Communication Networks
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Module - 04
Queuing Theory
Lecture - 18
Poisson Process (cont'd.)

So, we have been discussing the Poisson Process, In this particular class we will be also discussing again Poisson process again. We have already seen the mathematical implication of the non-simultaneity principle, and we have also seen the insight of independent increment. Now, we will be combining these two to get our Poisson derivation. So, that is the whole thing that we will be covering ok.

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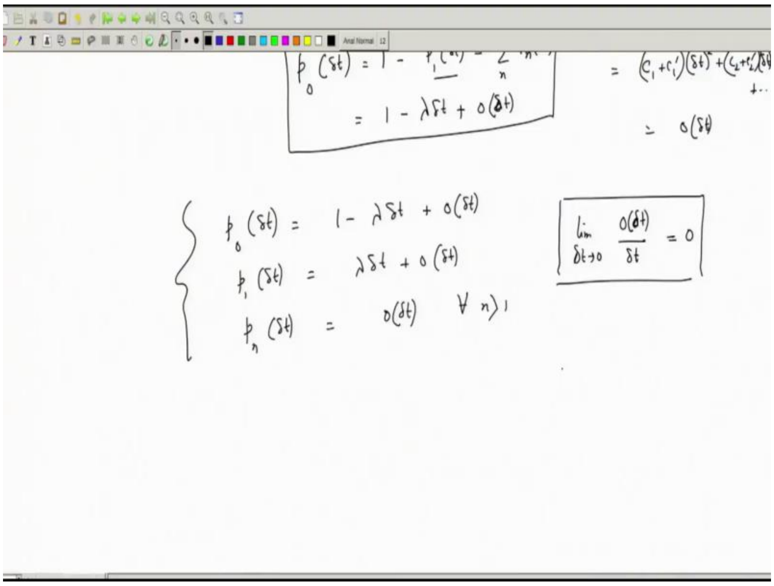

So, that was the summary of the previous class ok. So, that is what we have already derived, so this is something we have we could derive. Now, let us try to see from here what can we talk about the actual random process that we are interested in.

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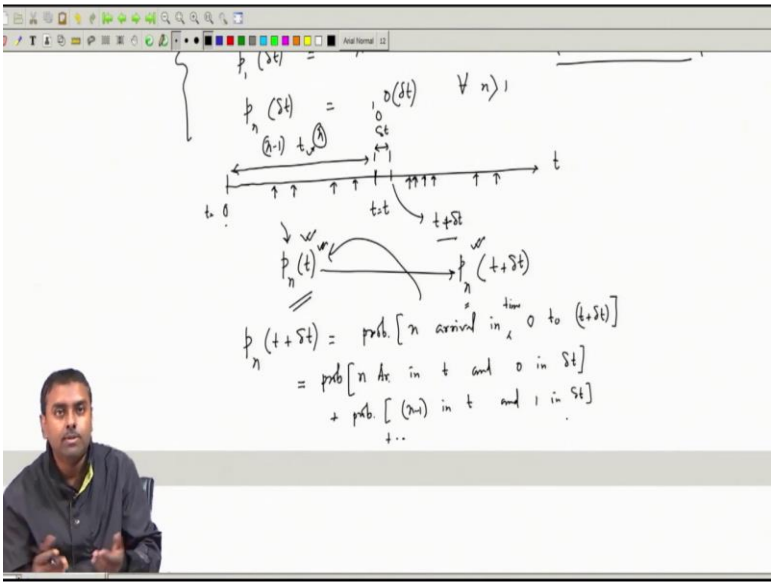

So, how do I define that random process, let us try to define that. Again I will draw the timeline I know there are arrivals that are happening, something like this ok. It has those three properties stationarity, of course, simultaneity has to be there that is fundamental and it has independent increment.

So, let us say we assume that is happening ok. Now at any, so this time let us say we have a definition of time equal to 0 t equal to 0 ok. And we have a definition of some t, t equals to t ok. So, basically, what we try to do over here we will define a time t. So, this is 0 to t, t and then over t, we will be defining another small amount of time delta t ok.

So, as you can see this is an independent increment that we are doing. So, we are going up to time t and then we are going t equals t plus delta t ok. These two intervals delta t and t are independent intervals or non-overlapping intervals. So, therefore, the principle of independent increment we will be able to put, how we will be putting we will see that ok.

