Communication Networks
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Module - 05
Queuing Theory (Contd.)
Lecture - 25
CTMC

So, in the last class, we have already seen how from the DTMC description we can go to continuous time description ok. So, we have derived the Chapman Kolmogorov forward equation we have also seen a very important matrix that characterizes the continuous-time Markov chain which is called the rate matrix or rate transition matrix.

So, that something also what we have characterized what are each of the terms of the rate transition matrix what is the row sum of that matrix? So, these are the things we have already characterized. Now, with that, we are well equipped to go towards actually deriving the theory of CTMC.

So, today what we will try to see is get the basic governing equation of CTMC for of course, again stationary case and homogeneous case of course. So, that will do, and will try to see the application of that.

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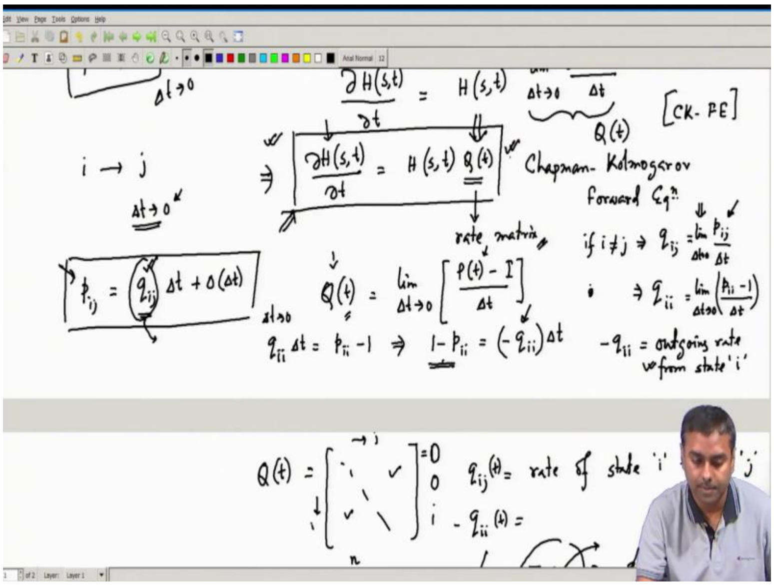

So, in the previous class if you see that was our characterization of rate and as well as this is the Chapman Kolmogorov forward equation ok. So, this is how HST has been characterized. Now, let us try to see how we move from this transition to state ok. So, it is a Markovian process.

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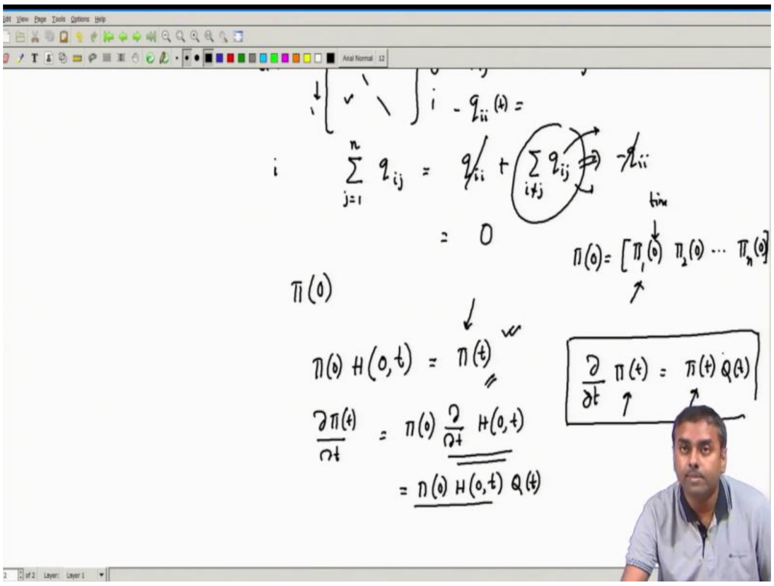

So, suppose I am at some 0 I have or let us say not let us not talk about 0 let us say even 0 is ok. So, at zeroth time I am at pi 0; that means, this is a state probability. So, of course, pi 0 is a vector which is having pi 1 0 pi 2 0. So, this is the time index at time 0 and this is the state index like this as many states are there. If there are n number of states this will be the case.

