Signal Processing Techniques and its Applications

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Lecture - 12
Z – Transform

So, today, we will talk about the Z-transform. So, we have talked about the signals; we have talked about the systems and their discrete implementation we talk about.

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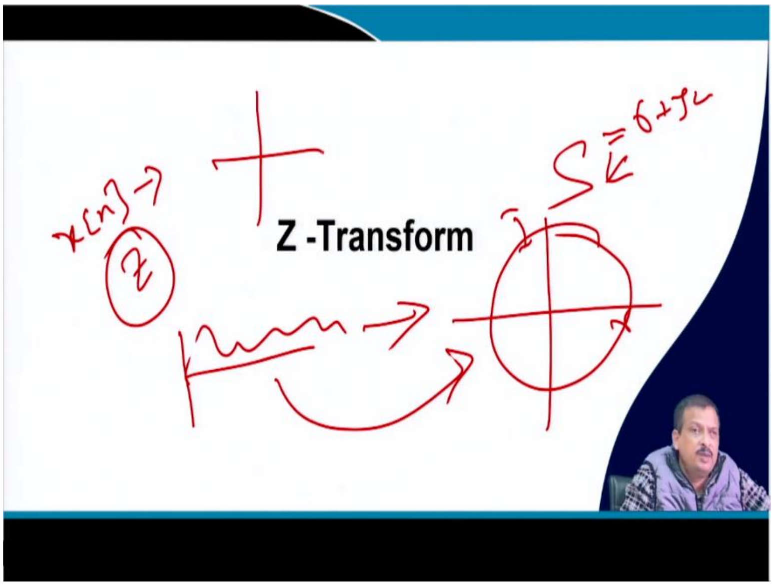

So, today, we will talk about the Z-transform. Now, before we start Z-transform, do you know what transform is? Why do we go for transform, Z-transform, or any kind of transform? There are a lot of kinds of transformations. What is transform? Why do you go for Z-transform? You have already learn learned about the Laplace transform.

So, why are these transformations required? So, transform means a change of representation. So, suppose something is there; let us say this is an object. I can see this side of the object, but I cannot see the other side of the object.

Now, if I want to see this side of the object, what do I have to do? I have to take it and rotate it. So, rotation can be a kind of transformation. So, I am representing the object in a different plane. So, in that case, that is called transformation. So, let us talk about the Laplace transform. When you talk about the Laplace transform, what have we done? Now, Laplace transform, any time domain system or time domain signal, when I want to make it frequency domain which is called S-domain. So, I want to decompose, I want to transform a continuous time domain signal to another plane, which is called S-plane, the laplacian plane that is Laplace transform.

So, I am transforming the time domain representation to another. So, for any vector, let us talk about the signal; a signal is nothing but a vector. Let us talk about a vector. So, one representation of the vector is there; I want another representation.

Say the time domain signal is there. I have a time domain signal like this. Let us say I have a time domain signal like this, but I want to represent this signal in another domain, which is a complex domain called the S-domain, which is called the Laplacian transform. What is a complex domain? That means I am saying this time domain signal can be represented in a complex coordinate; one is real, and the other is an imaginary axis.

Then, I plot this signal in this representation; then, I call it the Laplace transform. So, for continuous signal, we go for Laplace transform. Now, when I have a digital or discrete signal, it means time is not continuous. So, I have an x[n]; here, we use Z-transform. Same as Laplace transform in the continuous domain, what is S? S is nothing but a something j plus something. So, that is nothing but a complex. So, this is an S; S is nothing but a complex number.

Similarly, Z is also a complex number, so any discrete signal when I want to represent it in a Z-plane. So, the z-plane is a complex plane; I have a real and imaginary axes. So, when I have a discrete signal, I want to represent it in the z domain, which is equivalent to the equal frequency domain. So, from the time domain signal to the frequency domain signal, we call Z-transform ok.

