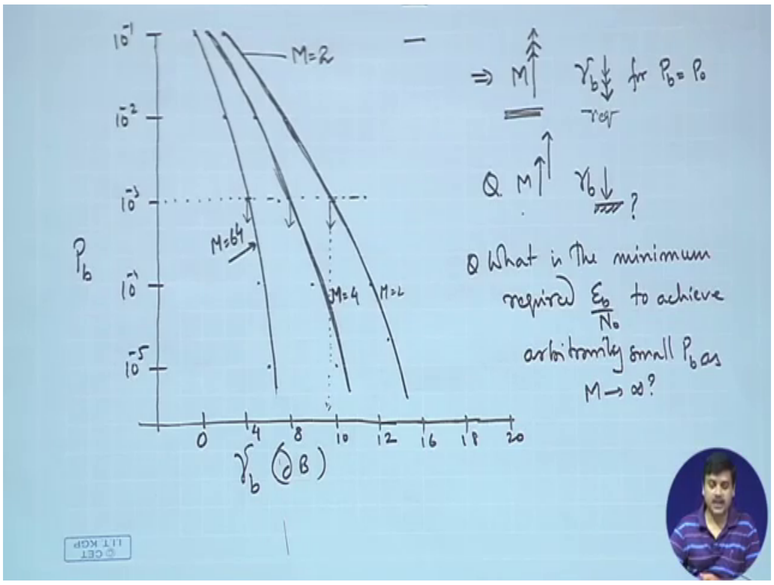
Modern Digital Communication Techniques
Prof. Suvra Sekhar Das
G. S. Sanyal School of Telecommunication
Indian Institute of Technology, Kharagpur

Lecture – 49
Performance of Digital Modulation Techniques (Contd.)

Welcome to the lectures on modern digital communication techniques. So, in the previous lecture, we were discussing about the performance of orthogonal signals, M-ary orthogonal signals and what we found is an interesting result in which is in contrast to the situation of M-ary PAM is that as we increase the constellation size we found that the required E b by N naught decreases or we could say that given a given a particular require yeah given a particular P b the E b by N naught decreases or given a particular E b by N naught the P b decreases.

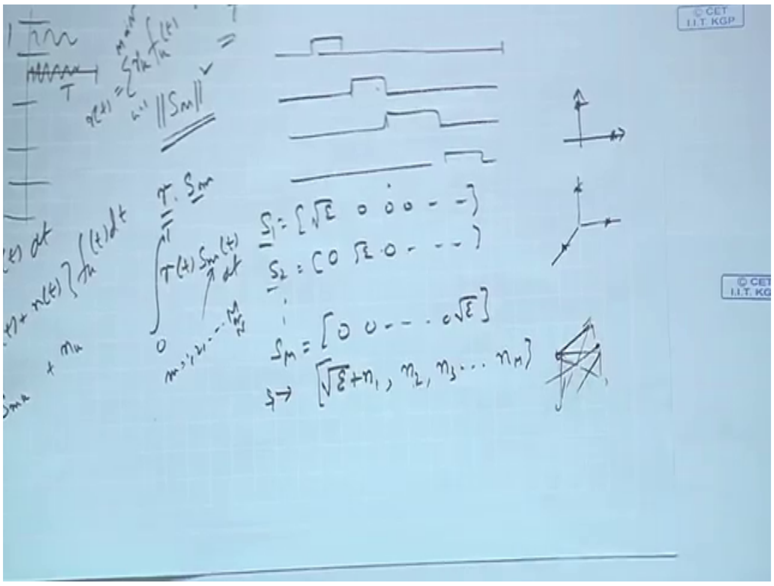
(Refer Slide Time: 01:02)



Which can be visible from this particular figure that we have drawn is that as we increase the constellation size if we keep the constant P b and we want to increase M from M equals to 2 to 4 to 64; the required E b by N naught decreases, right, on the other hand, if you have to be with a different way that I want to keep the constant E b by N naught and I keep increasing M from M equals to 2 to M equals to 4 to M equals to 64; what we find is that P b keeps on decreasing. So, the question that remains with us is; if we keep on increasing M indefinitely then what is the impact on E b by N naught; that means, on this side or what is the impact on probability of array.

