Modern Digital Communication Techniques
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Lecture - 48
Performance of Digital Modulation Techniques (Contd.)

Welcome to the lectures on modern digital communication techniques. So, after discussing about transmission techniques and the receivers where we could detect the signals, we are now discussing the performance of digital communication systems or digital modulation techniques and we are evaluating the performance by means of error probability. And we have explained on several occasions why this error probability is significant and just to remind the importance comes because the receiver has to make a decision between a few options. So, the receiver cannot make any choice which is lying outside the setup options.

Accordingly the decision made by the receiver could be the exact signal or the exact waveform that was sent or it could be some other waveform and hence we have the issue of whether there is a mistake in detection or whether it is a correct detection. And accordingly you would like to calculate the chance of making an error the less the chance of making an error the better is the performance of the communication system. We have studied the error probability of binary pulse amplitude modulation, and we have extended the concept to the study of M-ary pulse amplitude modulation, and what we have found is that as we increase the order of modulation; that means, as we increase the size of the constellation, we are getting decrease in the performance in the sense that for a given finite amount of energy or average power of the signal, as we increase the M that is the order of constellation we are getting a decrease in sorry an increase in error probability.

So, what we have also seen there is a tradeoff between the bit rate and error probability in case of modulation suggests pulse amplitude modulation. What we have seen is if we fix the symbol duration that is t and increase the k which is bits per symbol, then effectively the number of bits send per symbol duration that is per unit time keeps on increasing, that is k upon t s which is bits per second keeps on increasing. Whereas, for the same situation if we keep the average energy constant; that means, you are not spending extra power compared to the earlier situation where k was a lower value; that means, you have increased k, but kept the average transmit power the same, in that case what we found is that probability of error increases.

So, there is an increase in bit rate as well as increase in error probability. Now if we have to maintain a certain quality of service; that means, if you have to maintain error probability below certain threshold, then as we increase the number of bits per symbol duration; that means, as we increase the modulation order we should also increase the transmit power in order to maintain the same probability of error. So, that was visible in a in a curve like this.

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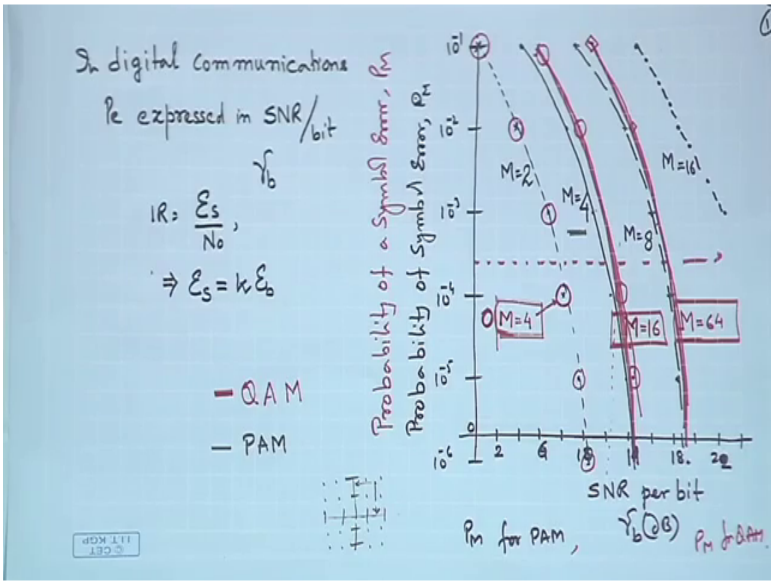

So, in this particular picture we not only had PAM, but we also had QAM and we said the performances very very similar simply because QAM can be constructed by 2 p a ms. So, what we find is M equals to 4 and M equals to 16 M equals to 4 for p a m has similar performance as M equals to 16 for QAM.

The reason being 16 QAM can be constructed by 4 p a m. So, if this is 4 PAM you have another 4 PAM and we had reminded you that this could be used to construct a 16 QAM and whenever a signal is received suppose a signal is received here it is component is taken on the i PAM as well as it is component is taken on the q pam. So, once it is component is taken on the i PAM, then you can find the appropriate PAM signal on the i channel and you can also do the same on the q channel and accordingly we derived the error probability for QAM.