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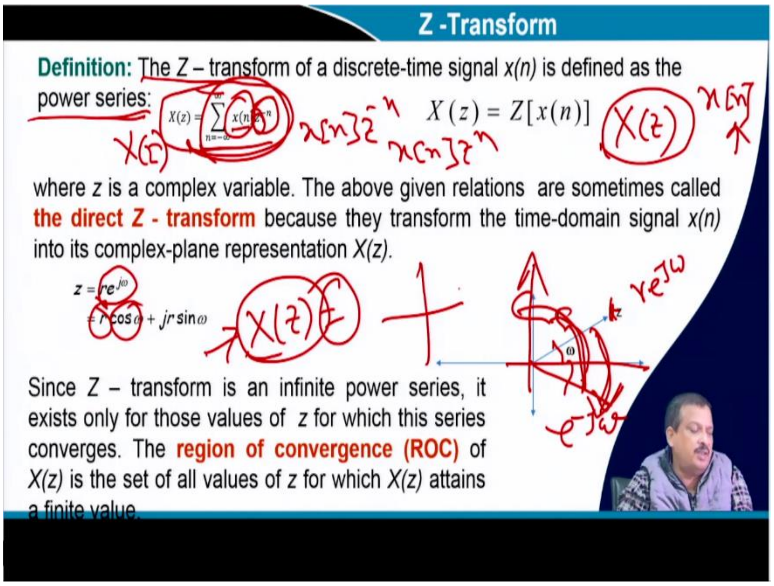

So, let us say what is Z-transform. So, the Z-transform of a discrete time signal x[n] is defined as a power series. This is the definition of Z-transform. So, what am I doing? I have a digital signal, x [n], and I want to transform it into a z-plane. What is the formula for transformation?

n equal to minus infinity to infinity x[n] z^{-n}. So, if n is positive, it is x[n] z^{-n}; if n is negative, it is x[n] z^{n}. So, each sample is added up. So, X(z), so in the z-plane representation of x(n), I go to the z-plane, where z is a complex number, and when I do that, the new signal, which is represented in the z-plane is called X(z).

Now, the question is, why should I do that? As we have already said, if I want to see this phase of this object, I have to rotate it. So, I have done rotation because I want to see this face phase. Now, why do we do Z-transform; why is it required? Why do we go for Laplace transform; why is it required? Is it for the sake of mathematics? No. So, what is that? The Z-transform is required to analyze a discrete system called the pole-zero concept.

Similarly, Laplace transform is required; I go for the Laplace domain when I want to analyze a continuous system. Here, also, if I want to make it analyze. So, analysis will become easier with a discrete system when I go to the z domain. So, that is why I require. So it will be easy to analyze discrete signals for discrete systems. So, the z domain is called frequency domain representation. So, that is why I do it. I have to establish what kind of analysis. So, at the end of the lecture, you can understand why this Z-transform is required.

If the Z-transform is not there, what we cannot do? So, initially, I said X(z) is the z domain representation of x[n]; x[n] is a discrete signal X(z) is the z domain signal, and z is a complex variable. It has a real axis, a real or imaginary plane. So, z in any point can be represented by $re^{j\omega}$, or z is a complex n number, which is $re^{j\omega}$. So,

$$e^{j\omega} = (\cos(\omega) + j \sin(\omega)) * r ;$$

r is the amplitude and jω is the rotation ω is the rotation.

This axis is positive; on this side, it is negative, and when anticlockwise, it is called positive. So, it is (-)jω, which means clockwise. So, if it is ω is positive, this side is $e^{-j\omega}$. So, that is the requirement for the z domain, So, when I use Z-transform, we want to represent a discrete signal in a complex plane called the z domain. Now, is the z when the Z-transform will be valid? I want to make it transform, but transform has to exist.

If you see here, it is an infinite power series, X(z) is nothing but an infinite power series x[n] multiplied by z^{-n}. So, it is an infinite power series. So, when will it exist? If X(z) is finite, how can I do it? if it is an infinite value, then what representation can I get? So, when we do transform? When X(z) is finite, this means that in this series, this sum has to be [FL] and be a finite number. That means it has to converge.

Any transformation must have to be converged when I want to do it; unless it is infinite value, I cannot see anything there. So, for which z value, this X(z) has a finite value, or the transform is, or series is converged, that is called the region of convergence; ROC, region of convergence. For those z values, the z value is nothing, but a z is a plane. So, it is an x-y plane, which is the real part, and every z has a real and imaginary part. So, there is an X(z)-plane. So, I transform that discrete signal into the z-plane.