First, let us try to do the statistical description of what kind of statistics we want to get from here. So, the statistical description that we want to emphasize here is as we have been discussing that is the number of the count. Poisson is the distribution for counting ok.

So; that means, the fundamental random variable that we are talking about over here is the number of arrivals or number of events that have occurred, so it is the count. So, now, we define a sorry. So, it is these things that we will define. What is this? This is called n number of arrivals that have occurred in t starting from 0.

So, the time I will always count from 0, from 0 how many arrivals up to this time have occurred. So, P n t is the associated random property that or random thing that we are trying to characterize ok. If I characterize this as actually giving me all the counting things ok.

Because it will give me the count up to some t ok. So, that means, whatever t now I put I will be getting an associated count that is probability ok. So that means, that should be a good criterion for our characterization ok. So, now let us try to understand this P n t in a slightly different manner.

This P n t if we now take time up to t plus delta t. So that means, that there should be some P n t plus delta t, what does that mean? That means up to time t plus delta t only n arrival has happened ok. Now, if I have to solve these things I need to form an equation governing equation for this time evolution so, from here to here, this evolution that I will have to characterize ok. So, let us try to characterize this to this. So, let us say this by definition P n t plus delta t.

What does this mean? That means, this is the probability that n arrives in 0 to in time sorry 0 to t plus delta t. What does this mean? So, how do I relate this to this one; let us try to do that so, n arrival between 0 to t plus delta t. Now, I have to do this independent interval.

Can I, now say that if overall n arrival has to happen what are the possibilities which are there? It might happen that n arrival has happened within this t and 0 arrival over here, which will count for overall t plus delta t n arrival has happened or it might happen that n minus 1 arrival over here and 1 arrival over here and so on, as many possibilities are there.

So, therefore, I can write those all those things are mutually exclusive if n arrival happens over here and 0, if that happens never simultaneously n minus 1 arrival and 1 arrival will be happening. So, they are mutually exclusive, so their probability will be added.

So, therefore, I can write this probability to be a sum of the probability that n arrival happened n arrival in t and 0 in delta t plus probability, that n minus 1 in t and 1 in delta t and so on. All those possibilities n minus 2, n minus 3, up to 0 over here, and all n over here are the only possible cases. So, they are mutually exclusive means these events are mutually exclusive. So, probability gets added we know from the basic probability principle ok.

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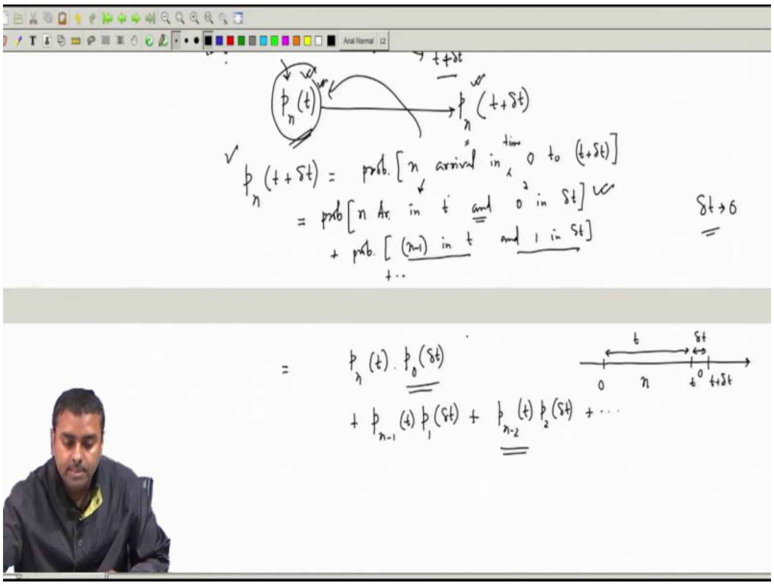

Now, let us try to see how I evaluate this, what is this? This is coming with the and; that means, it is a joint event. Probability that n arrival happened over here and 0 arrival happened over here. So, if I just draw the time frame over here for reference. So, this is 0 and this is t and this is t plus delta t; so that means, this part is delta t and this part is t. So, basically at this time, no arrival has happened over here 0 arrival has happened, ok?

