Communication Networks
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Module - 04
Queuing Theory
Lecture - 17
Poisson Process

So, in the last class, we have already discussed the Stationarity of a Random Process and we started describing this arrival process as a stationary random process. Then we started giving two fundamental properties of this particular process, and this process will later on see that it it talks about the process which is called Poisson process or Markovian process. So, we will start describing about these ok.

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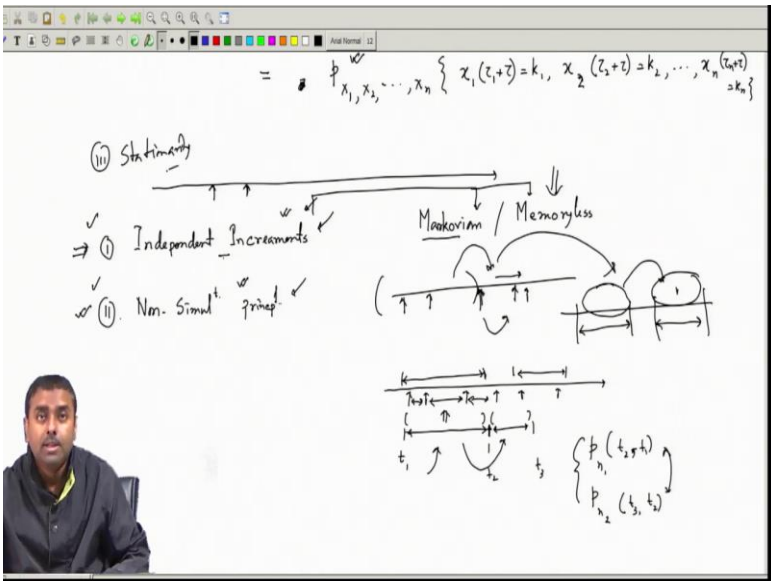

So, let us try to understand. So, if we go back to the last lecture what we have started describing is two things one is called independent increment and the other one is the nonsimultaneity principle. So, already we have seen what we mean by nonsimultaneous and some basic definition of that ok we have given. So, these are two very fundamental properties of this Markovian or Poisson process.

What are they? One is in any way fundamental; that means, depicting the event occurrence says that it is nonsimultaneous; that means, whatever happens, no two events in this world can simultaneously occur. So, this is one principle, the second principle is not so fundamental probably, but for this Poisson process, this is required where it says that it is a description where things of the future depend only on the present and do not depend on past ok.

So, with that, you will see just these two things, these two definitions that we have given from this two definitions we will derive the Poisson process that is the strength of this particular process. So, that says that the Poisson process does not require any extra description. So, it has only one important criterion which is that first point that is very typical of Poisson because of that point it has become Poisson ok.

Because this is fundamental to any event or any arrival ok? So, any arrival of this Non-Simultaneity principle will always be true, but independent increments are the property of the Poisson process and we will probably I do not think this course has the bandwidth to prove that Poisson will be able to show that this memoryless property that we have talked, Poisson is the only process in continuous time of course, which describes memorylessness. No other distribution is possible and for discrete it is geometric.

So, other than these two, there are no other processes that you will be able to describe which has memoryless properties. So, that is a very very fundamental thing and this, therefore, is very fundamental to this memorylessness, Let us try to understand these two things a little bit more mathematically. So, in mathematical terms what we are trying to say, first we are in any time instance where there are events that have occurred.

So, we are trying to get some intervals, the first criteria of these two intervals are they must be independent or nonoverlapping intervals. So, that is a very fundamental thing that they should not be overlapping ok, they might have common boundaries. So, I can start from here. So, basically, I can define these two intervals with a common boundary, but generally what we should avoid is that they should not have overlapping instances and then what I can say is whatever we are observing.

So, random things are happening. So, I might get count over here ok, I might get some other things what are the inter-arrival time between the arrival that has occurred. So, these kinds of different things I can get all are random properties ok? So, if I take count let us say the one we were describing was count.

So, if I say count then how many counts have occurred over here statistically that has no relevance to how many counts I will be observing over here; that means, these two are statistically independent and that is why they are called independent increments.

So, we are incrementing the time from here to some other time, and those incremental times that we are discussing that statistically independent from the previous one ok? So, that is why we call it independent increment; that means, any statistical property we define. So, suppose the probability of count we want to define p n over this interval let us say this is t 1 this is t 2 this is t 3.