So, after having discussed this particular modulation scheme the performance of a PAM or QAM which is multi amplitude signal, we would like to look at the next important digital modulation technique which we have studied are the ones which is orthogonal signals.

Now, what we have studied for orthogonal signals if you would recall, that one way of creating such a signal was.

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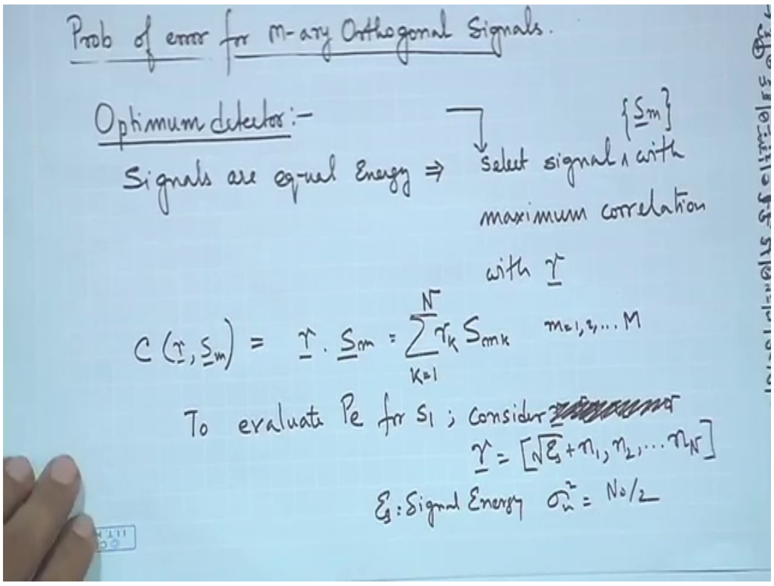

If you have the bandwidth you divide it into orthogonal sections where each sub bandwidth would be of size delta f and if the symbol duration was of period T, then if you would maintain delta f is equal to 1 upon T in that case these frequencies would be orthogonal to each other that is what we have found. So, by this we could create n dimensional orthogonal signals and whenever we are discussing the performance of orthogonal signaling, one could easily consider this as a possible realization in trying to connect to the outcomes that we get.

So, moving further we are now interested in calculating the probability of error for M-ary orthogonal signals. So, in study of the M-ary orthogonal signals we will take a look at the optimum detector.

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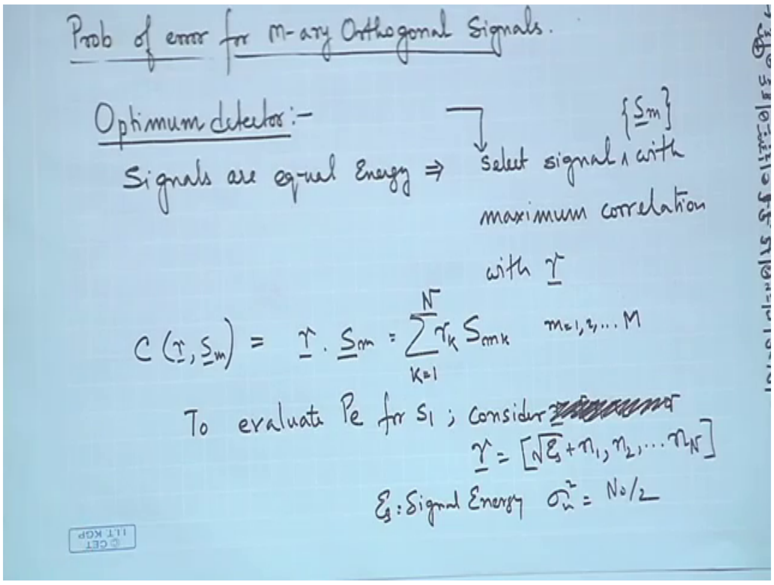

So, we have discussed what is meant by the optimum detector and now it is realized. So, there are several realizations and one of the realization as we saw while expanding the expression. So, just to remind we started off with the posterior probabilities and what we found is that it could be expanded using bayes rule where the denominator term, did not influence the decision and in the numerator term there was the multiplication of the likelihood function by the prior probabilities.