It is the joint distribution of that one, but because this is independent intervals or nonoverlapping intervals. So, therefore, I can say from the principle of independent increment that these joint distributions can be separated. So, therefore, this can be written as the first term can be written as, a multiplication of two terms; where it says the probability that n arrival has happened in t, which is this these things.

So, that is what we have targeted that this I will be relating with this one. So, P n arrival in t multiplied by P 0 arrival in delta t. If now I put delta t tends to or I should not even put that delta t tends to 0 later on I will be doing. Now, P 0 delta t, can you identify this? This we have already proven over here P 0 delta t, we will be able to put these things over here ok.

Similarly, the next one what is this n minus 1 in t? So, I can write that as P n minus 1 in t according to my definition and what is this one 1 arrival? Again because of independent increment, so they will be separated out. So, P 1 arrival in delta t is also something we know, and so on, P n minus 2 P 2, and so on. Now, this is just the probability value, the value multiplied by P 2 delta t; what is that? We can now substitute this thing for what we have already known. So, let us try to do that.

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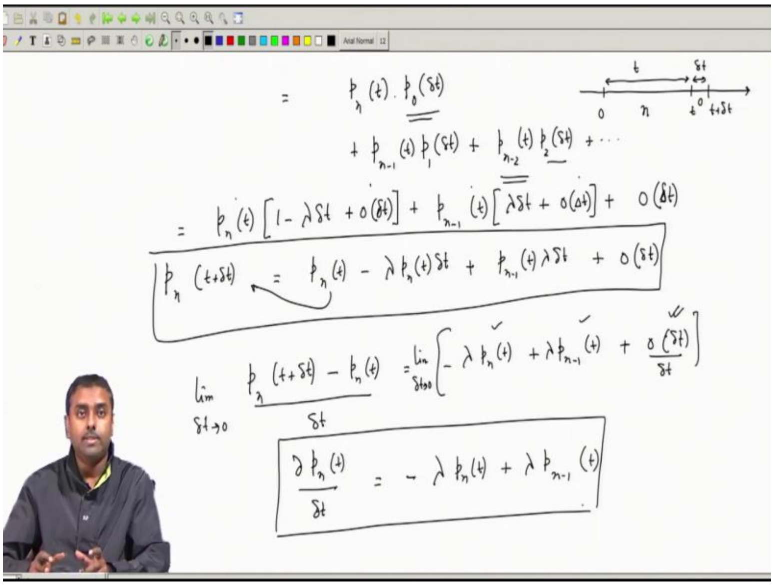

So, if we just do that. So, this will be P n t P 0 I can write as 1 minus lambda delta t plus o delta t, plus P n minus 1 t P 1 I can write as lambda delta t plus o delta t. And these things o delta t multiplied by something ok, and added. So, some coefficients are being added to o delta t and then summed over all o delta t. So, that will be nothing but o delta t only ok.

So, if I just write that again this P n into o delta t that will go inside o delta t. P n minus 1 into o delta t that will go inside o delta t, because o delta t is this is just o delta t as we have depicted earlier also. It is just multiplied by another coefficient and again added to another o delta t; so, that will be again o delta t.

So, I can write this as P n t minus lambda P n t delta t plus P n minus 1 t lambda delta t plus overall o delta t. This is ok? This is fine, right? So, now, let us see what has happened, I have got this fundamental equation. Can I define a can you see that I can define a partial derivative or differential equation through this?

How do I do that? Just P n t take it on this side and divide by delta t take delta t limit delta t tends to 0. So, basically, I will do P n t plus delta t minus P n t which will be minus lambda P n t and also I divide by delta t. So, this delta t gets canceled plus P n minus 1 t into lambda this delta t also gets cancelled, plus o delta t divided by delta t. Now, I take the limit delta t tends to be 0 on both sides.

Now, this is not dependent on delta t these two terms. So, they will remain intact according to the definition of delta t o delta t this will vanish. So, therefore, I can write that the derivative of P n t d t is equal to minus lambda P n t plus lambda P n minus 1 t, a very nice differential equation has been formulated.

As you can see it was a very simple principle from those three things we could derive this whole particular differential equation. Now, we will see how to solve this differential equation to get into the Poisson distribution. So, over here before solving the differential equation, of course, we need to have the limits. So, let us try to understand the limits what we are trying to solve is this one P n t.