So, over this interval t 2 minus t one or t 2 comma t 1 t 2 to t 1 and p of let us say over here n 1 and this is n 2 which is t 3 comma t 2. So, these two are independent ok what does; that mean? Mathematically they are a joint distribution that will be separated they will be multiplied. So, that is exactly what will be happening.

So, that is the mathematical definition we get from here, you will see that in our derivation we will be using this property. And independent increment later on you will also see that gives us the memoryless property we will also prove that ok. So, in this part we will do we will try to see that independent increment what is the consequence of that and we will try to say that this independent increment is responsible for this memorylessness.

We will also be able to prove that ok once we do the derivation of course. So, this is one thing next let us try to understand this Simultaneous principle what is the associated mathematical description of that, that is something we will be doing now ok?

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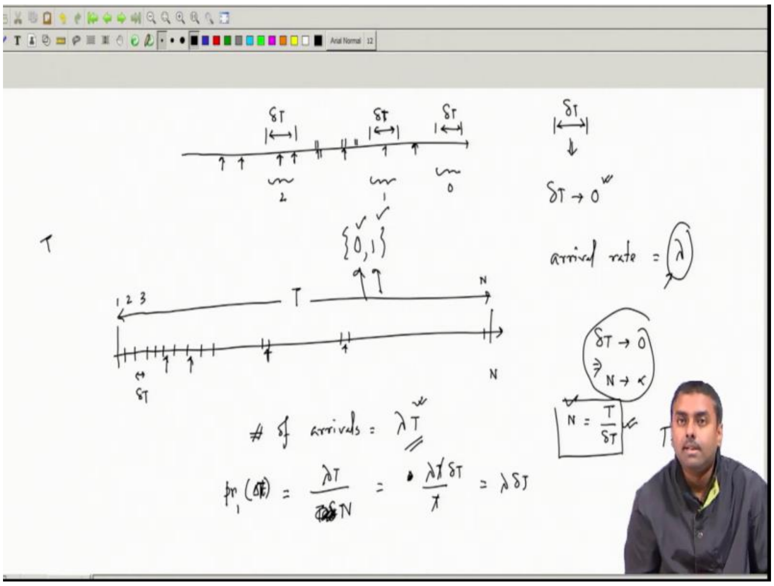

So, now let us try to see this non-simultaneous principle and what this means. So, again mathematically let us try to define it. Now let us call a probie time interval of interval delta T ok that probing can be happening at any instance. So, I can probe it over here for an amount of delta T, I can also probe it over here for an amount of delta T, I can probe over here for an amount of delta T ok.

So, I define this probing interval of delta T and I try to gather the statistical property ok what are my statistics over here we are gathering? It's the count. So, that means, within delta T how many arrivals have happened? So, over here if I just do a bookkeeping over here 2 arrivals have happened, over here 1 arrival has happened, over here 0 arrivals have happened, like this, if I keep on doing keep on doing this probing I will see a different kind of because its random. So, different kinds of things I will be getting ok.

Now, because of this nonsimilarity principle. So, you can carefully see how we are coming to this nonsimilarity principle and its mathematical definition. If we start making this delta T tends to be 0 ok. So, if you do this limiting condition; that means, my delta T that probing interval is becoming infinitesimally small.

Now because of non similarity principle when delta T is vanishingly small can I observe anything other than 0 and 1 arrival can I observe? Because earlier for this finite amount of delta T where it was not we took some value and it was not going to 0 ok.

So, at that point I could have said any value is possible I can have 3 count, 4 count, 7 count everything is possible. But when I start making delta T vanishingly small then I can see because of the nonsimultaneous principle, I will have only either 0 arrival or 1 arrival. So, whenever it is probing it will its so small that either it will probe nothing or it will probe only 1 arrival because even if 2 arrivals are very close my delta T can be even smaller than that. So, that it will skip the other one.

So, therefore, I can only have either 1 arrival or 0 arrival if I make this delta T tends to be 0. So, this is the essence I will try to bring in ok? So, let us try to see how I bring in this principle ok? So, for that let us say I have a stationary process, if I have a stationary process what is the arrival rate? That must be constant over time and that we have called lambda.

So; that means, lambda number of arrivals are happening for a unit, when I make this delta T to be 1 let us say 1 second or 1 millisecond whatever is my time unit. So, over that lambda number of arrival will be happening whatever the value of lambda if lambda is 2; that means, 2 arrive remember its average arrival. So, it will not be every 1 second you take everywhere 2 arrivals will be there, it just that will be the average you take multiple such things you average it will be coming close to that lambda value and this will not vary over time.

