Communication Networks
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Module - 05
Queuing Theory
Lecture - 23
DTMC cont'd

In the last class, what we have discussed is, we have discussed a little bit about this DTMC and the associated property. What we skipped over there was the inherent Markovian property that we had actually assumed. We have just very briefly given a discussion about that, ok, there was a Markovian property which was already assumed inside.

Let us try to explore that Markovian property a little bit more generally, ok, because that will be very helpful towards our deriving the CTMC which is our target, ok? So, we will try to do that now. Let us try to see this.

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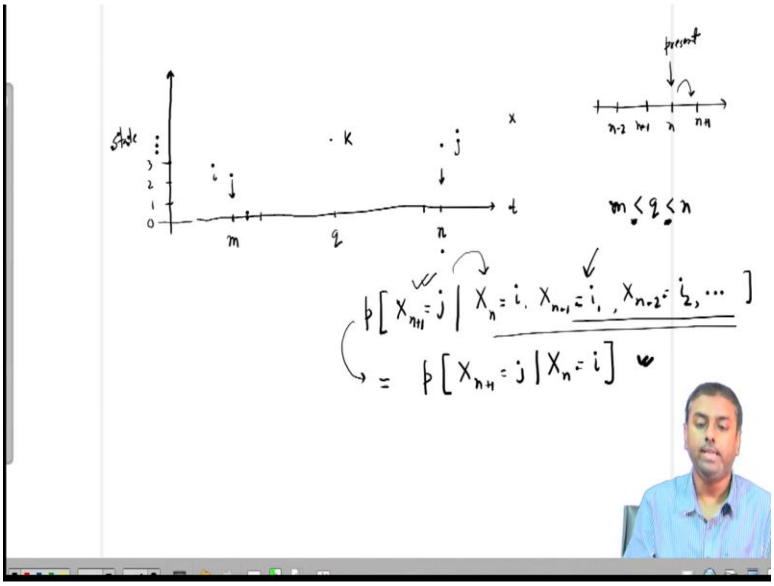

Let us again draw the time frame. This is the time instant; it is a discrete one. So, of course, it goes one by one. So, m then intermediate it is in some q and then it goes to n, ok. So, now, we are trying to take the time description. So, according to these things, the index m is less than equal to q, less than equal to n, or our description we are taking strictly less than, ok.

So, basically, time is progressing, n is a later time. So, this you can talk as future probably, this might be some intermediate step and this is the present state probably, ok. So, that is what we are trying to see. What will happen? Now, this is the time, of course, on the other axis you have the state, ok. So, the state let us say we have some countable states because we have already talked about discrete states, right?

So, its state will be like, it might start with 0, 1, 2, 3, and so on, ok. So, like the hippie case it was only having 1, 2, 3; 3 cities, ok. So let us say, I observe in present him to be in the ith state, ok. And then intermediate, he is every time he is changing state, with some transition probability, ok. So, he will be in the next one he will be some i 1 state, next one he will be in some i 2 state. So, let us say in the qth this one, I observe him to be in the kth state.

And then in the finishing, I see him in the jth state, but just giving arbitrary state, that can take any value, I can take any value from the state space, k can take any value from the state space, j can also take any value from the state space, ok. So, that is not our whole argument, our argument will be for any possible case how do I characterize this whole thing, ok? So, now, what we are trying to see is that the first thing that will try to see is something called the Markovian property.

So, what is Markovian property? We have kept on talking about this Markovian property or memorylessness property. So, this says that it does not depend on the past; it only depends on the present, the future events. So, basically, it says, that probability let us say I am trying to calculate this event, that suppose X n equals to j. So, this is the future event n. Given the past event, ok.

So, I can list all these past n minus 1, n minus 2, q up to m, and all those events, ok? So, I can keep on listing them, ok. So, let us say X n equals j. So, I just mean, I will forget about this picture right now, for the time being, I am just giving some examples that, maybe I will try to describe the event in n plus oneth time. So, I draw another timeline.