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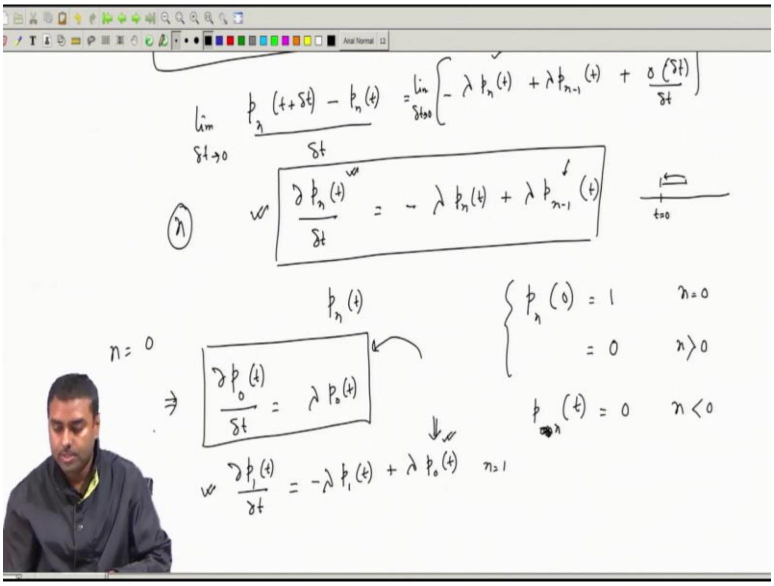

Let us try to understand what the value of P n 0 that is the time I am solving. So, the boundary condition is time equal to 0 because anything will be starting from time equal to 0. So, therefore, that is the boundary condition. So, P n 0 how do I calculate this P n 0 or how do I say what will be the value of P n 0? So, let us try to understand, what is P n 0. At time t equal to 0 I start and the interval that t I was taking is 0; that means, it comes over here only.

So, within this, if n is equal to 0 what is this probability that in 0 times 0 things has arrived that is the sudden event because in 0 time nothing can arrive., therefore, these are certain events, so that must be having probability 1. But this will be equal to 0 if n is any value greater than 0 because no arrival can happen within 0 time.

So, therefore, I get this whole boundary condition P 0 0 is equal to 1 and P 1 0 P 2 0 P 3 0 and all other things will be equal to 0, so this is the set of boundary condition that we get. Now, you might be asking how many of these things we have got this is true for any n right.

So, therefore, as many n you have. So, n can go up to infinity, so the infinite differential equation we have got; is the first equation. So, when I put n equal to 0, but if I put n equal to 0 that is the first equation if I put n equal to 0 what do I get over here I get P minus 1.

So, now that is another thing in the boundary condition I have to put P minus any value any n ok or I should say P n any t you put that must be always 0 if n less than 0 because it really does not make sense, the probability of negative arrival that does not make sense physically. So, therefore, those things should be all accounted for 0 for any value of t no interval you can take where negative arrival will be happening.

So, that is impossible. So, therefore, t might be 0 any value of t whatever you take always negative arrival whenever n is less than 0. So, that will be all 0, so therefore, this becomes 0. So, at n equal to 0 I can write this P 0 t plus this vanished, I get the first differential equation the next one will be n equal to 1. So, for n equal to 1 and so on for any value of n that will be happening.

So, now there are multiple differential equations I have to solve. But can you see the second differential equation that depends on the first solution? So, therefore, it needs a recursive solution. So, the first one has to be solved first whatever I get I substitute over here then for P 1 you will be able to solve, once you solve P 1 then P 2 definitely will have P 1. So, recursively you keep on solving this differential equation. So, that is exactly what will be doing will be showing what will be the solution to this one, let us try to do that ok?