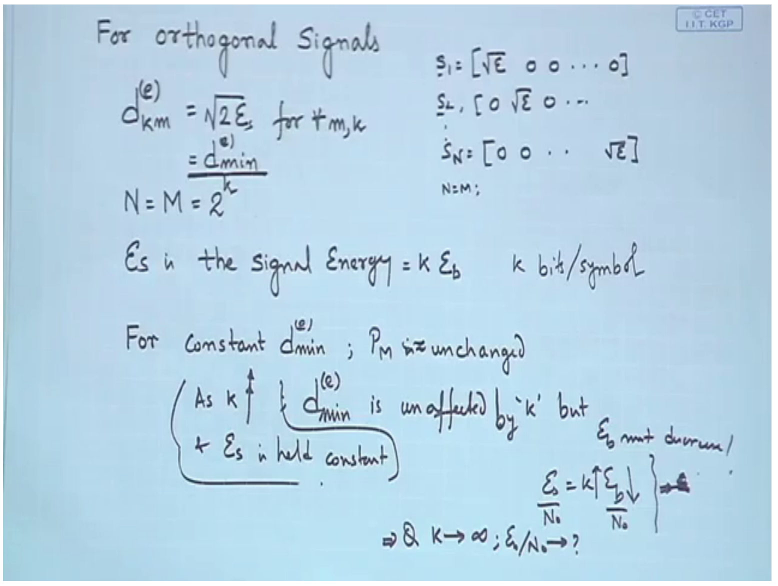
So, technically we could say that as we keep increasing M we could in the limit almost make P b vanish, right. So, if that happens, we should also be able to travel on this direction. So, what is the E b by N naught that we reach is the other important question to partially answer that what we could do is let us write down for orthogonal signals, the signal structure as we have here. So, for this signal structure, we know that the distance between any 2 constellations the Euclidean distance is root over 2 E; that is it is the constant distance and this is what we have been trying to explain in the previous lecture that all the constellations are equidistant from each other.

(Refer Slide Time: 02:50)



So, for 2 dimensions, you can easily visualize this; that means, for 2 dimensions you could easily visualize this is 1 constellation point, this is another constellation points in 3 dimension, if this is x, y and z plane, then possibly we could have it this way, but beyond that it is left to our imagination. So, if we imagine them to be some orthogonal dimensions, a particular signal is equi distance from all others; if you take any others signal that is equi equidistant from all others; that means, the distance between the constellations is the same; that means, that is the minimum distance of the constellation that this point, we would like to recall the discussion we had in the lecture, before the previous one is be explained that why the minimum distance of constellation place a very important role in or place dominant role in the error probability expressions, right.

(Refer Slide Time: 03:31)



So, taking that and we did write the expression in terms of de mean square that is the distance between the 2 constellation points. So, minimum distance; so, here what we find is that since that distance between the constellation points remain the same. Therefore, the probability of error should not be affected that is the premise. So, for K increasing; as we increase K, as we increase M d remains constant, right. This is the basic thing that we have whereas, the E s is the signal energy that is K times E b because it is the signal energy there are K bits per symbol and E b is the energy per bit.

So, if there are K bits; K multiplied energy per bit is the energy per symbol where there are K bits per symbol where there are K bits per symbol. So, now, something could be a bit clear in the sense that as we increase K, we are actually increasing Es. So, if you are increasing so; that means, if we keep E b constant as we increase K we are actually increasing Es. So, if E s increases that would mean d increases.