But that transform is only useful when X(z) has a finite value, or I can say that for those z values for which X(z) is finite, those z values are called regions of convergence in that z-plane; understand? So, that is why when I do a Z-transform, I have to think about the region of convergence.

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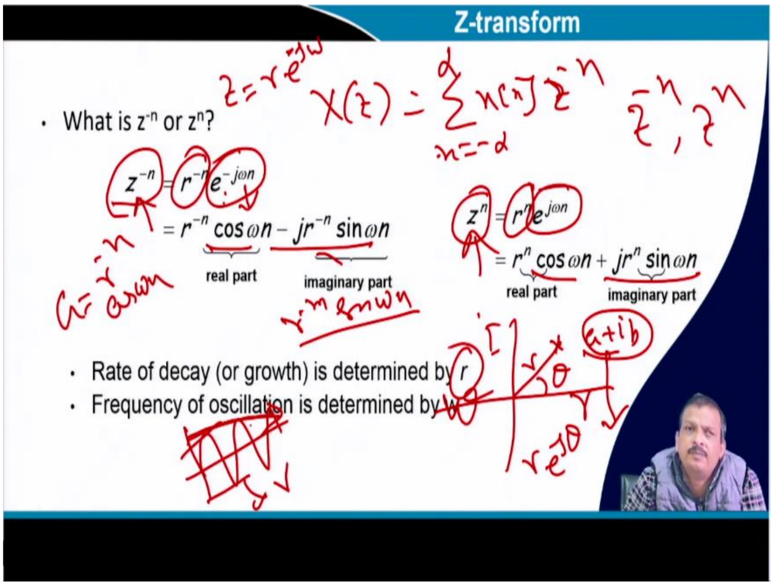

Now, I come to the concept of the z. What is z; what is z, z^{-n}? You know the X(z) is equal to minus infinity to n equal to minus infinity to infinity x[n] z^{-n}. So, if we either z to the it could be z minus n or z^n, when z is n is positive; it is z^{-n}, when n is negative is z^n.

So, what is z? I said z is nothing but a plane; z-plane. So, I have an x-y axis, and z has two values; one is a real value, and the other is an imaginary value. So, any value, suppose I have a real axis and an imaginary axis. So, this point is a+ib. So, I can also represent this with r and θ. So, I can say instead of a+ib. I can say it is $re^{j\theta}$; θ is the angle.

So, in the same way, I can represent z^{-n} if it is let z equal to $re^{j\omega}$. So, when I say z^{-n}, so r^{-n} $e^{-j\omega n}$, let us say $re^{-j\omega}$. So, I can say minus jωn.

Now, when the n is positive, I can say r^n $e^{j\omega n}$. So, $e^{j\omega n}$, you know that it is nothing but

$$e^{-j\theta} = \cos\theta - j\sin\theta \ \ \& \ e^{j\theta} = \cos\theta + j\sin\theta.$$

So, z has a real value and has an imaginary value. So, I can say if I want to represent a+ib, a real part is r^{-n} cos(ωn) and the imaginary part is r^{-n} sin(ωn). If I want to represent in a polar form, so I can say $re^{-j\omega n}$, when z is z^{-n}.

I can say this is positive when z is z^n. So, what is r? r will determine; what is z? z is nothing but a cos and ω cos and sin. So, cos and sin is nothing but an oscillation. Physically, z is nothing but an oscillation; cos θ + jsin θ, I have an oscillation and oscillation has an amplitude and a frequency.

So, ω, which determines the frequency, r, determines the amplitude of the oscillation. So, frequency is determined by the ω and r. So, I can say every z has an oscillation; the amplitude will depend on r, and the frequency will depend on ω. So, whether it will be decay and how much it will decay depends on the value of r.