So, this is the state matrix we have seen a simple relationship between the state and the transition if we multiply the state with the transition matrix state vector with the transition matrix we go to the next one which is pretty straightforward. So, if I know pi 0 and I know the corresponding transition H 0 t s I am putting 0.

So, that is why I told you it can be even generalized to means s also. So, this might take me to pi t ok. So, this is well understood this will be happening always. So, basically, currently, I am at this state this is the transition probability. So, if I multiply I see where I will be going at time t.

So, that will be our probability state probability at time t this is a well-understood fact. Now, if I differentiate this with respect to t then what will happen, pi 0 does not depend on t. So, therefore, pi 0 comes out ok.

So, this will become d dt of H 0 t. Now, I will put this from the Chapman Kolmogorov forward equation by putting s equal to 0. So, what do we get over here? Del del t of H s t is nothing but H s t Q t. So, I can write that this is something I can write. Now, what is this? Again from this definition that is pi t? So, therefore, I can write del t of pi t is nothing but pi t into Q t, which is the fundamental state equation or differential equation that describes the state, ok.

So, what it says is the time differentiation or rate of change of the associated probability is nothing but the current probability into the Q matrix the rate transition matrix ok, this is what is fundamental to CTMC. Now, what we can do is something we have done earlier also if we take a homogeneous assumption, immediately Q t becomes free of t it becomes Q. Because that does not vary over time.

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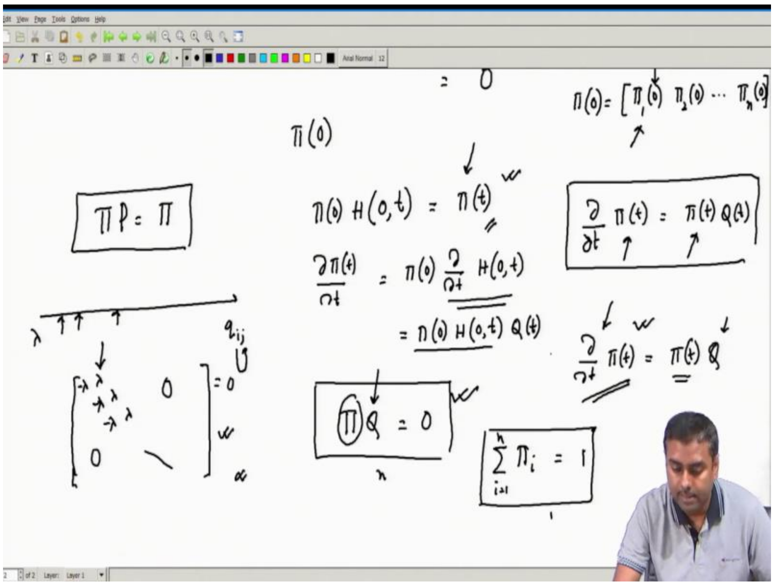

So, I can write this del t of pi t is nothing but pi t into Q. Now, if we take another assumption of stationarity then immediately the state also will not be changing over time. So, that will become some pi and this rate of change of those associated probabilities also will not be changing so, that becomes 0.

So, I can get a very simplified equation by Q equals 0 almost like DTMC in DTMC we had pi P equals pi. So, the similar equation we are getting over here, which is pi Q equals 0 and of course, the associated state probabilities have a normalization. So, pi i is equal to 1 i equals 1 to n if there are n states.

So, again set of linear equations so, n number because this is an n-dimensional matrix. So, n number of linear equations over here and 1 equation over here you can solve that linear equation and you can get the associated probabilities state probabilities. So, now, we can derive our continuous-time Markov chain associated equation also and this will be our governing equation. Remember this equation is only true if it is in stationarity or it has been stabilized ok.