So, I have this nth timeline, n plus 1 is the means this nth instant this is the present. So, this is my present, where I am sitting, n plus 1 is my future and all other n minus 1, and n minus 2 has already occurred in the system. So, these are the past. So, I am trying to predict this event at n plus 1, that it will take state j at n plus 1. So, this is something the associated probability I am trying to predict, given all the things, that we have observed.

So, let us say X n I am in i and then X n minus 1 i was in, i minus i i 1, then X n minus 2 i was in i 2, and so on. So, I know this entire history and the present. So, for a Markov property, what happens actually, this probability n plus 1 j, this condition only depends on where I am in the present and it does not matter where it was earlier. So, this earlier value whatever it is, this probability is independent or irrespective of that, ok.

So, in Markov chain, you can always talk about these things. That any conditional probability, where all the past has been taken into account, this condition probability will be always equivalent to the conditional probability that it is just means condition on the immediate present, nothing else, all other things are not important. Because whether they are there or they are not there, the associated probability will be the same associated behavior will be the same.

So, my prediction probability will remain the same, even though I have some other observations in the past. So, those observations even if the changes also, are not going to influence. So, these actually these things actually come from the memorylessness process which we have discussed also. So, this is the actual mathematical statement of the memorylessness process or this is the actual mathematical definition of this kind of system, ok, where we can always say that history I can delete.

I do not have to really care about history. It's good enough that, I summarize the present, I see the state and it only future what will be happening it only depends on the present where it is and the associated transition probability, nothing else, ok? That is exactly what we have also taken while deriving that pi p equals pi, ok. So, same thing we have assumed.

So, these are called this kind of things are called the Markov process. It's a special kind of random process, you might be asking what is this process, is it different from whatever random process we have, so far understood like let us say Gaussian noise or AWGN process you have talked about? How it is different? So, there is a fundamental difference between these two processes.

Actual random process the way I have defined in some of the earlier classes, that it should have all the time instances and it should have that joint distribution. So, the entire joint distribution should be described, ok. And if there are correlations, it will have that. So, therefore, the entire joint distribution probably cannot be separated. It will be a proper joint distribution and there will be a correlation, so, they might not be separated out.

Whereas, the Gaussian process that we know, is called the independent process; that means, it can be separated, with respect to every time instance. So, therefore, the overall joint distribution will be a multiplication of the associated distribution at every time instance, ok. So, that is the Gaussian process or independent process we talked about. Markov is just one step forward to that. Markov still depends on at least the previous state.

Whenever we are talking about the future, it still depends on the present state. So, that means, it has one dependency, just the previous state it depends on. So, the Markov state or Markov process goes one step forward towards generalization, the most generalized process. The most generalized process is very difficult to understand. So, that is why for the noise we have seen, it's a completely independent process.

And it was actual noise that was completely described by a completely independent process, if it is actual noise, ok. So, where your autocorrelation function will be impulse; that means, it's not even correlated to the next one, ok. Whatever close sample you take, it is not correlated to that, ok. So, that kind of independent process is described by AWGN.

So, that is why probably your spectrum becomes flat and all those things, because its autocorrelation is impulse and all those things. Whereas Markov has some correlation, it at least depends on the previous state. So, this is the Markov process, but still a long way towards simplification. Because it still summarizes everything in the previous state only. So, from the previous state, I can easily predict the next state, I do not need to go to any other states previous to that.

So, the Markov process is those process, which is characterized by these particular conditions. Wherever this condition is true, then only I will be able to do whatever analysis, so, far we are deriving. If these conditions fail, that all these theories will fail, we cannot write those theories for or we cannot apply these theories to whatever we are developing for any process which is not Markovian.