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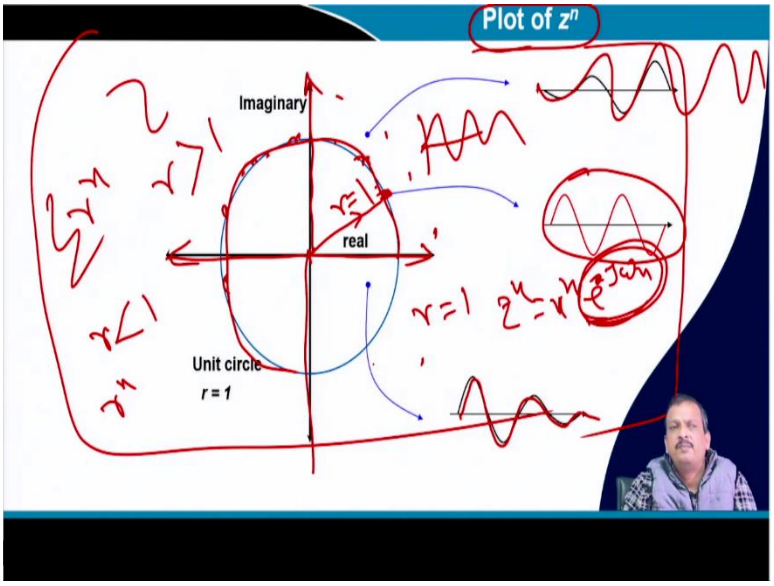

Now, all of you have heard of the Laplace transform unit circle. What is the concept of a unit circle? So, that is why I said plot of z^n let us say what if I want to plot. So, this axis is my real axis; this axis is my imaginary axis. So, the axis is the imaginary, and the axis is the real axis.

Now, you know the stability; the system is stable if it is within a unit circle. Now, if you see the z value is less, this blue circle is the unit circle. What is the meaning? Meaning is requal to 1. So, when r is equal to 1, you know z^n is equal to r^n $e^{-j\omega}$ or plus jωn; let us say z is positive.

Now, when r is equal to 1. So, it is nothing but $e^{j\omega n}$. So, it is a constant oscillation. So, when the z value dip it is here, this is the z-plane, and this entire plane is the z-plane; so, when the z value is here, but the r value is equal to 1, r is equal to 1; then I said any point on this unit circle, it is nothing but $e^{j\omega n}$.

This different point means different ω, but it also means constant oscillation. So I can get a constant oscillation of different frequencies. If I go through the unit circle, now, inside the unit circle means r is less than 1; what is the meaning of inside the unit circle? r is less than 1.

So, when I say r is less than 1, that means the oscillation has not decayed because r^n, the infinite sum of r^n, r is less than 1; so, I can say it will decay down. So, the oscillation will decay down. Now, if I say outside the unit circle, that means r is greater than 1; here, here, here, here, here.

So, when I say r is greater than 1, then I can say that in an infinite series, r^n is going to be infinite. So, there is a constant gain in the oscillation; gain. So, the oscillation is increasing, and the amplitude of the oscillation is increasing. If n goes to infinite, then it will be an infinite signal. If it is an infinite signal, the value will be infinite.

So, the oscillation will depend on the value of r, z where you are present in the z-plane. So, when I say stable system, what is the concept of unit circle? A unit circle means that oscillation will decay if it is within the unit circle. So, it is not self-oscillatory in nature. So, it will decay down. So, I can say the system is stable. Unless, what will happen? Outside the unit circle, it creates an infinite amplitude oscillation, and I do not know there is no control. So, that is the concept of the unit circle that exists in the s-plane that exists in the z-plane in both cases. Is it clear?

