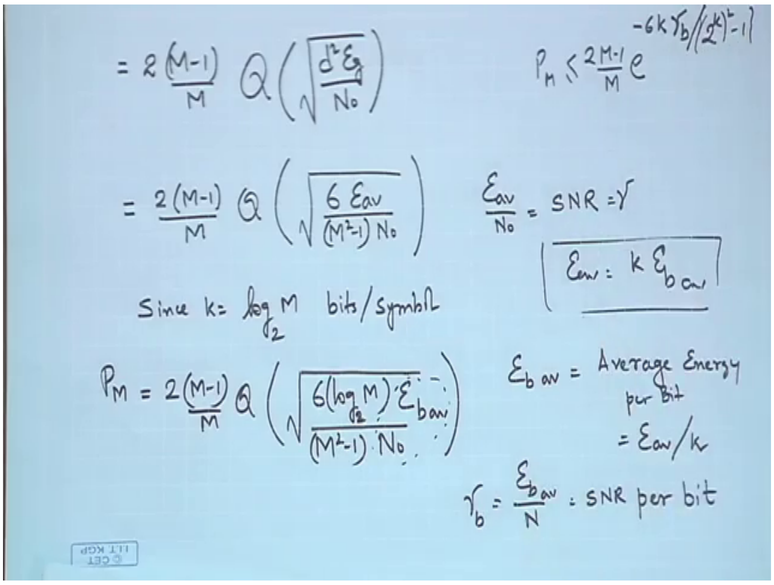
Modern Digital Communication Techniques
Prof. Suvra Sekhar Das
G. S. Sanyal School of Telecommunication
Indian Institute of Technology, Kharagpur

Lecture – 47
Performance of Digital Modulation Techniques (Contd.)

Welcome to the lectures on modern digital communication techniques. In the previous lecture we have just seen how to calculate or how to compute the expression of probability of error. First for BPSK or binary pulse amplitude modulation, and then we have extended the method to bind to a emery pulse amplitude modulation. So, what you would be interested to see now, is the graphical representation of the resources. Because typically with the results are in terms of Q function or in terms approximated as exponentials, which are not very easy to visualize. So, would like to see the graphical results which will give us some intuitive feeling of what happens to the probability of error as you change E b by N naught or as you change the parameter M.

So, as we see this particular expression here.

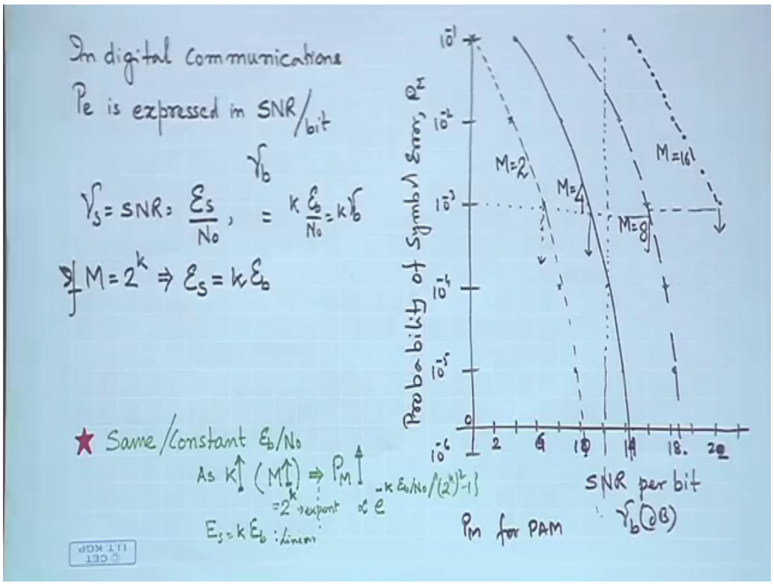
(Refer Slide Time: 01:12)



That we discussed in the previous lecture what you find that probability of error for the symbol error this is the probability of symbol error for emery P M is given by 2 M minus 1 upon M and we seen the reasons for this or to be we are talking both sides of the tail and M minus 1 upon M, because we have M minus 1 complete PDFs. So, inside the Q function you have E b by M naught, because in digital communications we are interested in E b by M naught that is average SBR per bit or average energy per bit because the basic unit of communication is bit. So, we want to compute how much of energy or SNR per bits used in order to actually come communicate.