So, if you take the N symbol averages it will be always similar at every time instance. So, let us say this lambda is my arrival rate right? Now let us try to understand suppose I have taken a very big time let us say that is capital T ok. So, this very big time I divide into small-small units of delta T ok. So, basically, I am defining my probing times as ok and if delta T tends to 0 then how many such intervals will be there? That will go towards infinity. So, if that count is capital N. So, N goes to infinity.

So, if I keep counting. So, this is my first interval second interval third interval and so on, if the last one is the N th interval. So, N will go towards infinity of course, that will be happening because delta T tends to go to 0. So, therefore, the number of arrivals and sorry number of intervals will be increased. And this T is very big I should say ok. So, now, let us try to understand how many such intervals are there. So, the N I want to characterize N must be equal to T divided by delta T of course, that defines these things. So, whenever N goes to 0 sorry delta T goes to 0 N goes to infinity.

So, this is my basic definition ok. Now let us try to understand if T is sufficiently large then within this time capital T how many arrivals will be happening? On average lambda arrivals happen per unit time. So, in T time how many arrivals will be happening? So, the number of arrivals if I count that should be lambda into T because lambda arrival per unit second or unit time. So, in T time it will be lambda into T on an average if T is sufficiently large, then this averaging will come to be true and this will happen to be lambda into T.

So, that is why I have to take a very big T ok? So, I can almost surely say as long as T is going towards infinity or very big then I can say surely with the law of average or we should say this frequency definition of probability principle or the definition of average, this many arrivals will be happening lambda into T number of arrivals will be happening inside this.

So; that means, if I now try to say if my delta T is very small how many either it can have 0 arrival or T arrival within this delta Ts because they are infinitesimally small. So, there cannot be any other things which is possible. So, either there will be 0 no arrival or there will be one arrival ok. So, this kind of instance will be there ok?

So, therefore, if I now try to calculate the probability that one arrival has happened in delta T ok or delta capital T sorry what is this value? Probability one arrival has happened that is actually if you now count. So, probability is also probability has a frequency principle.

So, the frequency principle means all possible cases that should be in the denominator and numerator the favorable cases. So, what are the favorable cases of one if you count over here all these infinite almost infinite numbers of cases, if you try to just see how many arrivals are happening which means one arrival is happening or a favorable case to your probability definition that will be there are lambda into T arrivals overall.

So, that must be distributed over here. So, lambda into T favorable cases must be there. So, lambda into T. And how many total possible cases and total possible intervals how many are there? That is N which is T delta sorry T by delta T. So, this is N which I can write as lambda T divided by T just put this definition delta T T gets cancelled.

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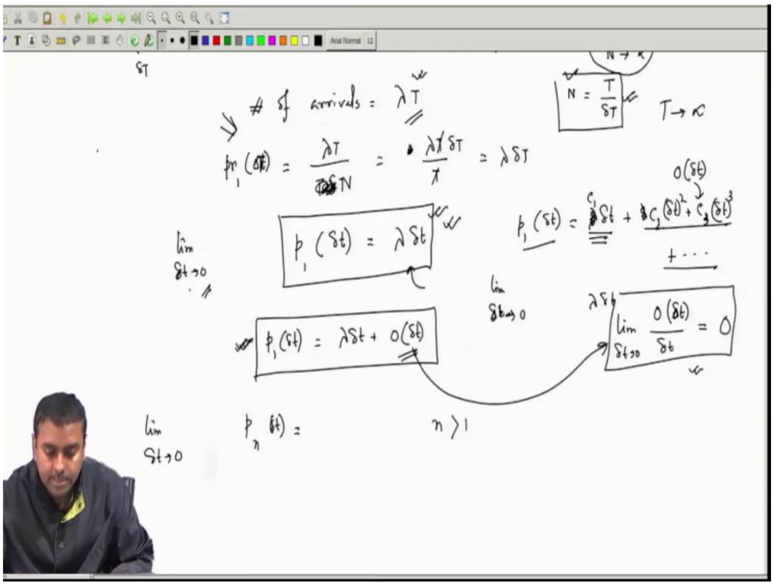

Immediately, as you can see what we have got if we now generalize that probability that one arrival happened in some instead of capital T now I give a general definition, in some small delta t time that will be always lambda into delta t this is our first definition ok that is what we are saying.

But over here I have another condition that I have to put a limit this delta t tends to 0 this can only happen when delta t goes to 0 otherwise not otherwise this calculation is wrong because then there will be a lot of 2 arrivals, 3 arrivals which we have not accounted in this derivation. We have done this derivation this entire thing assuming that only two cases are there either 1 arrival can happen or no arrival can happen and accordingly we have taken all these 1 arrival counted as a favorable case.

