Modern Digital Communication Techniques
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Lecture - 51
Performance of Digital Modulation Techniques (Contd.)

Welcome to the lectures on modern digital communication techniques. So, in the previous lecture, we have seen the models of channel. So, it is a same thing that we have been doing before, but we have a specifically looked at some of the things which we will need now when we look at the celebrated expression of channel capacity due to contribution of the great communication scientist Claude Shannon and just a word of note that because of the contribution which we made the whole field of information theory was developed and it is a very interesting field and it is involved with lot of mathematical expressions, it gives generally bounds on performance of certain systems.

We will take a look at one of the important results, we will not be very rigorous in our development because our idea is to look at what is the end result is and how is it relevant for the study of communication systems. So, continuing with our discussion. So, now, we look at the development of channel capacity.

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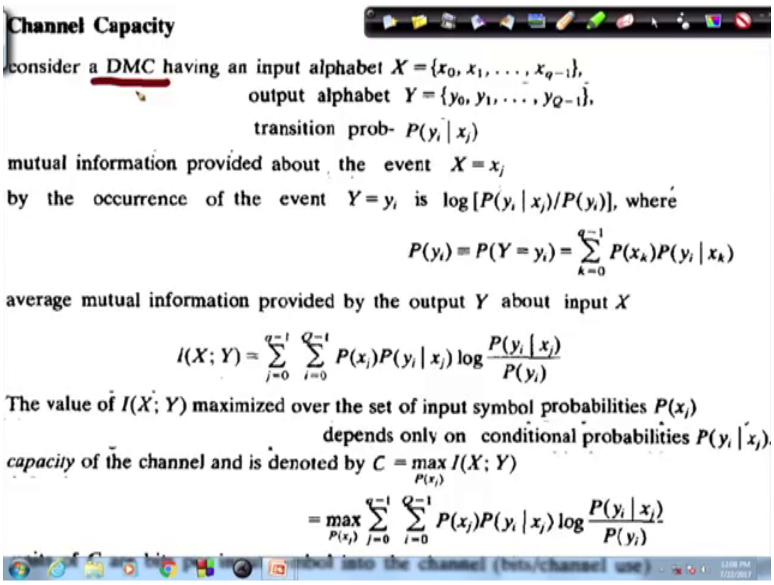

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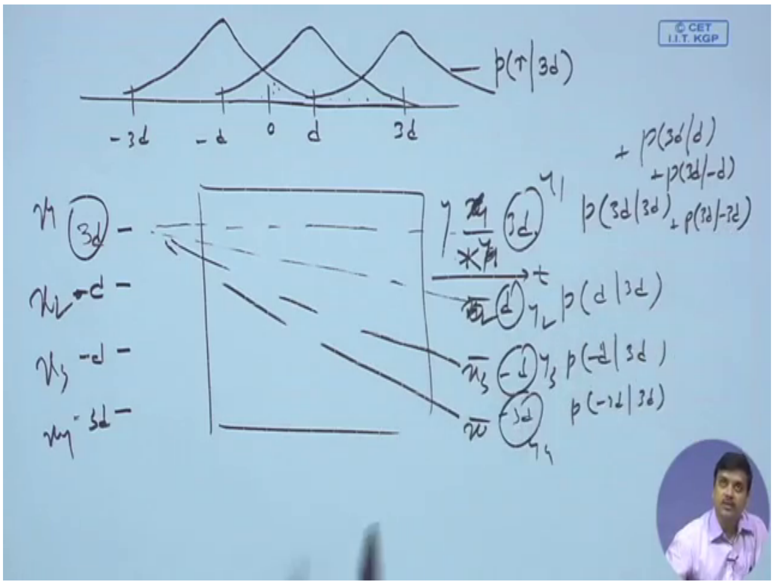

So, we consider a discrete memory less channel. So, a discrete memory less channel is something which we have been using before; that means, it has a set of possible symbols, right, discrete it is a channel and it produces a possible discrete set of outputs and the output at a certain instant of time now is not dependent on what happened before. So, it is not dependent on what happened before, right. So, that is the memory less channel and the previous channel that we had seen was the binary channel so; that means, it had 2 possible inputs, you could have m possible inputs if there are m points in the constellation.