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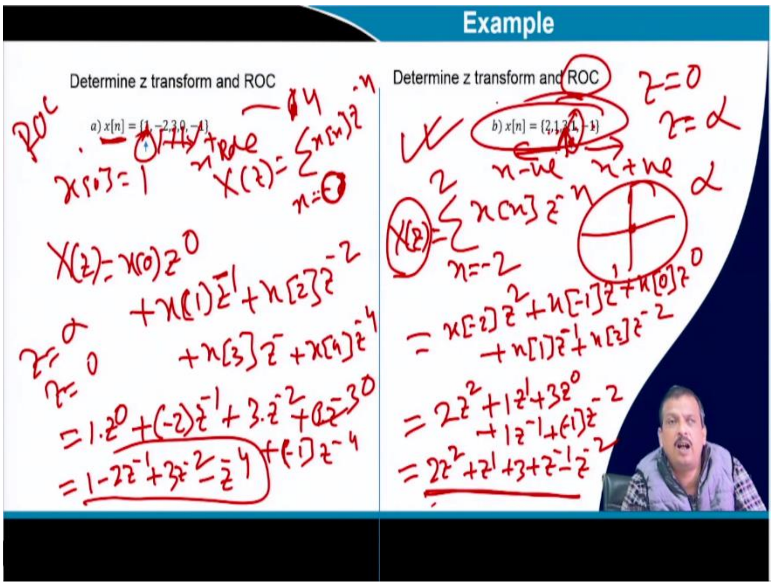

Now we go, for example, in Z-transform. Let us say I have one signal x[n] with a 0 value, which means this arrow indicates the 0 value. So, x[0] is equal to 1. So, at n equal to 0, the value is 1 and this side n is positive, positive. So, I can say X(z) is equal to x[0] because I know X(z) is equal to n equal to minus infinity to infinity x[n] z^{-n}.

Now, if you see this signal, it varies from the 0th sample. So, instead of minus infinity, we will vary from 0 to how much value there was. How many samples are there? 1, 2, 3, 4, 5. So, it goes up to 4; n minus 1, 0 to 4 in five samples. So, I can say X(z) is

$$X(z) = x[0] * z^{0} + x[1] * z^{-1} + x[2] * z^{-2} + x[3] * z^{-3} + x[4] * z^{-4}$$

How many? 0, 1, 2, 3, 4; after that signal does not exist, so I do not require; I do not have to write down that. So, I can say x[0] is 1 into z^0 plus x[1-2] into z^{-1} plus x[2] is 3, z^{-2} plus x of x[3] is nothing but a 0, 0 into z^{-3} plus x[4] is nothing but a minus 1 into z^{-4}. So, I can say X(z) is equal to 1 minus 2 into z^{-1} plus 3 into z^{-2}, this is 0 and this is minus z^{-4}. So, that is the Z-transform.

Let us say another example. Here, I have two sides: this side and this side both; this side n is positive, and this side n is negative. So, when I say the summation varies from X(z), it is equal to. So, how many samples are there? So, this number 1 sample is 1. Let us say the number zeroth sample is 3. So, the zeroth sample is 3.

So, this side is minus 1, and this is minus 2. I can say n varies from minus 2 to 0, 1 and 2, x[n] z^{-n}. Now, write it down. So, for n equal to minus 2, x[-2] into z^{-2}. So, z^2 plus x[-1] z^{-1}, 1 plus x of minus my x[0] z^0 plus x of I can say 1 z^{-1} plus x[2] z^{-2}.

So,

$$X(z) = x[-2] * z^{2} + x[-1] * z^{1} + x[0] * z^{0} + x[1] * z^{-1} + x[2] * z^{-2}$$

x[-2] is 2, z^2 plus x[-1] is 1, z^1 plus x[0] is 3, z^0 plus x[1] is 1, z^{-1} plus x[2] is minus 1 minus 1, z^{-2}. So, finally, it is 2 z^2 plus z^1 plus 3 plus z^{-1} minus z^{-2}.

So, that is Z-transform. What is the region of convergence here and here, ROC? So, for which X(z) is finite? Now, if you see that z is equal to 0, X(z) is infinite. If z equals infinite, then X(z) is equal to infinite. So, in this case, I can say the region of convergence is the entire z-plane. So, the entire z-plane except z is equal to 0, and z is equal to infinite.

Now, in this case, if you see if z is equal to infinite, it is 0, but if z is equal to 0, then this is infinite. So, I can say, in this case, the region of convergence is the entire z-plane except z equal to 0. So, I have to exclude those z values for which X(z) is infinite, except all the z values in the entire z-plane X(z) exist.