If some of this 1 arrival is clubbed into 2 arrivals, 3 arrivals have to be subtracted, but I have not done that. So, therefore, this calculation has some error if delta t does not tend to 0 how do I account for that what I can say is that this p one delta t I can is a must-be regular function. So, this can be defined as a polynomial ok.

So, always I can define it as a sum polynomial it can be anything I do not know. So, some polynomials whose coefficient has to be decided. So, how do I put that polynomial? So, I can put that to be some let us say lambda delta t plus some c 1 into delta t square plus some c 2 into delta t cube and so on I can define it ok. So, of course, it should not have any constant value in this polynomial, because if delta t is 0 I cannot get 1 arrival. So, basically, in a 0 time, I cannot have 1 arrival within that ok.

So, this is fundamentally known that if my interval is 0 how can I catch an arrival because that is a point ok? So, therefore, it should not have this polynomial, this generic polynomial should not have any constant of course, I should not put lambda over here. So, maybe, I will put c 1 over here c 2 c 3 something like that some general polynomial. Now what I have seen, is that if the limit delta t goes to 0 then this happens to be this one.

So, therefore, as you can see this must be the c 1 must be lambda delta t and this particular portion must vanish what does that mean? If I define these to be a polynomial o delta t then I can write this p 1 delta t to sum this lambda into delta t plus some higher order polynomial of delta t and what is the definition of that? That is this o delta t means this portion goes to 0 when delta t goes to 0 ok.

Or rather I should say that if I divide this not only this goes to 0 when delta t goes to 0 also I should see that if I divide this whole thing by delta t it first comes to lambda 1 ok. So, therefore, what I can do is this is a higher order polynomial of delta t square delta t cube and all those things. So, if I now divide this by delta t.

So, this is delta t the definition is if I divide it by delta t and then take the limit this must be vanishing. So, that is the definition of this order. Once I give that definition my overall things are clear, because now I can without putting this condition I can make this definition only thing is that for this one I define a separate thing.

So, that I am safe I can always say whatever delta t I take is the value only thing is that if delta t goes towards 0 this will anyway this particular part vanishes and I will get this definition my actual derived definition back. So, therefore, I can safely say that this is the now usable definition that we can create over here.

So, we have some probability terms remember with all these principles our target is to characterize the stochastic property. So, that is exactly what we are trying to do over here we are trying to characterize that stochastic property ok. So, we have got something now over here, that p 1 in some interval; that means, the probability that 1 arrival will be happening that count will be 1 in that arrival delta t is nothing, but this lambda which I already know that it is stationary.

So, that stationarity has to be another property that is the sum which is something I have already discussed even before these two fundamental property rights.

So, stationary you can always say that is the third property ok. So, stationarity has to be satisfied; once the stationary is satisfied then lambda I can fairly take ok and then and that should not be dependent on time therefore, I can define this equation which is calling that probability that one arrival happening within a particular interval delta t ok wherever that delta t is defined in time.

And it has a higher order polynomial which has this extra definition ok good. Now let us try to define the probability of n t or n delta t where n is anything greater than 1 it might be 2 it might be 3 it might be 4 something like that, in general, if I put this limit delta t tends to 0 where it will go what is p n delta t? That must be 0; because there are no favorable conditions or no favorable cases.

Again if I try to calculate this one what is the probability that within delta t 2 3 or 4 arrivals will be happening? All are 0 because nothing because of the nonsymmetric principle if my delta t goes to vanishingly becomes vanishingly small or goes to 0 then I do not get any 2, 3, or 4 arrivals within a delta t.

So, therefore, those should be 0. So, this must be all 0, but that is only true when delta t goes to 0. So, therefore, again similar definition I can give I can always define that p and delta t is a polynomial that vanishes to 0 whenever we put delta t tends to 0.