If prior probabilities are or equal then it turned out to be the maximum likelihood detector, in such a case we found the way to find the possible signal was the minimum distance detector. Then we expanded this particular metric and what we found it is it expands into the expression of the length of the received vector, along with that component of the received vector on the different signals as well as they was the biased term of that particular signals energy. So, when we consider this particular expression we look at it and found that the lengths squared of the received signal is common for all comparisons hence we removed it, and we were left with is the correlation of the received signal with the different signals. And the realization was correlation of the received signal with a particular one of the signal waveforms take away the energy of that particular signal.

So, effectively what we had was the correlation. Now if we have the M-ary orthogonal signals and if you recall this particular structure or even if you recall the situation where there was long time duration T and one could use orthogonal time slots and each time slot had the same energy. So, in this case our signals were as if is a vector and finally, so; that means, they had the same energy, but they had only one component and the rest of them was 0.

So; that means, they are all equal energy and at that point we said that if mod S m squared is same; that means, all of them are same energy then we need not take this term and we could easily concentrate on r dot S m that is a component of r 1 S m or we could take r of t S m of t 0 to t d t there is a correlation of r of S m t and we would select that M this or M because we had use both these things to mean the same. So, whichever value of small M would maximize this correlation was the choice of the signal that is what we said. So, it could be implemented in either of these 2 forms. So, in case of equal energy signals we would select the signal that is S m with the maximum correlation with r that is what we just explain in this particular short discussion.

So, we would compute the correlation matrix the correlation metric and the correlation metric is what we have just stated. So, in the vector form the correlation metric is the product of the kth component of the received vector. So, we are taking the component on the basis dimensions. So, getting back here each of them are basis functions. So, whatever is the received signal we will take a component of each one. So, the received r of t would be broken into r k times f k t k equals one to M or N either thing in this k it is the same in this particular situation.

Same applies in this way or any other constellation which we make sense. So, M is the index for the signals this waveforms and k is the index of the basis function. So, that is what we have. So, now. So, these are generic expression to evaluate the probability of error for S1. So, we are assuming that there is the S 1 that is being sent; that means, you are assuming any one sent. So, it will be the same for all other. So, it is not much of a problem. So, we let us consider with S 1 and this is approach we had taken while discussing the performance of P m signals as well. So, evaluate the probability of error for S 1 let us consider the received vector. So, it will naturally be the signal plus noise in the first dimension because there is signal already present in the first dimension and in the other dimensions there are no signals. So, output of the correlators will be the noise component on the basis function.

So, that is again straight ahead available from this because you are first decomposing r onto the different components. So, you are basically taking r of t f k t integrate 0 to t d t and r t was S m t plus n t this exercise we had done thoroughly before right and hence what you have is S m k integral 0 to t, S m f k and you had n k right that is the component of noise on the basis dimension. So, this is what we had so, that is what we are using. So, for the first dimension there is signal. So, there is the signal component which is root over Es and on the other dimensions is just the component of noise that we will see r vector and of course, need not mention over here Es is the energy of the signal if you squared this term you are going to get energy of the signal and noise power spectral density or the variance of noises N naught by 2 which is consistent with all our previous discussions.