So, if you have; so, if you look at the output of symbol mapper and at the transmitter and at the receiver, we see the decisions with respect to the symbols, right, not the bits yet; that means, the constellation point we have a discrete input discrete output and if its memory less modulation and the channel is without any memory for example, just AWGN channel, then we have a discrete memory less channel.

So, moving further, the input alphabet is given by this alphabet means is a possible symbols which you could mark it as possible constellation points in PAM or QAM in case of PAM this will be a real in case of QAM. This will be complex output alphabet again, we are talking about alphabet. So, all the noise is getting added, but still we are mapping the outputs to a particular constellation point and transition probabilities are of course, that is given there.

So, in this case, it is a probability of one symbol becoming another so; that means, suppose I have 3 d d minus d minus 3 d and here also, we have 3 d d minus d minus 3 d. So, we are talking about what is the probability 3 d remains 3 d probability of being correct probability of being error probability of being error probability of being error. So, we have transition probabilities of getting 3 d given 3 d probability of getting d given 3 d probability of getting minus d given 3 d probability of getting minus 3 d given 3 d. So, were going to have all these probabilities.

Similarly, probability of 3 d given d and so on and so forth, we are going to get all these transition probabilities and there is the definition of the term called mutual information. So, we will just use it without going involving into much detail. So, mutual information provided about the event x equals to x i by the occurrence of y equals y i, what it means is that I have observed. So, let us say this is x 1, x 2, x 3, sorry, this is y 1, y 2, y 1, y 2, y 3, y 4, this is x 1, x 2, x 3, x 4. So, only tells is it suppose, I have made an observation, right. So, the mutual information is of the event that we observed y 1 and x 1 could have been the selected signal.

So, instead of selecting y 1 and x 1 we say y i and x i any particular symbol. So, what we have is given by the expression log of P y i x i upon P of i y i x j P of y i. So, this is x j this is y i. So, for output is observed one of the output is observed; what is the mutual information with a particular source symbol, right. So, this is the expression that is used and P of y i is given by the event that the random variable y takes the value y i is P of y i conditioned on x k averaged over all possible x k; that means, for a particular output; that means, for this particular output the conditional PDF that 3 d has been observed given 3 d is transmitted plus probability of 3 d has been observed given d has been transmitted plus probability 3 d has been observed given x d has been transmitted plus probability of 3 d has been observed given minus 3 d has been transmitted.

So, this is the total probability of 3 d being observed this is also straightforward. So, the average mutual information provided by the output variable y has to be about the input x, right about the input x is given by. So, if you look at the extension is your averaging the mutual information over all possible. So, this is the mutual information expression that you have that is average over the joint distribution of x and y, right, over all possible x and y. So, this is averaged mutual information and the capacity expression is defined as the maximum value of the average mutual information over all possible source probabilities.

Now, this might appear a bit abstract, but will have to live with whatever description, we have here because we cannot spend much time into the details of how the steps have arrived. So, anyway well still try to explain. So, the value of I x y; that means, the average mutual information so effectively, what we are trying to do is we are observing a particular outcome and we are trying to find some kind of correlation with what has been said. So, if we know that the probability of observing 3 d is most when 3 d has been transmitted. So, that in turn would intuitively mean that the mutual information between 3 d and 3 d is should be very high, right.

So, as noise increases, as there is more noise, there is more probability of 3 d being observed because of other things other source symbols then the mutual information with other source symbols could increase. right. So, that is what it is trying to trying to convey.

So, the value of mutual information is maximized over set of possible input probabilities. So, what it says is that since you know the transition probability matrix that is P of y given x, right. So, that is P of y given x that is what we have. So, we would like to maximize the mutual information, we would like to maximize the correlation in some form would like to maximize the probability that what of detecting the right symbol from whatever we have observed and you can do it by altering the probability of the transmitted symbols; that means, if I know the transition probabilities; that means, if I know that given 3 d, there is a certain probability that it goes to d or minus d minus 3 d, then probably I could play around with the probability of transmission of 3 d and minus d and d and so on and so forth, right.

A quick example would be that suppose I have 4 possible source symbols and I have 4 possible the destination symbols.