If it is not stationary then it will be governed by this differential equation you can also do that in a transient Markovian Markov chain you will be always able to get the corresponding differential equation and then there will be a set of differential equations. So, you will have n number of differential equations that all have to be solved the way we have done for Poisson basically Poisson was similar to this one.

So, you can even solve the Poisson process by this analytical term also we have done it from the first principle, but thing same equation you can derive from this set of equations, and then recursively you can solve your Poisson process just the Q matrix has to be properly populated ok. Can we just do a little exercise over here can we just try to see what should be the Q matrix for a Poisson process?

What is the Poisson process? It's a pure birth process; that means, every time somebody is just arriving it is just an arrival process there is no departure. So, there is no death as such ok? So, people are just arriving and it's the count of that. So, the Poisson process is all about that. So, if it is a pure birth process with arrival rate lambda can I now talk about my Q matrix with the non-simultaneity principle and everything, ok?

So, now q ij I know pure birth means every time I due to the non-simultaneity principle only one arrival can happen only I can go to i plus 1 nothing else. So, whatever state it is there it can only go to the next state ok. So, if there is one customer the next time it can only go to which it can count can go only to 2, if there are the count is 2 it can go to 3. So, this is the only allowed transition or rate associated rate, and what is that rate, that is lambda.

So, therefore, if you see this is the diagonal one only this off-diagonal will be populated with the value lambda. So, only that off-diagonal will have to have value lambda and all other things other than the diagonal. Diagonal I will come to do the latter; the other diagonal will be all 0 ok. If that is the case I know this sum of rho what that should be? That should definitely that should be 0.

We have already characterized it. So, if that is 0. So, therefore, this diagonal term will be all minus lambda which is all, and what will be the size of that matrix it will go to infinity because the count goes up to infinity; Poisson when we started deriving it is an infinite matrix. So, put this q matrix.

So, now, you can see how to derive a q matrix for Poisson we have now derived the q matrix put this q matrix over here formulated that differential equation you will see exactly what we have derived for Poisson whatever differential equation we have formulated those will be derived over here ok, that is the strength of this theory.

So, we have done this theory in a different manner whereas, well while deriving Poisson that was directly done from intuition, of course, the same principle was taken into account. It was also taken into account over their stationarity was taken into account over here also stationarity we are kind of taking into account ok.

So, that Markovian was taken into account because Chapman Kolmogorov comes directly from the Markovian property that was taken in and non-simultaneous is taken into account because for defining the q matrix we are doing this non-simultaneity principle. So, all those things are actually taken into account and with that, we are trying to derive this ok. So, you can take this as a home exercise.

And try to see whether this is exactly matching once we get this q matrix whether it's exactly matching to our earlier derived Poisson this thing ok, Poisson equations, but having said that. Now, we have the governing equation this was just a check of that governing equation. So, basically, as you can now see all our targets will be get the correct system description and get a correct Q definition the associated rate definition this is what we will have to do.

So, our next onwards whatever will be deriving our time will be occupied with this thing only how do I get the Q what is from the system description how do I populate the q matrix terms or elements once we put those terms then it is just solving the linear equation nothing else ok. With this background let us try to solve some realistic things in queuing.