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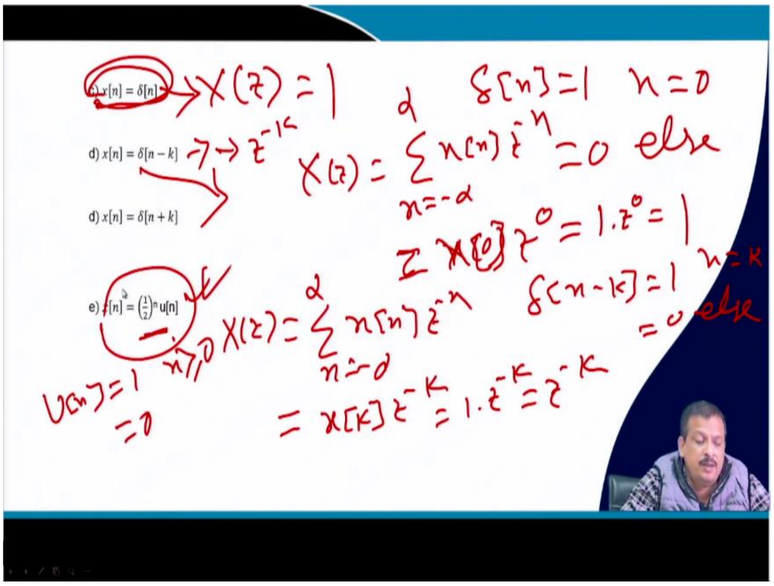

Similarly, if I go for x[n] equal to δ[n], what is X(z)? Is equal to 1; δ[n] only n equals because why is it 1? Because you know δ[n] is equal to 1 if n is equal to 0; elsewhere, it is 0. So, I can say when I say X(z) is nothing but a n equal to minus infinity to infinity x[n] z^{-n}.

So, when n is equal to 0, only this exists. So, this is nothing but a x[0] into z^0. So, x[0] is 1, z^0. So, it is nothing but a 1. So, that is why x[n] is equal to δ[n]; then, I can say X(z) is equal to 1. Similarly, if I say x[n] equals δ[n-k].

So, the same thing I will say X(z) is equal to n equal to infinite minus infinity to plus infinity x[n] z^{-n}. Now, what is this δ[n-k]? That means δ[n-k] is equal to 1 when n is equal to k, else it is 0. So, I can say the signal exists when n is equal to k. So, I can say x[k] into z^{-k}. So, it is a 1 into z^{-k}; it is z^{-k}. So, X(z) here is z^{-k}.

Similarly, you can do this also. So, what is U[n]? U[n] is equal to 1 when n is positive; n is equal to 0, 0 to a positive value, n is greater than equal to 0, elsewhere it is 0. Now, it becomes a series. You can solve the series and get n. I did a Z-transform, z domain representation of this signal. So, if I give you any signal or any system, you can represent it in the z domain; impulse response is given. So, instead of a signal, if I give you a system, the impulse response of a system.

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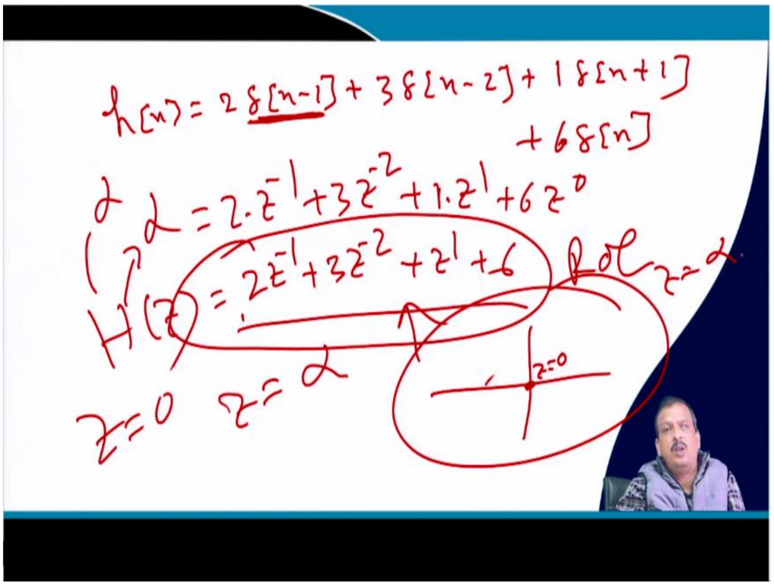

Let us say I said H[z], H[n], this is the system impulse response of a system is 2 into δ[n-1] plus 3 into δ[n-2] plus 1 into δ[n+1] plus 6 δ[n]. This is the impulse response of h n of the system. Then, what is the Z-transform? You know the delta Z-transform of δ[n-1] is nothing but a 2 into z^{-1}; you know that δ[n-k] z^{-k} plus 3 into z^{-2} plus 1 into z^{1} plus 6 into z^{0}.