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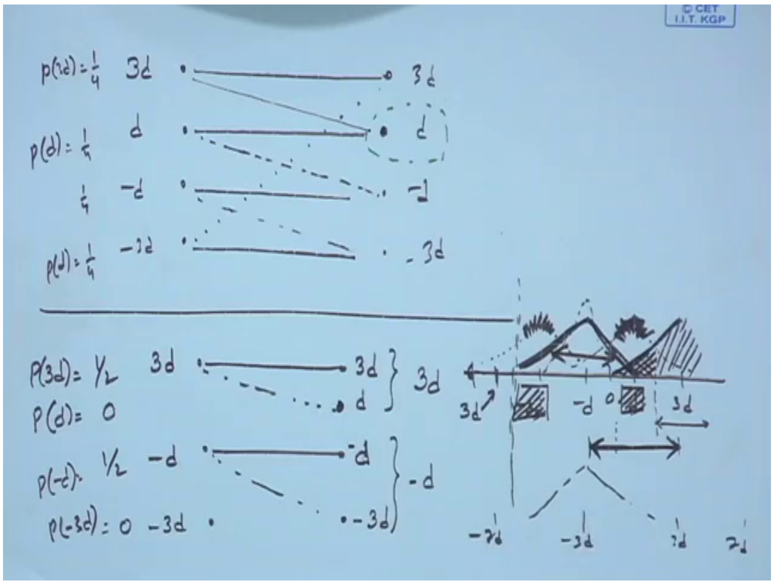

So, what we try to do is suppose, this symbol becomes there is; so, let this be 3 d d minus d minus 3 d; 3 d d minus d minus 3 d. So, there is a certain probability of it remaining the same there is certain probabilities that becomes T and suppose the channel in the whole communication system is such that it does not become minus d and minus 3 d in a similar manner.

Suppose, we take d, there is a probability of it becoming d and there is a probability of it becoming minus d, but it does not become minus 3 d or it does not become 3 d. Similarly for this and similarly for this; now for this there is a certain probability that it becomes 3 d, right, let situations be like this.

So, in this case, we could roughly speaking do something like this. So, we would say that suppose. So, now, if all are equiprobable, we started with equiprobable discussion at some point, if all are equiprobable, then if I send d 3 d and I get 3 d, it is a correct decision, it is the correct situation, if I send 3 d and I get d, we are getting into error same discussion follows for d to d and d to minus d and so on and so forth.

So, but; however, looking at the symmetricity of this particular constellation, we can probably say. So, in one situation, we have probability of 3 d getting transmitted is 1 by 4, this also probability of sending d is 1 by 4 and probability of d is let us say 1 by 4, all of them are 1 by suppose this situation. So, it says that we want to maximize the mutual information on all the source probabilities.

So, one possible solution could be that we instead of choosing all of them, if I choose to have 3 d being sent with the probability of half and minus d being sent with the probability of half, let us see what do we get. So, when I sent 3 d, it can become 3 d or with a certain probability, it can become d, right, there is no d getting transmitted. So, d becoming d is not a problem, see what happens. If so, now, if I receive d, suppose I have received d, suppose I have received d. So, what is the detector going to do if it receives d the detector is confused whether the d has come because of 3 d or the; it has come because of d.

So, if I do not send d; whenever the receiver receives the d, it knows that d has never been transmitted because probability of transmission of d is 0, I have modified this in the new domain with the new probabilities, I have selected a probability where I do not send d at all. So, being 0 means I do not send d probability of minus d is equal to half probability of minus 3 d is again 0, right.

So, now what you see is that 3 d sent will be 3 d or d. So, if I receive d, I am sure that 3 d has been sent. So, when it receives minus d, it is because minus d has been sent because d could also become minus d. Now since d is not send will always be sure that d minus d is because of this.

Similarly, minus d could become minus 3 d. So, when I receive minus 3 d, again it is not confused because it knows that minus d and minus 3 dare produced only by minus d and these 2 are produced only by 3 d, right. So, by assigning these probabilities, you avoid any confusion and you would maximize the mutual information so; that means, you would like to maximize the mutual information, over all possible source probabilities. So, it means you could choose any other combination, but if we choose this as one of the combinations we avoid any kind of confusion at the receiver, right. So, this is all that it means.

