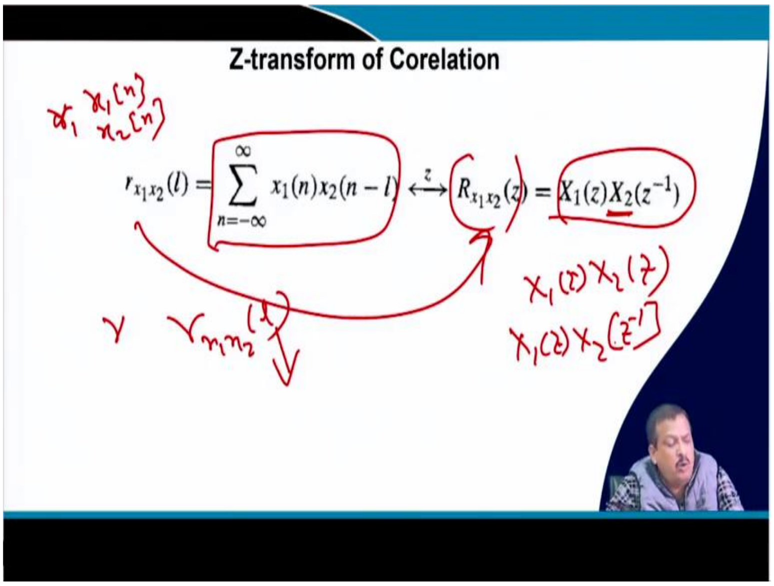

Similarly, z transform of correlation. So, convolution, we have said, now what is a correlation? So, what is correlation? Let us say I have a signal x1[n] and x2[n] are two different signals. So, R of x1 x2 is the correlation between the x1 and x2, and this is the correlation equation. Now you take the z transform on both sides you take the z transform. So, this becomes capital R x1 x2(z) instead of l because l is R x1 x2. l is also a discrete sequence. So, that becomes the z domain, which is nothing but aX1(z) multiplied by instead of convolution, it is X1(z) multiplied by X2(z) in correlation.

Since I have not held the signal here, it will be X1(z) multiplied by $X2(z)^{-1}$, understand? So, that way, I can calculate z transform correlation, also I can calculate in the z domain.

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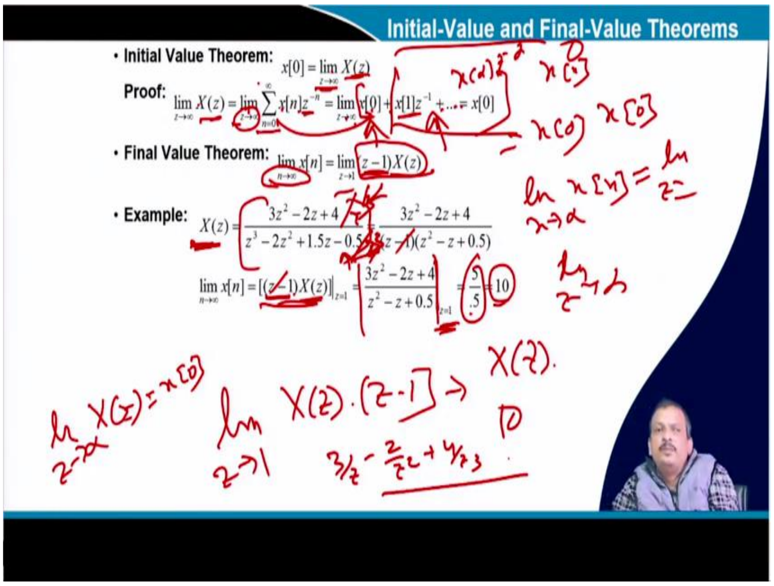

Then there is another important theorem, which is called the initial value and final value theorem both way think about. Suppose let us say it is an initial value means initial value. Suppose I have an x(n). What is the initial value? The initial value is nothing but an x[0] is the initial value.