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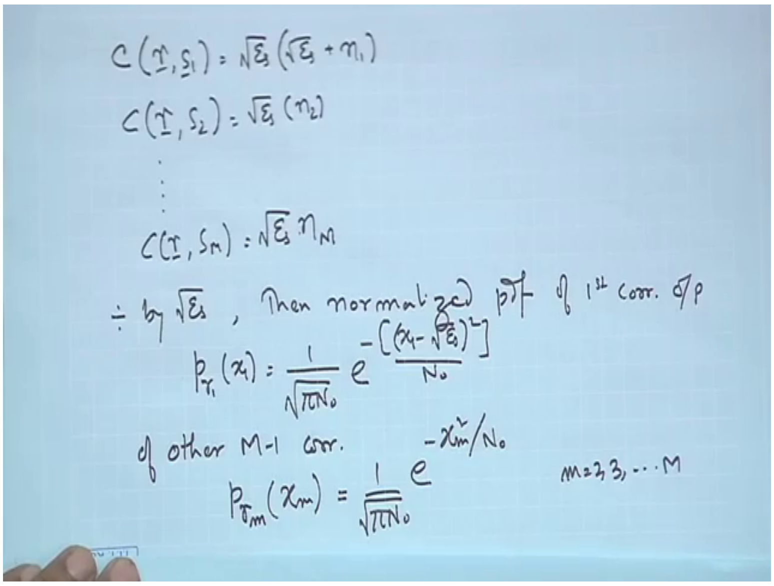

So, now let us take a look at what would be correlation output correlator output of the first correlator. So, if you look at the first correlator, the signal is root over Es right S 1 and this is the r component on the first dimension. So, this is the correlator output, the correlator output of the second correlator is r times S 2. So, r has a component n 2 and signal S 2 is Es on that particular dimension and similarly we are going to have this. So, if you look at this expression all of them has this particular thing common and hence we can divide by root over Es and then we are left with basically these components. Then they normalized p d f of the first correlator output there is what you have over here is given by Pr1; that means, component on the first one let x be that variable 1 by root pi n naught because sigma squared n is n naught by 2, there was a 2 pi sigma squared n 2 and 2 cancels and e to the power of minus x 1 minus root e 0.

E s because this is the mean of the signal and this is the Gaussian distribution with variance of n naught by 2, this we have been using before. For all other correlators what you can easily see that the normalized p d f should have a mean of 0, the normalized output should I have a mean of 0 and the variance will again be n naught by 2 we are ignoring this term because it is common to all. So, we can divide by root over Es then we can eliminate the term. So, we write P r m now we are using the m because for our particular case it is m are same. So, therefore, this is the situation. So, x m is the general symbol this the root 2 pi sigma squared e to the power of minus x m squared by n naught.

Because these do not have a mean this is the basically the 0 mean for all other correlator output. So, now, we have to use this to calculate the probability of error.

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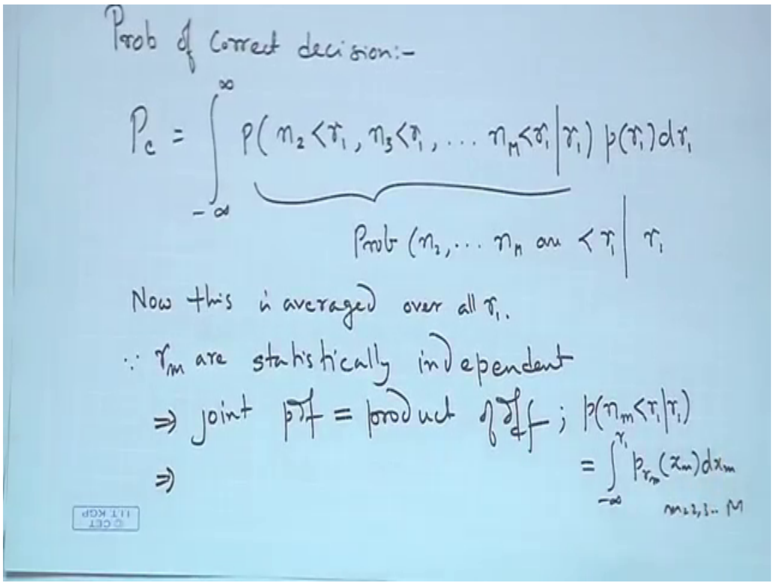

So, in order to calculate the probability of error it is better in this particular situation to go ahead with calculating the probability of correct decision. So, the probability of correct decision is the probability the joint probability that n 2 that is the second component is less than the first component of the received signal, the third is greater than the first component and simultaneously up to all components are less than the first component given r 1, and this has to be averaged over all possible realization of the r 1. So, accordingly the integration range of r 1 minus infinity to plus infinity right.