So, again in other words, you can say that it makes probability of making error almost 0, right. So, this is what it means that the value of i x y is maximized over the set of input symbol probabilities P x y. So, you choose these things in other words, you are choosing the constellation. So, look at the connection. So, in this way, you are saying that which possible symbols to be send which cannot be send. Now this is because of the transition probabilities, right. So, how it translates is like this that; suppose this is 0, this d; this is minus d this is 3 d this is minus 3 d, right.

So, suppose it is wrapped around wrapped around means here again you have, let us say 3 d, this wrapped around; suppose it is wrapped around. So, the conditional probabilities; how will they look like? The conditional probabilities would be like this, right; the conditional probabilities would look like this. So, sorry, this will be 3 d if it is wrapped around, right. So, this is this map to situation like this. So, when a 3 d is sent, the probability of it being detected as 3 d is area under the curve in this region, right. So, area under the curve in this region; so, I am going to get the probability of 3 d being received as 3 d and the area under the curve here would be the probability of 3 d being sent, but being detected as d and suppose it goes to 0 right it does not change. So, this is the probability that it becomes d, right.

So, now you can easily guess calculate or estimate even visually; what is the probability of error in these situations, right. So, now, going by this result all it says is do not choose this as a transmitting symbol, right and do not choose this as a transmitting symbol. So, if that is the case, our decision point would be somewhere here and what we have is this completely going away this is not present, right. So, if only 3 d and minus d are available; in other words, we are increasing the minimum distance between the constellation points. So, if we are increasing the minimum distance between the constellation points; we are improving the error probability. So, in some sense, it is relating to what we have discussed earlier.

So, again when minus d is sent since 3 d is not sent over here the probability of this becoming minus 3 d minus 3 d is very less and probability of it becoming 3 d is also very less because now you have if the matrix going the conditional probability matrix going out like this. So, this is gone and this is gone. So, you are left with this and this only, right.

So, since the constellations are sufficiently spaced apart, the probability of errors have been reduced by appropriate choice of constellations, we have reduced the possible constellations. Now this is because of the transition probabilities; and remember the transition probabilities are because of the variance of noise if the variance of noise would have been different right suppose the PDF is narrow the transition probabilities would be different and our result would have been different, right.

Again in the other hand, if you would have selected the constellation as let us say minus 3 d 3 d and maybe minus 7 d and 7 d somewhere there then possibly, we would have had a situation where the PDF dies out and we could have existed with all the 4 constellation points.

So, in other words, it is determined by the transition probabilities. So, given a set of transition probabilities, you are choosing the possible probabilities of source symbols, right; that translates to given a kind of noise power spectral density or given a amount of noise that is present in the system compared to the received signal strength that is the minimum distance. We are choosing the number of bits that we should send per symbol; that means, the constellation order. So, this is how you could somehow relate to whatever this result is ISIS giving us and what we have studied earlier in terms of constellations and error probabilities, right.

So, getting ahead; so, we have the capacity of the channel defined by the expression see which is the maximum value of i x y over all possible source probabilities. So, you have the mutual information expression, again it in terms of relative entropy which we are not discussing averaged over all possible x y combination, right.

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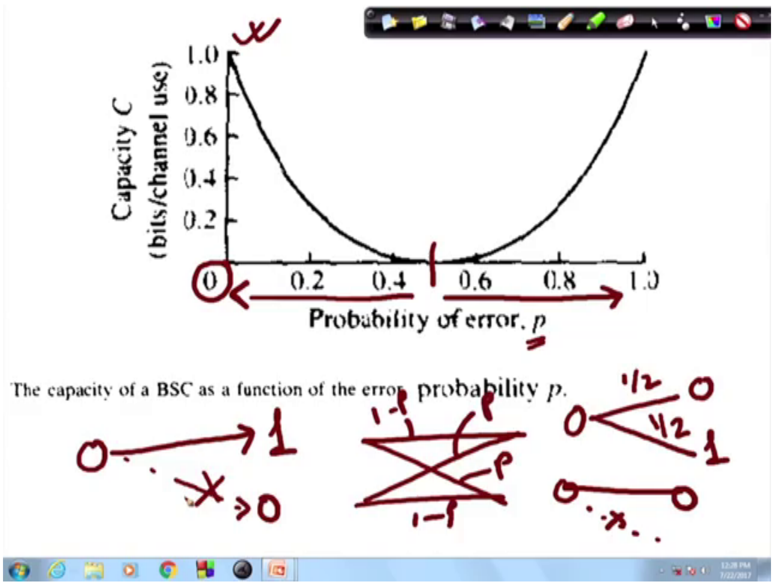