So, what is the initial figure of the initial value theorem said that if I know X(z), if I know the z domain representation of signal x[n], which is nothing but an X(z), then if I take the limit tends to infinite on that z domain signal I get the initial value of the signal. So, what is the initial value theorem for an unknown signal, if you know the z transform of that signal or z domain representation of the signal, then you can estimate the initial value of the signal. So, it is nothing but a x 0.

How do you do that now that x 0 is equal to the limit put limit z tends to be infinite on X(z)? You get the initial value; what is the proof? So, I said limit z tends to infinite X(z).

So, initial value theorem. So, z tends to be infinite. What is X(z) n equal to 0 to infinity x[n] z^n? Now, you can break down the series. So, limit z tends to infinity x[0] plus x[1] into z^{-1} dot dot dot x to the x of infinite into z to the power minus infinity.

Now, when I say put the z limit to infinite. So, those portions become 0, and only x 0 exists, which is nothing but an x 0. So, that is why I said the initial value theorem said that if I know a z domain representation of a signal or system whatever, then if I put limit z tends to infinity of that z transform, I can estimate the initial value of that signal x 0, this is the proof. Final value theorem: I can estimate the end value of the signal; how do you do that end value is very simple: n tends to be infinite here, as I said.

The limit of x[n] end value is if the n is infinite, then n tends to be infinite, which is equal to limit z, which tends to 1, z minus 1 into X(z). So, X(z) will be multiplied by a z minus 1, then I put the limit z, which tends to be 1, and then I get the final value of the signal. If it is infinite, then x infinite; let us say this is said that the example X(z) is equal to this one. So, X(z) I know; I know X(z). So, what are the initial and final values of this X(z) signal x n represented by z transform z domain in X(z)?

So, for the final value, what do I have to do? For the initial value, I have to say that limit z tends to be infinite, and I have to evaluate this function limit only for the final value; I have to multiply by z minus 1 into X(z), and then I have to limit for z equal to 1 understand. So, what they do is they make a factorize, and then they multiply X(z) minus 1, which will be cancelled. Both that factorize, and then I put the z equal to 1 value. It is 5 by 0.5, which is equal to 10.

So, I can say the final value is 10, then what is the initial value? I can put z equal to infinite. So, I have to multiply both sides. I have to divide it by z cube, and then I get. So, it will be 3 by z minus 2 by z square plus 4 by z cube divided by like that way you can do, and you evaluate the limit you can get the initial value.

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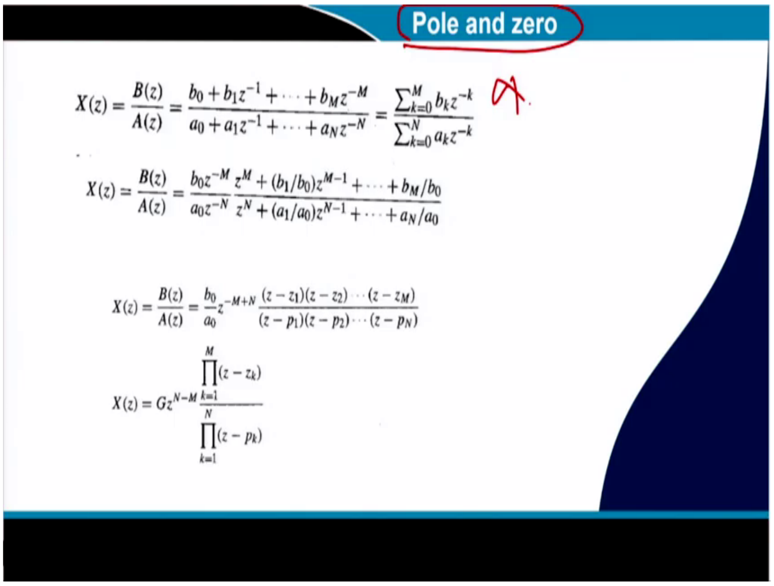

So, those are the properties of the z transform. In the next class, I will talk about the concept of pole and zero and why this z transform is required to find out what kind of problem I can solve using the z transform. Why should I use the z transform that I will discuss in the next class.

Thank you.