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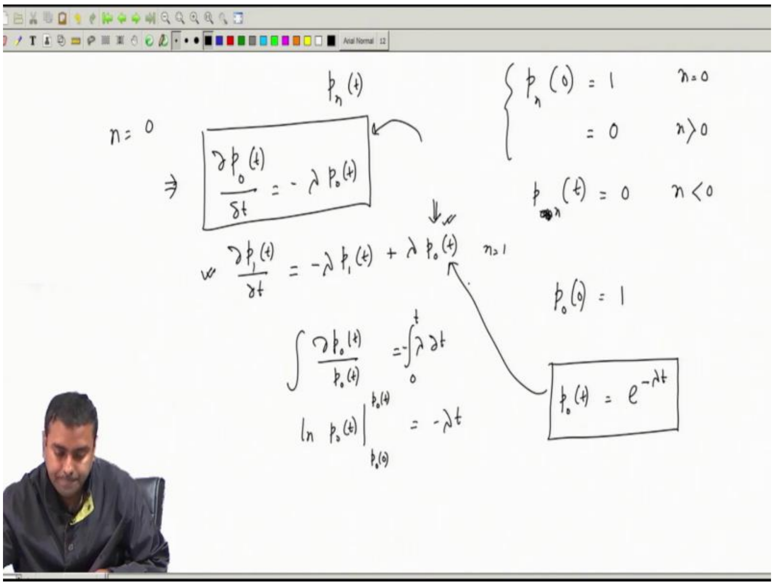

So, this is easy ok. What will be the limit over here? At this point, it will be P 0 0 ok, so that should be 1. So, basically, this will happen to be l n P 0 t and the limit will be P 0 0 to P 0 t. So, if you solve it this will become 1. So, you put that and then you will get P 0 t is equal to sorry this should be minus, isn't it yes. So, that should be minus, so that minus as such that is our first solution P 0 t is equal to e to the power minus lambda t. That is, we can see the first glimpse of Poisson being generated. Now, substitute this over here ok.

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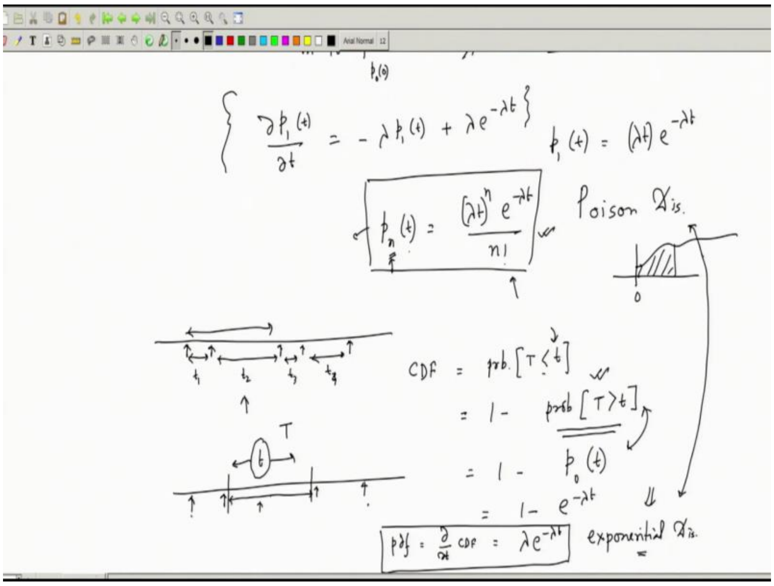

Now, again your 11th standard differential equation solution method you put over here, you will see that P 1 t will be becoming lambda t e to the power minus lambda t. And if you keep on doing this that will be the case ok. This is famously known as Poisson distribution.

So; that means if a particular thing has all those three properties it follows Poisson distribution. So, this is the fundamental derivation of Poisson distribution. So, we have now understood that the arrival process if it is having some independent increment and follows stationarity and nonsimultaneity of course. Then it must follow the counting associated counting process; that means, between 0 to t how many have arrived.

So, this associated probability that n arrival has happened within t that probability is dependent on with this equation or that probability depends on this equation. Of course, the distribution is dependent on time, so therefore, it is a random process and it is characterized by this Poisson distribution.

So, any amount of time you fix then accordingly you will get the associated probability ok, for 1 arrival 2 arrivals 3 arrivals. So, all the count-associated probability you will be getting is a distribution of course, so it is normalized already. So, you can always see that the summation will be over n from n starting from 0 to infinity of course, it is a sided distribution.

Because negative arrival does not make sense we have already been told that this limit condition says that, that p any value other than positive or 0 will be always 0. So, the probabilities are all 0, it will be just having these values ok. So, once we have derived this let us try to now see if can we get some other insight into this arrival process. So, now, we have got the counting process or the associated statistical property of the counting for these kinds of arrivals.