So, this is averaged over all possible r 1 and now since we have always being using that if these are orthogonal signals. So, we have been using them to be orthogonal let us say. So, if they are orthogonal then if they are orthogonal in that case r ms are statistically independent because they are Gaussian distributed and first one has a mean of root Es all others are mean of 0. So, if they are orthogonal and Gaussian; that means, they are statistically independent. So, statistically independent means that this joint distribution of course, conditional turns out to be a product of the individual or the marginal distributions of course, conditional for all the m terms from 2 to m right. So, where any individual marginal p d f condition marginal p d f is obtained in this fashion as written over here.

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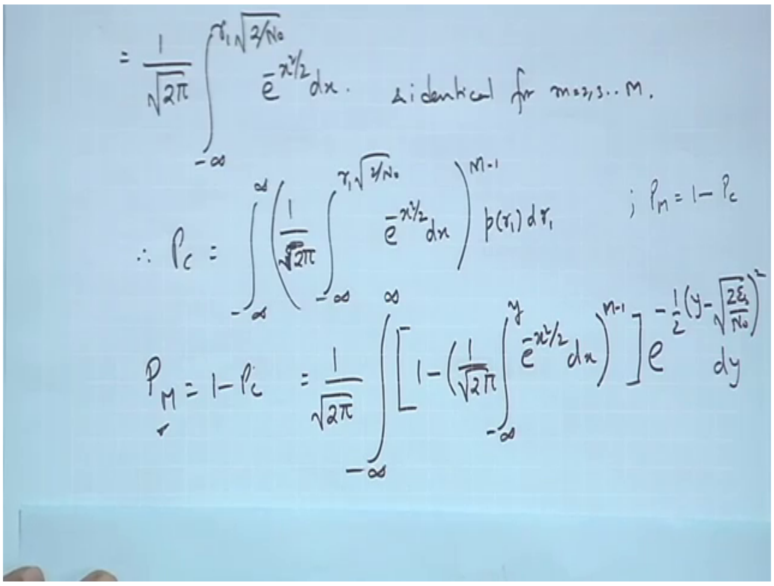

So, which is simply. So, it is minus infinity to r 1 because you want this variable to be less than r 1 so minus infinity to r 1.

So, this should strains from minus infinity to r 1, p of x m conditioned on r 1 right. So, that is what you have over here and that turns out to be minus infinity to this range with change of variables. So, these this is the expression for any particular other p d f and it is same for all. The simply same for all and you can easily guess by looking at these correlation outputs all of them have a similar output. So, it is the same p d f and hence the probability of correct decision which you have written over here can now be written as the individual raised to the power of M minus 1 because we have 2 to M the first one is of course, the one itself so, for all others must be less than the first component. So, this joint distribution becomes a product. So, that is what we have over here in this particular expression and then in order to calculate the probability of error it is simply P probability of error is one minus probability of making correct decision.

So, hence probability of making correct decision is 1 minus P c which is this whole expression which you can easily see. So, 1 minus this and it is averaged. So, this is the average probability of expression average problem of error which is given by 1 by root 2 pi e to the power of this particular expression as we have. So, it is averaged over all possible values of r and that is what is the expression here it is. So, this expression is cumbersome and it has to be solved almost numerically in most of the cases other than very simplistic situations.

So, what we will do in this particular case is instead of calculating the exact expression of error, we will use the results which can be found by solving this particular expression and we will focus on the performance of such scheme, that is error probability versus the E b by n naught or the signal to noise ratio.