So, you have to be very careful, that which process I am analyzing; is that process Markovian? So, that is the first thing you will have to ask, is it a Markovian process? That is the first thing you will have to ask. The next thing that you will have to ask is, is my transition homogeneous? So, these are the conditions which will be required. Is my transition probability changing over time or it is remaining constant? Is my system stationary? Is it an ergodic system?

So, these are the four questions that must have the affirmative answer, then only that restrictive system you can do an analysis. So, our trunk switch will have all these conditions true and that is why probably will be able to do an analysis. Otherwise, there will not be any condition that gets violated, you will not be able to do this analysis.

So, suppose the arrival process, if that changes with time, you will see that the Markov chain does not remain homogeneous and then immediately the analysis will whatever analysis will be following that is not the simplified analysis that will be deriving next on, ok.

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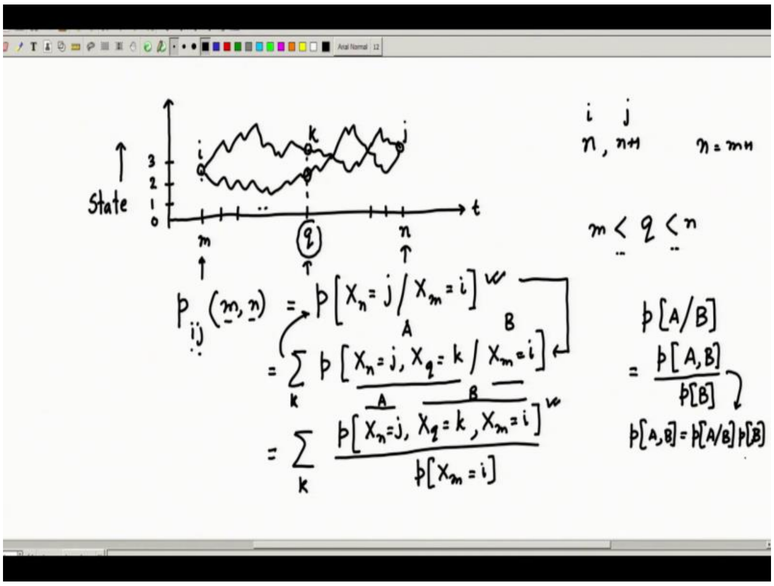

So, let us understand what is the Markovian property in mathematical terms. Let us try to understand what we were actually discussing, that is this particular figure where we have time in 1 dimension state in 1 dimension and what we are interested in now is a multi-stage or multi-state, sorry, multi-stage transition probability. So, earlier whatever we have discussed, that was one state.

So, transition probability. So, if it is given in n, what will be happening in n plus 1? So, if this was in I, what is the probability that this will go to J? So, from n to n. So, time if times are stamped, with integer values or countable values like 1, 2, 3, 4 like that in nth time you are in some this one, well what is the probability that you will be going to some other state in n plus oneth 1, ok.

So, that transition probability matrix we have already understood. Now, what we are trying to do is, at a particular instance m you are somewhere, let us say that state is in some i, ok. So, I can take any value. And then, at a later time instance, not necessarily the next one, m plus 1, it might be any value. So, at a later time instance, what is the probability that I will be in j?

So that means, it's actually accounting for any arbitrary stage transition. So, I can go to any time and then can I predict, ok. So, that given over here it is there, what is the probability, that it will be in some other state, ok? So, this can be characterized by these probability transition values. So, what is this? This is a scalar parameter. It says that at m it was in state i and what is the probability that at n, it will go to j?

So that means, it says given that, it is in state I, what is the probability that in nth instance it will be going in j. So, I can write also as, the probability that X n will be j. So, nth instant the state corresponding state will be j, given that currently, I am in this state. So, that is exactly what it is saying. So, this is the probability that we are trying to characterize. And of course, for all i j values all possible values of my states, ok.

So, I will be actually putting a state transition matrix, ok? But this is a multi-stage transition matrix remember. So, therefore, any m and n, I will be able to characterize it. So, now, my transition will be generally, means more generally characterized. This is what we want to do, ok. So, now let us try to see, if we can make some sense or put the Markovian property into it, and let us try to see what happens to this particular matrix.