So, can we now characterize these arrivals a little bit more, what will be now trying to do? Whenever there is arrival there is also something. which is random is called the inter-arrival time, which is the time between successive arrivals.

So, this might be t 1, this might be t 2, this might be t 3, this might be t 4. So, this t 1 t 2 t 3 t 4 all these things this means the amount of time between two successive arrivals not between this and this that is not inter. Inter-arrival has to be with the next arrival or the previous arrival that is only inter-arrival.

So, these things are also random because if the count is random when they will be coming if that is random then of course, their inter-arrival will be random can you now characterize this random process? So, now we can see this is a time and it can take any value any real value. So, it is a continuous time distribution this was a discrete distribution this is a continuous distribution. Why this was discrete? Because this was defining count how many.

So, that is always countable as 0 or 1 or 2. So, it is a it is a discrete distribution. So, this has associated P m f whereas this will be having associated P df ok, probability density function. So, that is exactly what we will have to characterize now.

How do I characterize that, can I characterize it from this? Because this is already giving the counting process; so, there must be some relation if the count is having this distribution probably inter-arrival has to have some relationship with that counting process. So, that is the thing that we will be trying to exploit now.

In this counting process what we will be trying to do is we will be trying to evaluate the CDF, what is the CDF? So, let us say this inter-arrival time that random variable I define as capital T. So, CDF means the probability that this capital T is taking value CDF means it takes value from the lowest value to a particular value. That means if the sum value t is less than equal to that. So, 0 up to sum t value small t.

So, this small t is the parameter that is exactly what we are trying to do. Or it can be written as 1 minus probability because it is, so if it has a distribution ok. So, it is starting from 0 to t. So, that integration, of course, overall things should be 1 because it is probability, so it's normalized. So, therefore, this portion that is greater than that value should be 1 minus this one, ok?

So, this is the portion we will have to now evaluate ok. So, let us try to understand what is this from the count process. This is inter-arrival time, so let us take an arrival. So, inter-arrival is this, this inter-arrival is greater than some value t; that means, suppose this is the inter-arrival ok.

So, inter-arrival is greater than some value t, what does that mean? So, if this is my t and these are two inter-arrival. So, inter arrivals are it can be whatever it is it might happen over here, it might happen over here it cannot go inside. That means the inter-arrival is greater than t what does that mean, that I know that is certainly within this t there is no arrival? That is inter-arrival because inter-arrival means two successive times between two successive arrivals.

So, in between there cannot be any arrival. So, within t if inter-arrival is greater than t, certainly I know that there should not be any arrival. So, this probability that inter-arrival is greater than t that entire probability is just to ensure that within t no arrival has happened.

That means I can write 1 minus the probability that 0 arrival has happened within t. So, this is how I can link the counting process to the inter-arrival distribution. So, my CDF will be 1 minus P 0 t, what is P 0 t? 1 minus e to the power minus lambda t this is something we have already derived over here; that was the first derivation e to the power minus lambda t ok.

So, this was already derived. So, therefore, I can easily write these things. So, if CDF is this, what is P df? P df is the differentiation of CDF concerning its parameter t or the random variable. So, therefore, the P df must be differentiated this will be lambda e to the power minus lambda t, this is famously known as an exponential distribution.

So, what we could see is that, if the arrival counting process is followed by Poisson according to our definition of that particular arrival; then the associated inter-arrival time; which means, when the next arrival will be happening is exponentially distributed. These two distributions play a huge role in whatever we are going to derive next.

And we will in the exponential distribution because it is taken from this Poisson and Poisson was proven from the memory-less property. So, now will be defining memoryless property in the next class; so, what we will be trying to see is that because of that independent increment a fundamental thing that has happened over here is memorylessness; that means, the future event will not be dependent on the past it will be only dependent on the present itself.

This is something through the exponential distribution that will be proven without proof we will also say that the exponential and Poisson two classes are complementary actually. These classes are the only ones which are having this memory-less property because this is the distribution we have got from this particular independent increment.

It will be able to mean it can be proven that this is the only distribution that can be derived from its unique ok. So, that is something we will prove next once we have proven that then we will see how this particular distribution can take forward towards our queuing analysis. So, that we can do the trunk switch design ok.

Thank you.