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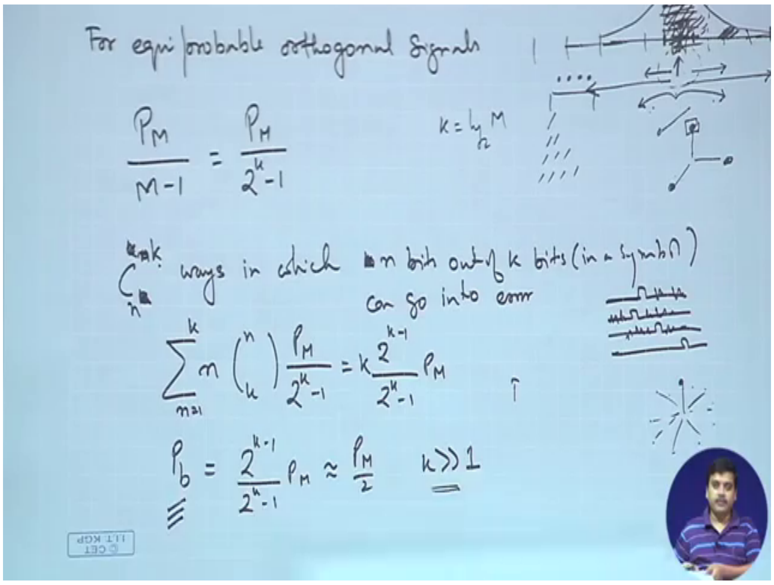

So, for equiprobable signals, we could write P m upon M minus 1 which is equal to P m upon 2 to the power of k minus 1 that is M is equal to 2 to the power of k and there are choose n out of k bits; that means, since there are k is equal to log base 2 of M. So, each symbol duration consists of k bits right. So, all what we are trying to do is a symbol can go into error by 1 bit going into error or 2 bits going into error or 3 bits going into error or 4 bits going into error.

So, because these are if you look at it just try to it is a bit difficult to imagine suppose we have constellation points which are these. So, this signal if it is sent could go into error if the correlation on this dimension is less than the correlation on other dimensions and the correlation on any of these dimensions have equal probability to be a better than this or to be less than this. So, they are not differentiated and that can be again understood from what we have over here. If you look at these are very very similar you can also be understood based on this because if we have sent S 1 what we receive is if S 1 is sent what you receive is root E plus n one at the output n 2 n three up to n m this is what we have said.

So, now probability of this being greater than this is similar to the probability of this being greater than this and so on and so forth. So, what it means is that all the signals are equidistant from each other right and because of equi probability we have this situation. Now why we are discussing this the reason is that in the other scenario where we had pam. So, let us take a few more constellation points. So, we have 8 although it is not equi spaced. So, if we were choosing this particular symbol let us say if you are transmitting this symbol then we said that the conditional p d f goes like this; that means, given that you have send the symbol probability of the received signal line in this range is area under this curve which is pretty large. Probability of the signal line let us say in this range is much smaller compared to probability of the signal line in this decision interval right.

So; that means, probability of further probability of this signal being detected as this is even less because the p d f falls area under this curve is even less right. So, what we mean to says that if I have sent this signal, the highest probability for the signal is probability remaining over here. And if at all it has to go into error it has a higher chance of going to the neighbour and going to the neighbour far away is even less. So, going to this neighbour is probably equal to going to this neighbour, but going being decided at this neighbour is much less than probability of becoming this neighbor. Now probability of becoming any of the neighbours is general less than probability of remaining by itself. So, that is how we design communication systems.

So, that and remember we had considered grey coding; that means, if one makes a mistake by going into the neighbouring symbol there is at most one bit error so; that means, probability of making one bit error is much more then probability of making 2 bits error, then probability of making three bits error and so on and so forth right. Whereas, in this case since the constellation points are such that if I would consider n dimension this is not n dimension this is just imaginative if these are all orthogonal and they are equidistant, then this symbol could be detected with equal probability and all the orthogonal dimensions so; that means, there is the probability of getting one bit changed is the same of probability of getting to bit changed, and the probability of getting all bits changed.