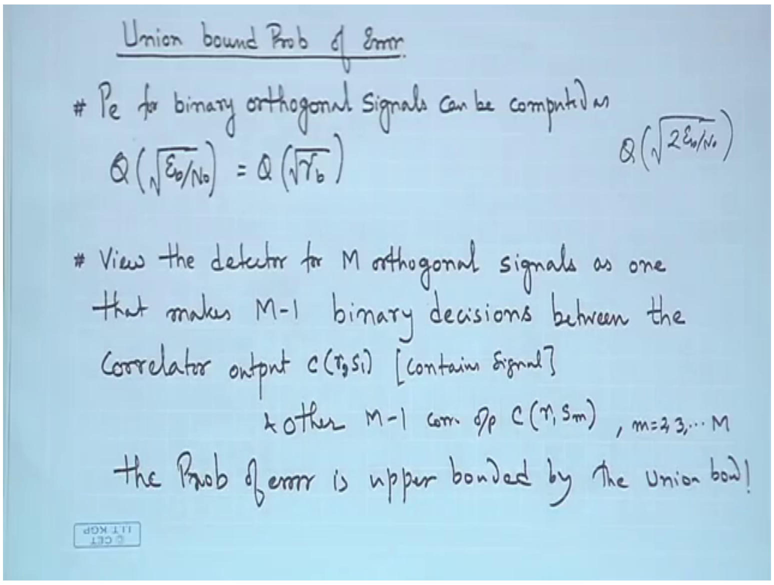
So, d increases; that means, probability of error decreases so that is a straight forward think that we have. So, if I am going to increase K by keeping E b constant; that means, I am increasing the total energy because E s is equal to K times E b right E s increases what we have is DKM is root over 2 E s. So, this increases the distance increases, if distance increases probability of error decreases or you get a better probability of error result that is what is clear visible over here right for a constant E b as we increase M; that means, as we increased K the probability of error decreases simply because distance between constellations increase.

Well, the same discussion can be argued in a slightly variance variant way; almost meaning the same thing. So, for the constant d min P M is unchanged we know. So, as K is increased right bits are increase. So, going by this if K increases while this is kept constant; that means, E b must decrease, right. So, in other words, it is saying that if I am increasing the constellation size while keeping while keeping the signal energy constant; that means, while keeping d constant; that means, while keeping unchanged P b then your E b requirement decreases. So, that is the other view that we have. So, if we have E s as constant; that means, sorry if we have distance is constant because if E is constant d is constant right if E is constant d is constant, it could mean that probability of error is unchanged; that means, you are talking of this line. So, if that happens as K increases if this is constant then this must decrease.

So, as K increases; that means, as M increases from M equals to 2 to M equals to 4 to M equals to 64 K equals to 1 K equals to 2 K equals to 6, what we clearly find is that as we keep this constant, we traverse this line because P b constant d constant E s constant. So, K increases along this direction along this direction. So, therefore, what we have is E b decreasing this is E b by N naught N naught is constant.

So, this is constant. So, gamma b decreasing E b decreasing means gamma b decreasing. So, both the things could be explained by looking at the expression and are going in terms of the minimum distance of the constellation points from each other. So, this is a qualitatively argued, but we are still left to find out that what would be the minimum E b by N naught that we require to proceed on that one has to; one could resort to the union bound on probability of error. So, that is what we are going to look at in this part. So, what we could do is we could calculate the probability of error for the binary orthogonal signals, right.

(Refer Slide Time: 08:30)

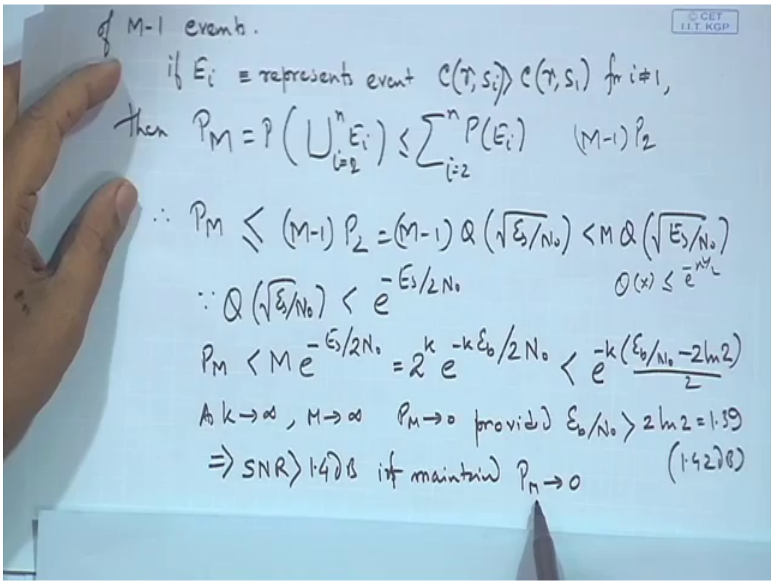


So, we did not give the result that is could be used by what we have done or it is not very difficult its basically using the correlations that is c of r S 1 and c of r s two. So, if S 1 is sent what is the probability that outcome C r s 2 is greater than probability outcome of a correlator C r S 1 that is what we have done and for 2 it is possible for more it becomes of bit cumbersome and it is equal for both the situations. So, the result that you obtain if you do that is Q function of root over E b by N naught to just a reminder for binary PAM it is Q of root over 2 E b by N naught just for your comparison say there after things change .