So, I can say the system Z-transform is 2 into z^{-1} plus 3 into z^{-2} plus z^{1} plus 6. So, this is the z-plane representation. Now, if I told you what the region of convergence, ROC, is, if z is equal to 0, then also it is infinite. So, this is h z. If z is equal to 0, then also h z is infinite; if z is equal to infinite, then also h z is infinite. So, I can say ROC is the entire z-plane except z is equal to 0 and z is equal to infinite ok. So this is called Z-transform.

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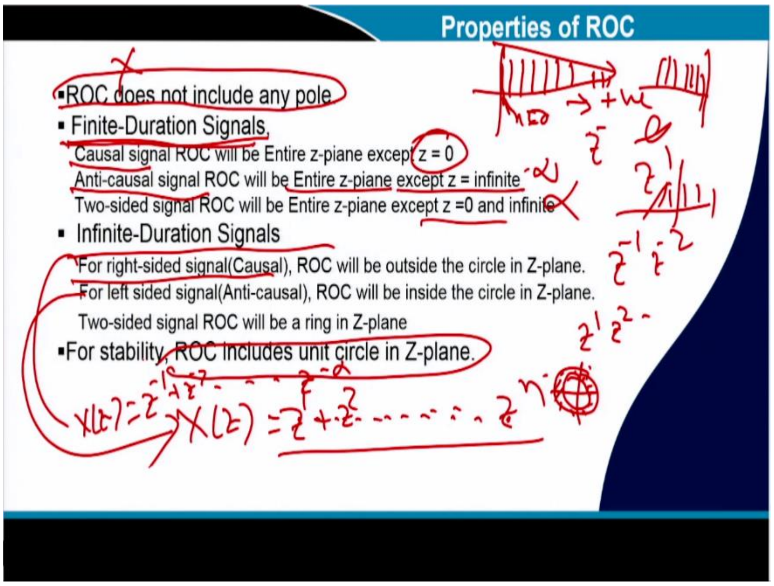

So, there is a property, some properties of the region of convergence, ROC. ROC does not include any poles. I will explain this concept at the end when I talk about applying Z-transform. Now, see that for a finite duration of the signal, if the signal is causal, I have already said what the causal signal is; what is the non-causal signal; that means, if the signal exists only this side, n equal to here n equal to 0; this side is n equal to 0, this side is positive.

So, here, n is positive, then I can say Z-transform is always z to the power minus something. So, that means, except z equal to 0, the entire z-plane will be the ROC. Similarly, an anti-causal signal that only exists on this side, then I said z to the power only plus will appear. So, I can say that ROC will be the entire z-plane except z is infinite; z is equal to infinity.

Now, if it is a two-sided signal, as you observe here, here, two-sided signal, the ROC is both ROC is entire z-plane except z equal to 0 and z equal to infinite for a finite duration of the signal. Now, if the signal is infinite duration, the signal is infinite duration, so X(z) is a series of infinite sum.

So, if it is a causal signal, if it is z^{-1} z^{-2} like that and in the case of an anti-causal signal system, it will be z^1, z^2 like that. So, z^1 plus z^2 plus dot dot z^n for this purpose.

When will it be finite? Think about it. Now, for this purpose X(z) is equal to z^{-1} plus z^{-2} dot dot dot z to the power minus infinity. When will this be finite? Understand. So, I can say that I can find out the region of ah. But if it is a two-sided signal, it will be on the ring instead of the entire Z-plane. We will give one example of that. It will be a ring. For stability, ROC includes a unit circle in the Z-plane.

So, in the next class, I will talk about the properties of Z-transform.

Thank you.