So, anyway, but the symbol can get into error by one bit going into error or all combinations of different bits going into error. So, we have to take out of k bits n possible bits can go into error is the number of ways it can go into error is, k choose n that is a combinatorial. So, this is what we have and since one bit can go into error with 2 bits can go into error and 3 bits can go into error. So, if you take up all such possibilities right we have to take all such possibilities along with the possibility that any of the bit goes into error. So, by expansion of this term what we have is P m upon M upon 2 to the power of k, because as k is greater and greater than one then this is almost equal to 2 to the power of k. So, you could solve this with this approximation and you could get that probability of a bit going into error would be approximately equal to probability of a symbol going into error by two.

So, this is an approximation that you can reach two, but this is just translating the symbol error to bit error probability. So, using that and the solution of this expression, numerical solution of this expression with E b by n naught; so, what we can have is the probability of error curves versus E b by n naught is what we are plotted over here, and it might be quite interesting to look at this particular result.

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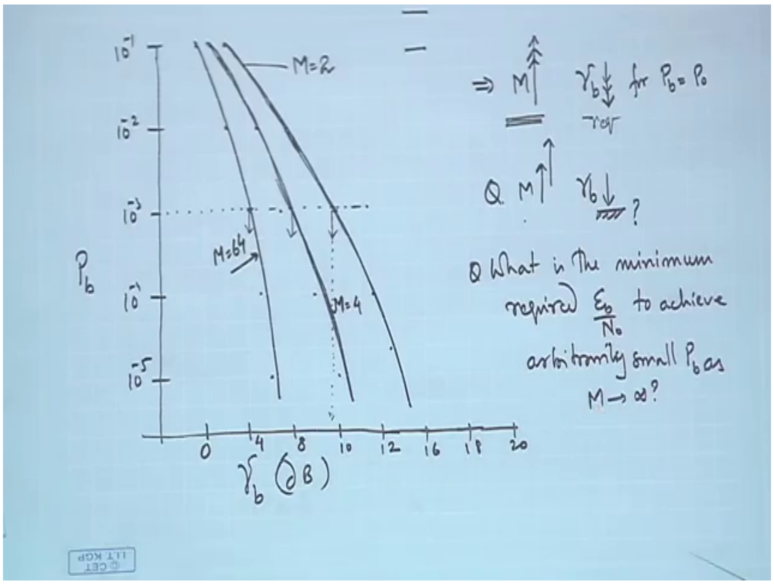

So, what we have with us here is this line this particular curve which is the probability of error for M equals to 2; that means, you have only 2 signals to choose from right whereas, when M is equal to 4 what we find is that the curve is changing in the left side direction; that means, it is going towards the left whereas, you might remember when we discussed probability of error for PAM like signals we had situation which was here; that means, as M increased you are going to the right.

Whereas in this case as M increases you going to the left. So, we have not done anything simply by solving that error probability expression and as a function of E b by n naught you generate this curve. It might be quite surprising that why we have this kind of a result and unfortunately by looking at the expression that we have here it might not be feasible to straight away comment that why such a thing happens. So, it might be we might address it in a few different ways and one of the ways generally as is addressed in the literature is through the union bound. However, we can take and try and take an intuitive view of why such a thing happens to get a feeling of well this could be a possible reason to remember the basics.

However the union bound and what we do afterwards would give a concrete impact of explaining why do we have this kind of behavior. So, let us briefly look at some of the things here. So, what we stated here is as M increases we find that as M increases as you increase M right the E b by n naught decreases for the same P b; that means, if I keep a particular probability of error right. So, as I increase m. So, I go from this M to this M and this M to this m. So, basically as I go there right. So, or as I increase M as I increase M with keeping the same P b, I am changing from M equals to 2 to M equals to 4 to M equals to 64 what I find that the E b by n naught that is required.

So, we should write required decreases right as M increases. So, quite contrary to what we have seen there. So, generally the question that we have just discussed is why should this happen that why at all this should happen right and the next question that we can ask that as we keep increasing M, what is the where is the limit of the curve does it reach a limit or we keep on going below is the next important question that we are confronted with. So, in the next lecture we will try to address this particular issue of orthogonal signals where one of the realizations could be M-ary FSK.

Thank you.