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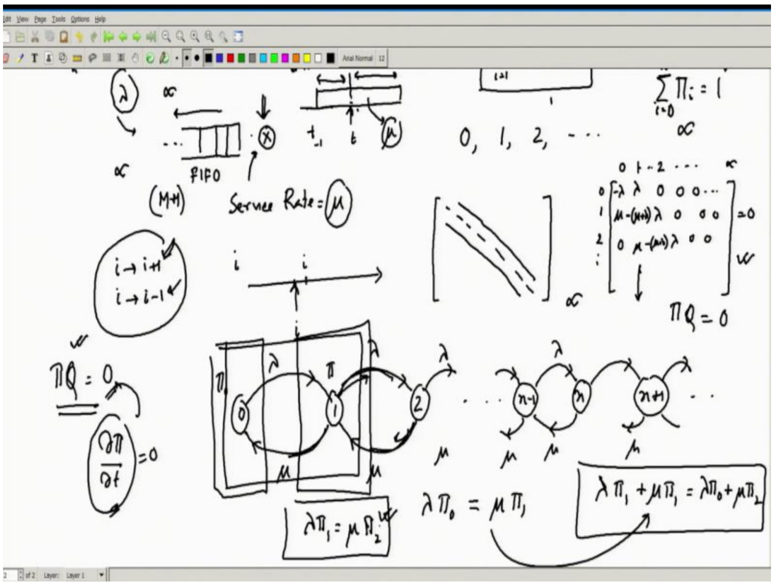

So, we have so far talked about queuing. So, we still have not solved any queuing let us try to solve a queuing system with a single server. So, only one server is there, which has some description what are descriptions that the arrival process follows a Markovian process ok?

So, it's a Poisson arrival with rate lambda and the service process that is also all and they are independent arrival to service they are independent and each service is also independent they are identically and independently distributed and they follow again an exponential which is a memoryless process exponential distribution, ok.

So, this also follows an exponential distribution with service rate that exponential parameter is mu ok. So, the arrival rate is lambda service rate is mu. So, these are the two things we have and we also know that there is only a single server q is FIFO first in the first side first out and the q length of how many customers it can hold is infinite.

So, these are the descriptions that have been given. Now, this q will try to analyze what we have done earlier all those mean timekeeping of the events related events that will be occurring. So, those things we have done we have also understood that if all these services and arrivals are Markovian then I do not have to keep track of those residual times.

So, I can actually describe this system with the state description which is the number of customers in the system. So, whenever you do this analysis queuing analysis the first thing you have to come up with the proper state description.

The state description should be in such a manner that summarizes the whole thing it is not leaving anything if it is leaving anything then the next state I want to describe will not be just dependent on that something else it is dependent on that you have not given in your state description.

So, I have to be careful while describing the state or defining the state. So, over here because of all these nice Markovian properties we have seen that it's memoryless, and the residual time is also similarly exponentially distributed.

So, basically, any time I try to observe I can count all of them as a number of customers I do not have to see how much residual time he has been served. So, that memory I do not have to keep I can just take them as a number of customers and a number of customers only described by the whole state.

So, therefore, my state description will be only the number of customers in the system or the number of packets in the system. So, this can start from 0 it can go to 1, 2, dot, dot, dot, dot up to infinity because now again another description comes into the picture because this can hold up to an infinite number of customers. So, therefore, the state can eventually go up to infinity I would have said that the buffer has a limits limitation in that it can only hold this much data.

So, let us say I have some M number of data it can hold then the state will definitely go up to M because beyond M it will be all dropped. So, it cannot be held in the system. So, the system can only hold if the buffer is M of course, one can be in the server; so, M plus 1 up to M plus 1 only can go. Only that many customers or that many packets can be there in the system. So, again the description of the system comes into my state description as you can see.

So, this is the whole description of the state now I have defined the state can I now define the q matrix because it is a continuous time Markov chain any time it arrives any amount of service can be required so, any time it can do departure. So, all those things can happen. So, it is a continuous time Markov chain or Markov process.

Now, let us try to see if can I define my q matrix ok. So, what is the q matrix? Let us try to understand this now again. Remember that the nonsimultaneous principle will be always very handy. In the queue can you see at any time if it is in state i what can happen? So, suppose I observe the system at time i.

So, at this instant in the next future, what can happen can stay in that nothing happens nobody arrives nobody departs. So, it stays in i or it can happen that immediately next time one arrival happens then I will go towards i plus 1 that is valid.