So, that should be our next target. So, for that, what we could do is we can define between m and n some arbitrary stage or some arbitrary time instance q, ok. So, the condition is, that this q must be less than n and must be less than m. Of course, there can be equality signs, but we are right now we are not taking that. Of course, an equality sign will come if m and n are not more than one separated, that one of them must be equality.

Because in between I cannot get. So, if n is equal to m plus 1, then I cannot have a distinct q which is not either equal to m or n. So, that possibility is there an equality sign will come. But otherwise, if m and n are more than one separated or one time instant separated, then always q will be the intermediate stage, which means strictly intermediate; that means, it is strictly greater than m and strictly less than n.

So, that kind of stage I define. And let us say, I now try to see where this can be. So, let us define that state to be k. So, basically, it is going through some intermediate stage q, where it is observed in state k. This is our observation, ok. Now, can I again characterize this particular transition or this particular probability including the definition of k, ok?

So, what might happen? So, of course, it will start from i, i is known, from there a target state j it will go. But in between it can go through. So, it can do different kinds of transition it can take some k value and it can do again different transition and go to j. But there are multiple possibilities such possibility. So, it can also do other transition in between and then go to some other k and then go to j, these possibilities are there.

So, there are all possible intermediate k values, if I now sum over all these k that should be the overall probability, ok. So, I can write, if I now include, suppose if I now characterize these things that, X n equal to j and this X q equals to k; given that X m equals to i. So, if I say this thing, what does that mean? That means, now I am as if this is my future prediction at n, ok, where it will go, ok, along with that, I also know these, ok.

So, it is intermediate in the state k given that it was in mth stage it was in i. Now, if I sum this whole thing over all possible k, that should be equal to this probability. Because then all possible k I have accounted for. So, therefore, this must be almost like, from a joint distribution, you are taking the marginal distribution. So, this is the joint distribution, I have also taken the joint event that X q was k.

Then, I summed over all possible values of k, then I got the corresponding marginal distribution. So, this equation is a probability term it is true. So, I have no issues with that. So, therefore, I can write this equation or I can actually re-represent this equation in terms of the following equation. Up to this, there is no problem. I have not put any Markovian property or anything else.

I am just introducing the intermediate steps and I am trying to see what happens, ok? So, this is true. Now, this is a conditional probability. So, the probability that this is happening given this is happening. So, I can call this event to be A and this event to be B. So, basically, it's a probability that A event given B, ok. So, this can be written as probability joint probability of A and B both happening divided by probability B, which is coming directly from Bayes theorem, ok.

So, I can apply this Bayes theorem, inside the summation no problem with that. So, I can rewrite this, A intersection B or both joint events A and B. So, that should be all these things happening, the probability that X n equals j X q equals k, and X m equals i divided by the probability of just this, ok. So, up to this it is alright, what I will do? Now I will put the trick. I will again this top one, that is the joint distribution, ok.

So, now, I will define, this to be my A and this to be my B, I can always do that, any event I can declare accordingly. So, if now this is my A and B accordingly, this p A, B from the same Bayes theorem, I can write as p A given B, into p B. Just algebraic manipulation of this same equation, that is exactly what I will be doing now with the top one, ok. So, let us see what happens if I do that.