So, with this, we move forward and see, what does it lead to and if you evaluate this expression for a binary symmetric channel. So, for binary symmetric channel you have 2 values of y only this y takes 2 values. Let us say 0 and 1 or plus 1 and minus plus A and minus A; let us say x also takes a plus A and minus A; let d or minus d. So, you can easily compute this because you know the transition probability matrix and if it is a binary symmetric channel; the P; the probability of error can be computed given the noise power spectral density. So, you are moving from the discrete input continuous output channel to a discrete memory less channel. So, we can do these translations and we can compute these values.

So, once you compute this, the N result that you would get is that the channel capacity would turn out to be minimum for the case where your P is half; that means, you have the channel capacity for P, remember, it is the transition probabilities. So, probability of making an error is P probability of making an error is P probability being correct is 1 minus P, it is 1 minus P, right so; that means, A 0 remaining A 0 with 50 percent probability and A 0 becoming A 1 with 50 percent probability, right so; that means, when I receive A 0, there is 50-50 chance that A 0 was sent, right.

So, in that case it is equiprobable whether 0 was sent or one was sent. So, I would make no special decision; no special outcome if I choose A 0 or A 1, right. So, that is the worst possible scenario. Now if I know that 0 being sent and the probability of it becoming 1 is let say 10 percent only, the 90 percent of the time you can send signals successfully. So, that is what is conveyed over here. So, at a transition probability value of 0 capacity is minimum, you hardly make sense out of what is going on in the channel, right.

Whereas if your probability of error is less as the probability of error becomes 0, it tends to A 0; that means, 0 remains 0, it does not become 1. So, there is hardly any probability of error with channel capacity is maximum right true because there is no error you achieve the best.

Counter intuitively; if the probability of error increases towards 1; that means, whenever I sent A 0, it becomes 1 for sure and it never becomes 0 it never becomes 0 so; that means, the bit is flipped simply the bit is flipped, we are assured that the bit is flipped. So, 1 received means A 0 was sent and A 0 received means A 1 was sent, then again you achieve maximum value of channel capacity. So, that is what is displayed by this particular figure that we have over here alright.

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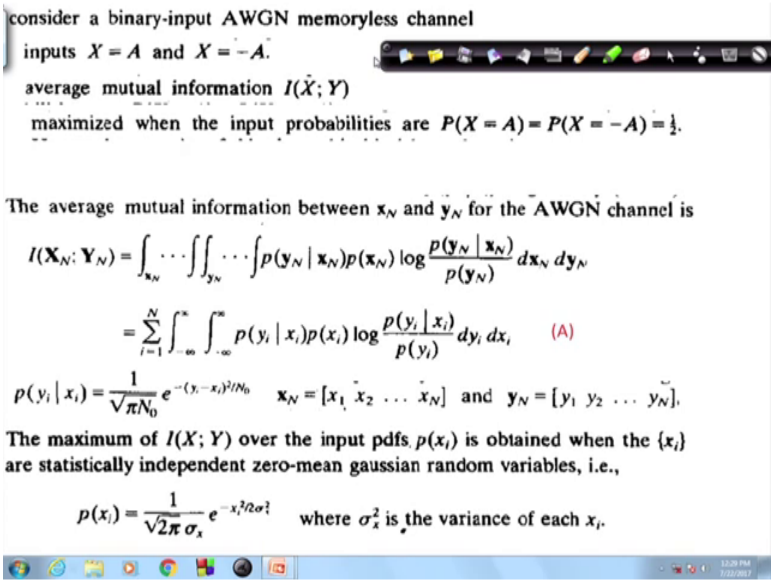

Let us quickly; let us look at the binary input AWGN channel. So, if we have the binary input AWGN channel input is A and minus A and you can calculate. So, let us just use this result that the mutual information is maximized for P the probability of selecting the source symbols being half. So, we are not saying P is equal to half mind it P is equal to half is not what we are saying over here. We are saying that probability of selecting the source symbols is half these are 2 very very different things and average mutual information between the out input vector and the output vector in an AWGN.