And if there might be also a possibility that a particular person who is in service his service gets over then I will go to i minus 1 nothing other than this can happen, only these are the possibility; either at a particular time instance, it can increase the state by 1 only because of non-simultaneity principle. Simultaneously two arrivals cannot happen simultaneously arrival and departure cannot happen.

So, it can only mean what it can do is it can only either increase the state by 1 which is the possibility or it can decrease the state by 1 these are the only two possibilities nothing else. If that is the case if I see this q matrix again I can infer that you forget about that diagonal we mean we populate it separately because we know the row sum is 0. So, therefore, off diagonal if you can populate immediately diagonal will come.

Because it complements that over here what will be mostly populated I know only these two transitions. So, off-diagonal this one, one off-diagonal and the lower off-diagonal that is the only thing which would be populated and what is the state dimension that will go up to infinity. So, it's an infinite dimension matrix. So, only the off-diagonal one will be populated. So, if I just start putting what will have value.

So, this is let us say it starts from 0 1 2 0 1 2 and goes up to infinity ok. So, 0 0 is the diagonal term I will not be concerned about that, but 0 1 0 to 1 means what? It is in 0 state one arrival happens. So, that must be populated so that is possible for one arrival can happen. What is the associated rate of arrival? That is lambda and all others will be 0 because I know due to the non-simultaneity principle it cannot be populated.

Then what will be 0 0? This sum should be 0. So, therefore, this must be minus lambda immediately I can do that. Again this diagonal term I will keep it what will be this 1 to 2? 1 to 2; that means, one arrival again with rate lambda 1 to 0 that might happen due to departure. Now, let us try to see if there is a single server one server is there that server.

What is the associated departure rate let us say this particular rate is the rate that some customers will be departing from there that will be populated over here. 1 0 because it is in state 1 and it is going to 0 what is the associated rate that is what we want to try to characterize. Let us see in the server what is happening.

So, whatever the server is, the server suppose I am at a time t the server might have taken the nth customer or let us say a kth customer at some earlier time let us call that t minus 1 ok? Some earlier time it has taken that customer and it is servicing it. At time t I see that it has serviced some amount of time what is the residual time that is still because of exponential distribution this is still exponentially distributed with parameter mu.

So, therefore, what is the associated rate that it will be completing the service associated rate is mu because with mu rate it is service the server services the things with rate mu. So, therefore, the associated rate is mu.

So, with my rate because of this exponential distribution that nice these things I could get that even whenever he arrives it does not matter, wherever he intercepts the packet the residual time rest of the time he can just forget about the past he can again infer that the residual time will be still having a service rate with mu.

So, therefore, the rate at which the customer will be departing is mu only. So, this will be mu then what will be all other things will be 0 of course, they cannot. So, what will be the diagonal term? Because this should be 0 summation should be 0.

So, that should be minus mu plus lambda and you keep doing accordingly. So, next this term will be lambda this will be mu and this middle term will be minus mu plus lambda and all others will be 0, and so on. It will keep on getting populated. So, that is my q matrix.

So, I have characterized my q matrix now I have to do this if I have to solve it this should be pi Q equals 0, and the summation pi i is equal to 1 I equals 1 to infinity sorry now 0 to infinity ok. So, these two there will be infinite linear equations if I just recursively solve them I will get my things done.

So, this is easy now we know how to solve it, but before doing this let us try to also make life a little bit easier, and that you will also from there you will be able to appreciate why this is actually called a Markov chain, ok.

We have talked about the Markov process, but it is also called what we have been saying that it is a Markov chain. So, let us try to appreciate what chain we have. So, basically, it must have a graphical representation and it will have a chain-like graph that is what it will be let us try to appreciate that part. What I will do? Now, this state transition can be graphically represented. So, each node will be the state. So, 0, 1, 2 up to infinity n, n plus 1 n minus 1 something like that.

So, this is the graph of states where nodes are represented by the states now, the transition which is the effective rate that will be represented by the arc or the link between them. Let us try to see what the transition is valid from here. So, there is a transition from 0 to 1 it is a directed graph of course, because the transition has a direction it goes from one state to another state. So, therefore, it is a directed graph. So, 0 to 1 is possible; 0 to 2 is that possible? No.