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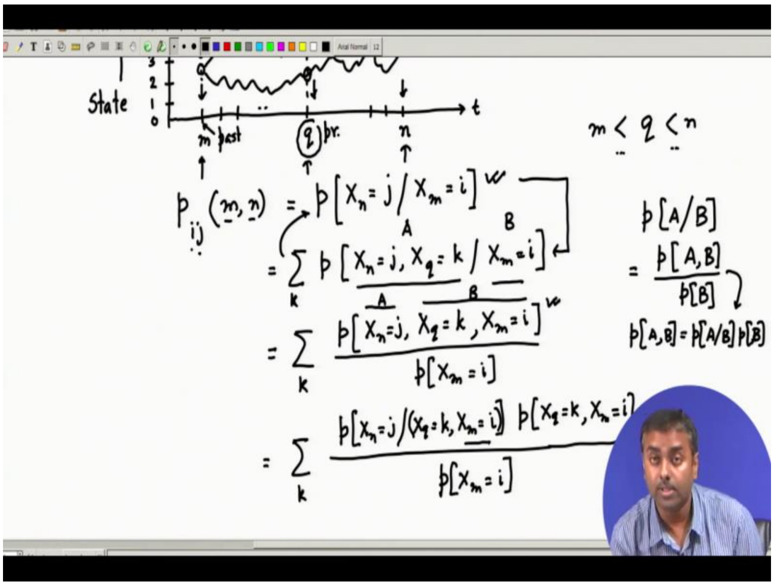

So, A given B means, probability that X n equals to j given both the events X q equals to k and X m equals to i, into probability this lower one. X q equals k and X m equals i divided by whatever is there. You know what I have done, by doing this trick, I have introduced myself, to launching the Markovian property. Can you now see, that I have created a conditional event, where I am trying to predict a future event?

So, if you just along with that you try to see the diagram. So, this is at n, given two events which is one is near past and this is far past, ok. So, I can take this anything past I can take that as present. So, near past I can take it as if this is my present and this is my past, ok? So, this is my past. If I take that one, then what will happen? Markovian property says, it just depends on the present ok or the immediate previous state.

So, if I declare this to be my immediate previous state. So, I know that, before that, beyond that the history is not really it does not matter. So, therefore, this particular part will not be required. So, I can as well write this particular thing. So, this whole thing I did actually to put this Markovian property over here and then to see what happens, ok.

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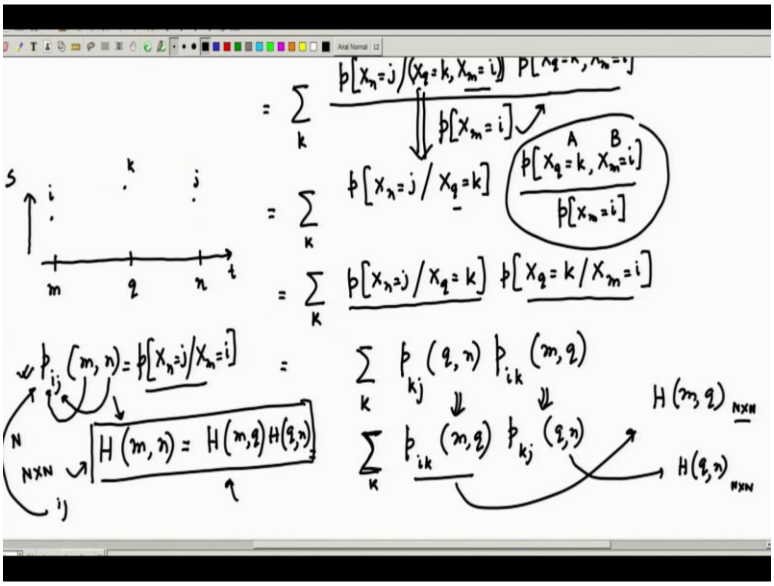

So, basically, I can write this. So, it does not matter, what has happened in further past, because I have an immediately previous state which is this one. So, as long as I know that summarizes all the past that is Markovian. So, Markovian says, that it does not have to be dependent on the previous things. As long as I have a current state or I have a nearer state.

So, that summarizes everything that is Markovian property, ok. So, if this is the case, I can always write this. So, therefore, no dependency is there. And this one I can take with him, ok? Again, you can see you can put another Bayes theorem over here. So, A into B. So, if this I declare, sorry, this I declare as A, this has B. So, A into B and or A comma B divided by p B. So, that should be A given B, ok.