Now, we could view the detector for M orthogonal signals as that which makes M minus one binary decisions between the correlator output C r S 1 and the other M minus 1. So, what you mean to say we select C r the output of the correlator of C r with S 1, compare to that with C r of S 2 that is one comparison we could again compare C r S 1 with another we could again compare C r S 1 with another and so on and so forth. So, we could compare with all of these correlator outcomes then the probability of error is upper bounded by the union bound.

So, what it means is that union bound of the M minus one events, right.

(Refer Slide Time: 10:10)



So, if E i represents the event that the correlator output of r with S i is greater than correlator output of r with S 1 for i not equal to 1; that means, correlator output S 2 is greater than this S 3 is greater than this S 4. So, any particular is the event E i. So, C of r S 1; that means, if fifth correlator output is greater than the first correlator output; that is the event if E I and so on and so forth.

Then probability of error is the probability of the union of all these events, right that is true because it is a orthogonal signal. So, you have to take the probability that the second is greater than the first, the third fourth and fifth and so on. So, you add up all these probabilities. So, this probability; so, we calculate the probability that the union of this events and this is bounded by. So, this is less than or equal to some of these probabilities right. So, this is this result can be derived. So, this is limited by the some of these probabilities right. So, of all these events probability of all these events individual events there is the worst case scenario, right. So, the probability of error is M minus 1. Now since all these probabilities are equi probable and there are M minus 1 events. So, P M minus 1 there is M minus 1 because there all equi probable.

So, this in our case would be given as M minus 1, P 2 indicating come binary comparison between any 2 and that probability of error; we have here given over here so; that means, M minus 1 times P 2 because there are M minus 1 such comparisons is the upper bound, there is probability of error is bounded by that M minus 1 Q function of whatever we have here is what we have use; their which is which can be further said to be less than M times because we are ignoring 1 for large values of M you may keep 1, but this helps us in finding some bounds, right .

So, now, what we do is we use the Chernoff bound for a Q function. So, Q of root over E s by N naught is less than or equal to E to the power of minus. So, basically Q of x, we use this is less than or equal to E to the power of minus x squared by 2. So, that is E s by 2 N naught, right that is what we have from the Chernoff bound. So, then using this we simply replace over here. So, M E to the power of minus E s by 2, so, we replace this expression in this. So, M is 2 to the power of K E to the power of E s is K times E b 2 N naught right, E s is K times E b because it is a signal energy, it is the bit energy multiplied by K bits per symbol is signal energy. So, what we have is this we can say is further; we can; we have 2 to the power of K. So, we convert this 2 E to the power of and this is the expansion that we have. So, we have this particular expansion in the numerator. So, from this what we can see is as K tends to infinity as K becomes very very large; that means, M becomes very very large; that means, we are in the curve.

We are going in this direction as M is increasing, right. So, as K becomes very very large P M tends to 0. So, again going back to this figure, so, you could also view it as M from 2 4 M probability of error goes to 0 that is what we have. So, we can see from here also, right.

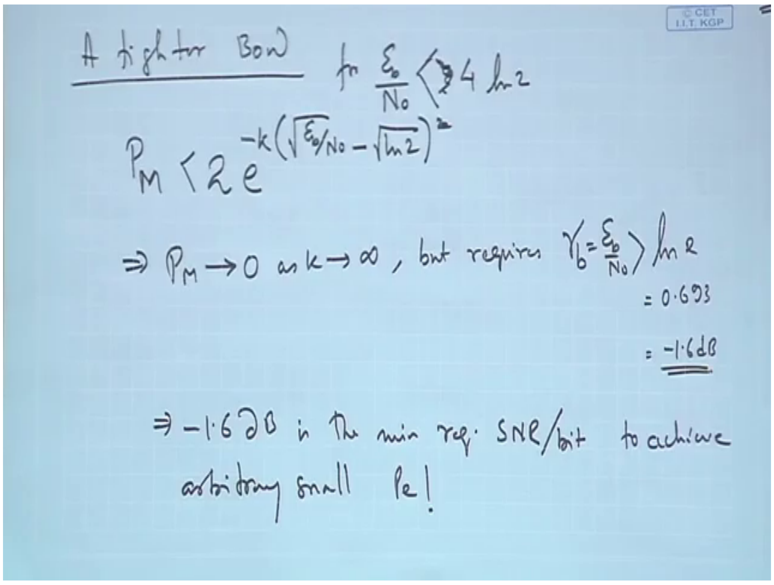
So, from this union bound, now you could visualize what we explained intuitively few minutes ago so, but the condition is E b by N naught should be greater than this number if it is not. So, what do we have? We have a negative number in side and negative number along with this would yield a positive number; that means, the probability of error will start increasing.