So, we have waveform channel, right. So, then we have this conditional probability matrix and where these conditional probabilities would be given by an expression like this. So, if you fill it in these expressions we can take few steps forward and we will be using these expressions, sorry, we will be using these expressions for conditional probabilities. So, if we carry on with a few steps; that mean, we have to solve this expression, previously there was no integration, there was summation because there were all discrete, but now since we have AWGN channel. So, waveform output continuous output will be using these densities in our expression. So, evaluation is little bit different, but the expression in philosophy remains the same. So, when you solve this expression using this transition probability using this.

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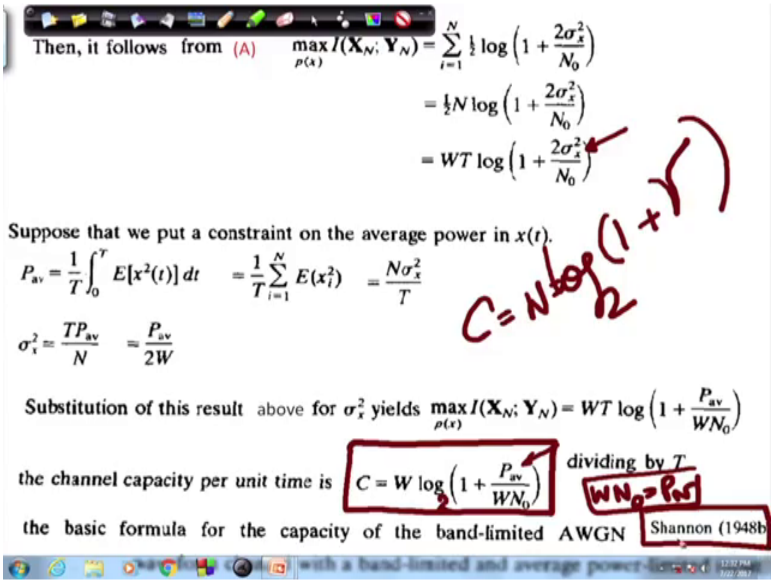

This conditional probabilities what you end up with is an expression which I will not go through a few steps, but finally, you would work out it as the channel capacity C is denoted by W which is the bandwidth of the channel T which is the symbol duration log this is based to 1 plus 2 times sigma x squared is a variance of transmitter times N naught, right and sigma x squared that is the transmit variance one could calculate as one upon T 0 to T expected value of x squared T.

So, this will be one upon T x summation over x i, x i at the constellation points; you can say. So, this would turn out to be if there are N such possible constellation points this is the result and you could summarise it with the average power by 2 W, right because you have 1 upon T as W. So, you could use this result. So, if you fit this result into sigma x squared what you will find that will end up with the expression of channel capacity which is given by this C is equal to W log base 2 1 plus average transmit power by W N naught.

Now, W N naught is the noise power P N because N naught is a power spectral density this is W in hertz meaning you have the total noise power. So, what do you have is W log base 2 1 plus signal power to noise power. So, in other words, sometime we will write capacity is equal to W is the bandwidth in hertz log base 2 1 plus SNR and this is in linear scale. So, finally, if you do a few steps and this is the famous result which Shannon gave us. So, well use this as a benchmark.

So, what it tells what we have not discussed is if we are given a certain transmit power, we know that power spectral density given the bandwidth we can calculate the noise power as N naught multiplied by W is a total noise power. So, average signal power by average noise power is the signal to noise ratio last 1 log base 2 of that is the bits per second so; that means, C upon W C has a unit of bits C upon W is equal to log base 2 1 plus gamma; that means, bits per second C is bits per second per hertz C is bits per second or bits per channel use sometimes you also call it bits per channel use.