Because means this is not allowed due to the non-simultaneity principle of two arrivals simultaneously 0 to 1 means it was in 0 state one arrival has occurred and it has gone to one state. From 1 it can go to 2 from 2 it can go to 3 and so on n minus 2 it can go to n minus 1 n minus 1 to n, n to n plus 1 n plus 1 to n plus 2 like this and also there might be a backward transition.

So, from one with the departure, it can come over here to 0 from 2 it can come to 1 and that is the only transition that is allowed. So, as you can see it actually constructs a chain-like graphical representation. So, that is why it's called the Markov chain. So, this chain is represented as a Markov chain, and in the arms in the graph, we actually in the link give some cost here we can give the rate.

So, this has a rate lambda this has the rate mu, this has a rate lambda mu this summarizes the Markov chain and all these equations. So, this pi Q equals 0 what does this tell actually? It actually tells that this 0 means this was actually del pi del t right rate of transition rate of change of as if the probability. So, these nodes are as if having this probability value. So, let us call that pi 0 pi 1.

And all those things and then what we are seeing is that with this rate it is either going out those probabilities as if probability some entity with some rate is going out to other states and its getting populated also from other states. This rate of change of this probability value as a probability is some entity that is getting out from one container this is that node 0.

So, pi 0 has some probability and it's going out and from container one probably some probability is also coming in and accordingly over time this probability is getting populated. It is like some container and some liquid is going out with some rate some liquid is coming in with some rate and I am getting the overall amount of liquid, ok.

So, you can think of it in this analogy as if some probability is going out with some rate and some probability is coming in with some rate and that net rate from any state is actually 0 because this is all 0. So, this part is all 0 due to the stationarity. So, therefore, if I construct a case of this state 0. So, the total outgoing and incoming rates must be balanced which is the stationary condition that is the governing of these equations.

So, what I can say over here is lambda into pi 0 is the overall rate of outgoing probability must be equal to that must be balanced by the incoming probability, which is actually mu into pi 1 that is the first equation we have got. If you see this pi Q equals 0 if you put the first equation you will see that the same equation will be formed over here ok? So, you will be able to see that ok.

So, if you put pi q equal to 0 you will see this equation will form the first equation for the first row; for the second row, you will see the second equation so on you will be able to derive this equation. So, basically everywhere for node 1 if you construct this one for node 1 all the outgoing rates must be equal to the incoming rate. So, all the outgoing rates for this one are outgoing.

So, which is actually lambda into pi 1 this is the outgoing one which is actually mu into pi 1 that must be equal to all the incoming rates which is lambda into pi 0 wherever it is coming from with that rate you multiply lambda into pi 0 plus from here something is coming. So, which is actually mu into pi 2 that is the second equation you will again see that the second column will be actually forming this equation and so on we can do that.

So, all these balance equations can be very nicely constructed from the chain itself we do not have to really go into that. So, if I can construct the corresponding chain I do not need to completely construct the whole q matrix and then construct each of these equations, not only that there is also a simple simplification over here. I can because it rate balanced of those probabilities. So, I can construct any combination of states as a combined state.

So, suppose let us say 0 and 1 I put as a combined state then from this combined state all the outgoing rates must be balanced again with the incoming rate. So, therefore, just this and this must be balanced. So, immediately I get an equation which is lambda into pi 1 must be equal to mu into pi 2 if you see I put this over here I get this equation back. So, I get this simplified equation if I can study the Markov chain properly.

So, next time onwards, we will be putting this system to the Markov chain description and then do the analysis steady state analysis. So, that will be our next task we will in the next class we will try to construct that Markov chain and very quickly will be able to construct the Markov chains which are linked to our trans switching. So, will do our trans switching analysis. Now, we are in a place to do the trans-switching analysis.

Thank you.