So, the probability of error will keep on decreasing subject to the condition that E b by N naught is greater than this which is equal to 1.39 or roughly 1.42 db. So, this tells that as long as SNR is greater than 1.4 db, you can try to make P M vanishingly small and how will you do it; you will simply increase K you keep increasing K over here and then you could make arbitrarily small that is you could make the probability of become of error almost becomes 0; that means, you could establish a very very reliable channel without much complexity, but the question is; what is the penalty paid for achieving this particular result. So, as you increase K what do we have as we increase K we keep on increasing the number of frequency bands.

So, as we keep on increasing the number of frequency bands we are tending towards using very very large amplitude. So, if K tends sorry bandwidth as K tends to infinity; you are tending towards infinite bandwidth. So, what we have in this case is the probability of error is traded off with the bandwidth; that means, if I could produce huge amount of bandwidth even the same constant transmit power I could achieve significantly low probability of error. Now if you contrast this to the PAM transmissions, what you would find is given a finite bandwidth; that means, t s symbol duration because bandwidth this universal proportional to t s 1 upon t s is equal to 1 upon t s, then if I have to reduce the probability of error I have to keep increasing the power in definitely.

So, there probability of error or bit rate is exchange with power, here the same is exchange with bandwidth, right. So, that is what we. So, we have a bandwidth error probability trade off in this case, right; however, one could guess that the bandwidth efficiency would be very less right. So, continuing on this there is title bound then what we had just discussed. So, going in to the details of the title bound is a bit constant at this particular point of time.

(Refer Slide Time: 16:45)



So, will use the result simply over here, if it permits then we can provide you with the necessary steps at a later time, but I doubt that it is possible given the fix number of lecture that is possible over here still we will try to provide you with additional material which you may go through.

So, a title bound is for E b by N naught less than 4 l n by 2. So, as of now let us take that somebody is derived this. So, for lower SNR regime I mean this approximation that we have used is not a very good approximation in the low SNR regime. This approximation is good for larger values of E b by N naught. So, for low SNR region the probability of error can be upper bounded by this expression where this condition is true only when E b by N naught is less than l n; l n of is sorry is greater than l n of 2. So, again I am stating that this result, we are using from some other derivation using a title bound of the expansion. So, this tells us that as long as E b by N naught is greater than log natural logarithm of 2 which is 0.693; that means, greater than minus 1.6 dB you can keep increasing K.

That means you could make probability of error go to 0 as K increases in definitely, but you have to maintain gamma b greater than minus 1.6 dB. So, what we have is that minus 1.6 dB is the minimum required SNR per bit to achieve and arbitrarily small probability of error and remember this is achieved by making bandwidth go towards infinity.

So, with this, we have analyzed the performance of a few the digital modulation techniques in AWGN channel, but since we have done them separately, but although we have taken insides at appropriate stages of time where we have seen how probability of error are bit rate a traded off with power and bandwidth in individual case. It is very important that we come compare all of these in one single discussion or in a common platform.

So, to discuss these things in a common platform, we require to understand or at least use if not completely understand something some important terminology in the domain of digital communications on this channel capacity. So, we will discuss the channel capacity at least somewhat quantitatively, but I do not know whether the time that we have will be able to do justice to that in this particular course; however, this the particular discussion on channel capacity which we have going to; which is going to taken in the next lecture will be mainly on information bases which will not be regular part of evaluation; however, will use the expression of channel capacity that we finally, use that we finally, get for evaluation at the end of this particular course, but at least; we should keep the notion of channel capacity; what is the meaning and why should we use it and we will use that as a benchmark for comparing the performance of digital modulation techniques that we have seen so far in the upcoming lectures.

Thank you.