This is what has happened, ok. Now, let us try to see, what is the consequence. Can you identify this term now and this term? So, again if I just, draw the time diagram over here. So, I have a time, I had some m instance, I had some intermediate q instance, I had some this n instance, ok. So, over here I am drawing the state, it was in I, and at this instant, it is in k, this instance it is in j, ok.

So, we had described something called p I j m, n what does; that means, that in the given that mth instant it was in i, what is the probability that it will go to j in, sorry, in j in nth instance, ok. So, this was the definition that p X n it will go to j given that at X m it was i. Can you see this definition is almost coming over here, The only thing is that now it's actually going from K to j.

So, can I write this, p it was in k in instance q, and then in an nth instant it goes to j, alright? This exactly follows from the definition, ok? So, I can write this. And what is this? Again, similar thing. So, probability, that it was in i, going to k from m to q, ok. Just shuffle them around, I will put this one first I k, ok? Now, from this particular structure, can we identify something?

It is a summation of all terms, where over here it has coefficient I k and over here k j. Does not this look familiar as if a matrix multiplication is happening, ok? This is the case. Basically, if I have a matrix, of this particular term with all i j and I have a matrix with this particular term with k j then this tells that these two matrices are being multiplied and the corresponding terms are being populated.

So, therefore, now if I represent the whole matrix, ok. So, let us say, I, these things I populate and I write a matrix H m, n, ok. So, this is the matrix, where it is how many dimensions, as many states are there. So, if there are N number of states, then it will be N cross N. So, that is the dimension of that matrix, ok. Each element i j actually corresponds to this term, ok. Now, this one if I write the corresponding matrix as H m, q and this one if I write H q, n.

Then what I can write, that this is also will be an N cross N matrix, because the intermediate step wherever it goes, has the same number of states to be populated. So, this is also an N cross N matrix correspondingly all i j will be there, ok? So, this is also an N cross N matrix. So, the matrix multiplication I can write, which is the fundamental equation that we get in return.

This is a very important property, remember, I could not have written this particular thing. If Markovian property was not true. If I had not written this step, where I was putting the Markovian property, I would not have realized this matrix multiplication. And then, multi-step transition would not have become this kind of multiplication, ok? This only happens in the Markov process.

So, in the Markov process only, I can get this kind of multiplication. One particular matrix if I define, that gets multiplied by the other one, ok. So, that is why probably you might now be realizing that whatever we have done so far, to derive that pi q equals pi. So, we were putting this P matrix one after another being multiplied. So, every state was multiplied by p and then going to the next stage state.

And then that was independently again being multiplied; that means we were having this luxury of any two or three-stage transition is just a multiplication of all single-stage transitions that happens only because of this.

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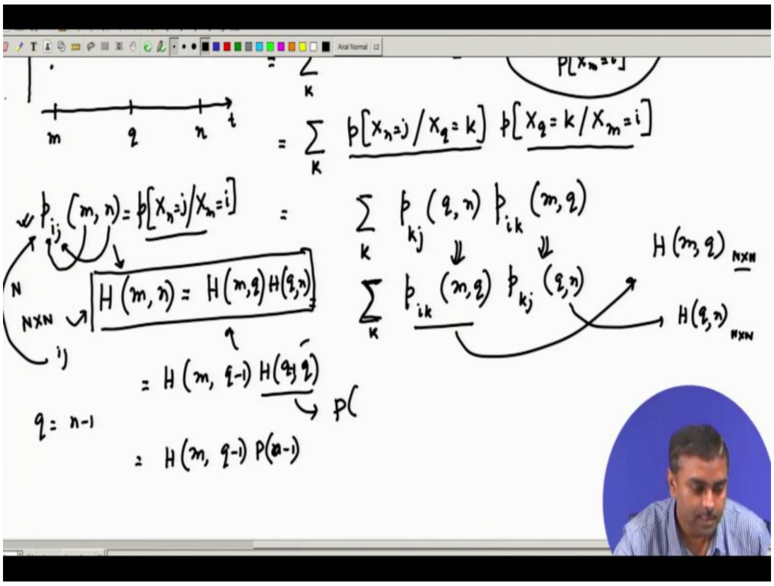