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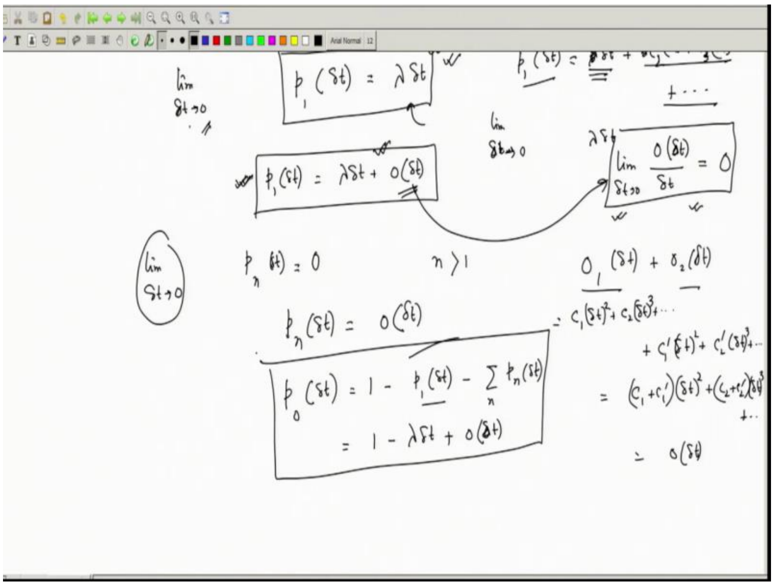

So, therefore, again p n delta t I can define with the same principle as o delta t where o delta t has the same definition as this one without giving this.

So, now we have two definitions, probability that 1 arrival is happening is this one probability that n arrival n anything greater than one that is this one let us now try to see only thing that is left is 0 arrival. So, what is the probability that 0 arrival will be happening in delta t? Because we have calculated all other probabilities it's just 1 minus all those so, 1 minus probability that 1 arrival will be happening and minus summation over n probability that n delta t will be happening ok.

Now, if I just put 1 minus lambda delta t minus o delta t and this is also a summation of o delta t and this o delta t all together if you sum them that will remain another o delta t because what will be happening? Let us say sum o 1 delta t and o 2 delta t we sum all that will be happening they are polynomials of second order and higher their coefficients only will differ.

So, if this I write as c 1 delta t square plus c 2 delta t cube dot dot dot dot and this if I write as c 1 dash new coefficient delta t square c 2 dash delta t cube dot dot dot dot. I can always write this sum as c 1 plus c 1 dash a new constant delta t square plus again c 2 plus c 2 dash a new constant of delta t cube which is nothing, but another o delta t. So, therefore, I can write this whole thing that one delta t coming from here and all delta ts are summation I can just write as another o delta t ok.

So, therefore, that happens to be my definition of p 0 t p 0 delta t. So, what we have done almost just by this nonsimilarity principle we could characterize the whole thing ok we could mathematically define the whole thing.

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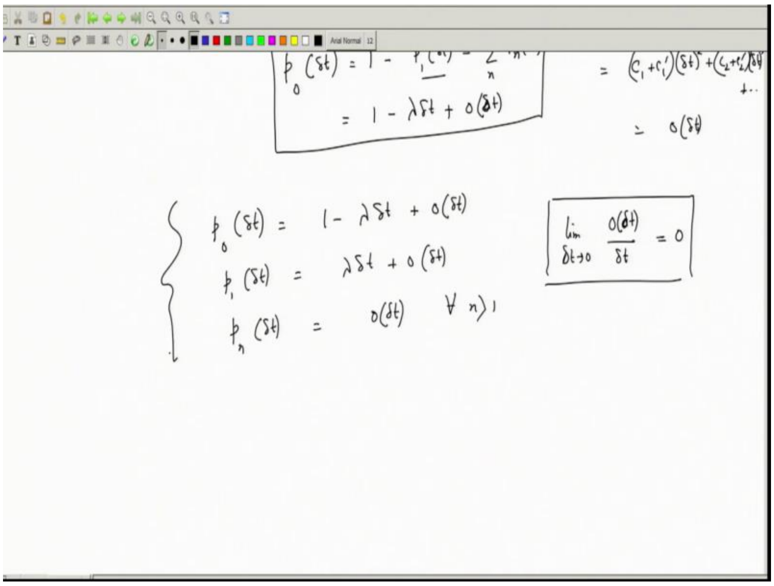

What is the definition if we just summarize? So, p 0 delta t is nothing, but 1 minus lambda delta t plus o delta t p 1 delta t that is lambda delta t plus o delta t and p n delta t is equal to o delta t for all n greater than 1 with an additional definition of limit delta t tends to 0 o delta t; sorry divided by delta t is equals to 0 that is the overall mathematical definition of the nonsimilarity principle.

So, now we have understood two things, we could define the nonsimilarity principle taking the assumption of stationarity, and we could also mathematically define earlier we have seen that independent increment. What will be doing in the next class is taking these two things and trying to derive the associated arrival process which follows these things which is stationary, which is nonsimultaneous these two things are ok that is by definition and it has this independent increment ok.

We will see that just by taking that independent increment the Poisson distribution will be automatically derived from these three principles. So, that will be our next discussion. So, till then goodbye.