So, given a particular bandwidth W, you could use this expression. This particular expression to calculate the number of bits that could be sent per second per hertz, this is known as the spectral efficiency of the channel right channel. So, with this one is equipped. So, now, using this expression we could study the impact on channel capacity given a particular bandwidth W. So, what we have over here is suppose you look at the expression C upon W is equal to log base 2 1 plus gamma; gamma is the x axis, right, this is given in dB. So, it tells you as you keep on increasing signal to noise ratio if right what happens your capacity your spectral efficiency increases.

So, how do we connect? So, as we increase the signal power to noise power right we can increase the number of bits per symbol for PAM and QAM, right. So, this will tell us the expression that we have seen the maximum number of bits per second per hertz that you can send with almost negligible amount of error, right. So, this is a philosophical interpretation of course, there is a detailed derivation as I told you it requires a huge amount of time for that. So, we will just use this description.

So, if we put in the value of transmit power and noise power into this expression, we are going to get the number of bits per second, per hertz or in other words we can calculate the number of bits per symbol that we can compare with the QAM or PAM modulations in order to see whether we are doing a good job or a poor job.

The next important result that we have with us is the situation when we keep on increasing the W.

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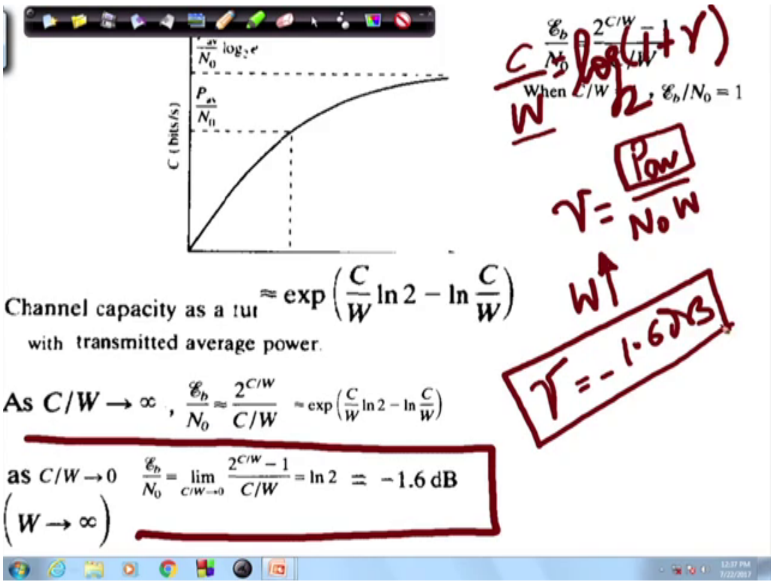

So, if we check this particular expression that C upon W is equal to log base 2 1 plus gamma, we stated that if we keep on increasing this gamma we keep on increasing C, if we keep W constant, right, if I keep W constant. So, this is related to when we studied PAM and QAM, we said that if you keep time duration constant; that means, symbol duration W is constant that is the bandwidth is constant which is one upon T. So, if you increase the transmit power you can increase number of bits per symbol and this expression is going to give us the upper limit do it, right.

Now, let us look at what would happen if we say that let P average transmit power by N naught W is the expression for gamma. So, if I say; I want to keep this constant write; that means, I am keeping P average constant, but I am increasing W, right indefinitely then what happens, right. So, what we will see is I will just show you the end result which we may do a little bit details in the next discussion, but briefly over here that if I calculate the limit W tends to 0 what I will end up with is a value of SNR which is minus 1.6 dB, right, I will do the derivation in the in the next class.

So, what we see here is that from this expression also; we are getting to the point the limit of signal to noise ratio that must be maintained to do a meaningful communication where W may be expanded to infinity while power average P average maybe kept constant. This we would remind us about the orthogonal signals where we said that if you would keep increasing k the number of bits; that means, you will keep increasing m;

that means, you keeping increasing the bandwidth what is the limit on signal to noise ratio that you reach and we did do the calculation earlier and we found that you could reach a value of minus 1.6 dB. So, let us look at this result in the next discussion.

Thank you.