Because I can actually elaborate on this, I can write this H m, q I can further divide it, or maybe q, I can take the term q as n minus 1, if I do that, then this becomes m, q minus 1 and this becomes H q, sorry, q minus 1, q. What is this? This is my one-step transition matrix P. So, I can put that. So, I can write H m, q minus 1 into P, ok.

If I do not want to drop the time, I can still write that it is, because if I say I do not know whether it is homogeneous or not, I can still keep that it is P n minus 1, ok. Whatever way you designate. If you designate this to be P, sorry, not n minus 1. So, that should be q equals n minus 1 sorry sorry, let me just write it properly.

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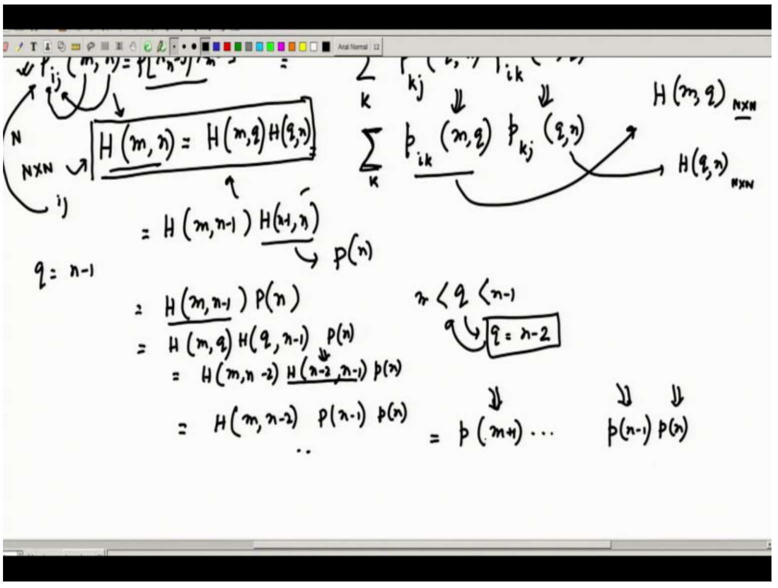

So, now if you designate this to be P n, then it is P n, you can write also this to be P n minus 1. So, if I designate it as P n, this will be the definition, ok? Again, this I can open up. So, again this can be written as H some m, q, and H q, n minus 1, P n because any matrix, I can sub-divide into some intermediate stage, where again this q is between m and n minus 1.

Again, q I can write as n minus 2. So, basically q I can put to be n minus 2 because q can take any value. So, I can also do q equal to n minus 2 because that is catering to this satisfaction. If I put that, then you will see again from here, if I put q minus 2. So, it will become m q minus 2, H, sorry, not q minus 2, n minus 2, again I am writing wrong I think.

So, this is again one step transition, from n minus 1 to, sorry, n minus 2 to n minus 1, that I can write according to my definition P n minus 1. So, this will eventually become, H m, n minus 2 P n minus 1 P n. I can continue this. So, this will just happen to be p m plus 1 dot dot dot all p, p n minus 1 p n. So, basically that entire matrix multi-stage transition will just become a multiplication of each single-stage transition.

If they are homogeneous, then they are all actually p and it will be p to the power n this is what we have we could do if it is Markovian. If it was not Markovian, we could not have done this thing.

So, in this multi-stage, I have to take into account all the previous stages, what has happened and all transitions will get mixed up which is not happening for Markovian property only. So, this is a very important property. This will give us towards the derivation of the continuous-time Markov chain later on you will see that from the next class.

Thank you.