And there again you have some further multiplicative factor, but there is k log M base 2 and in the denominator there is some more factor. So now, moving down further with this we have define what is of course, E b by E b a b a v that is the average bit energy average bit energy in this part which we discussed earlier.

(Refer Slide Time: 02:28)

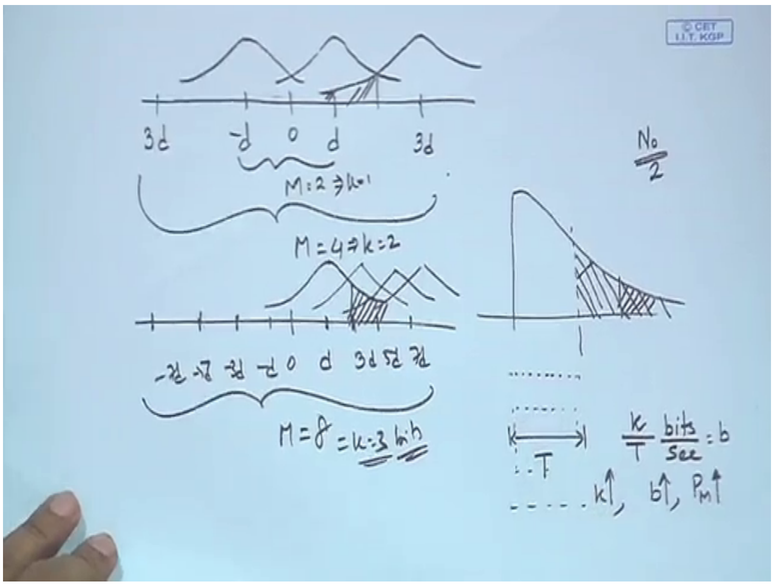


So and we have all already said that the probability of error expression is rather expressed in terms of average SNR per bit rather than SNR and this usually is denoted by gamma b, and gamma s is usually used to denote the signal to noise ratio of the symbol. So, this you could write it has k E b by N naught which is k times gamma b this what you can consider.

So, what we see over here is, say this access that we have drawn is the probability of symbol error P M which would just to write and this axis is the SNR per bit which is gamma b 4in decibels. So, you are looking at in decibels form is generally the SNR is represented in decibels, the first curve that we have over here is the result for M equals 2 binary PAM, which we have discussed before and it can also be obtained from the emery PAM result. So, then we look at the result when M is equal to 4; that means, 4 point constellations minus 3 d minus d, d and 3 d. So, in that case the probability of error results looks like this. And as we increase m; that means, as we increase the constellation size the probability of error curve shifts on the right. So, it can be read in 2 or 3 different ways.

So, one of the ways to look at it is, as I increase the size M right the SNR per bit required to maintain the same probability of error increases. You can clearly see that this increases right. The other notion that you can also see is given a SNR per bit there is a certain probability of error required for M equals to 2 which is approximately 10 to the power minus 7 over here for the particular case that I am considering, as I keep my SNR per bit constant and I move to higher order constellations, the probability of error increases, the probability of error simply increases. So, if I would draw a line along this direction I can only find that if my SNR per bits maintain constant the error probability increases. So, what it means the message that it conveys is that if we are increasing the constellation size that is M the energy per bit that is required to maintain the same probability of error is going to increase. Or if I am going to maintain the same energy per bit, but I increase the constellation order I will be getting into more and more errors right. So, why does it happen? So, one can intuitively analyze this and then one can see exactly how does it happen.

(Refer Slide Time: 05:53)



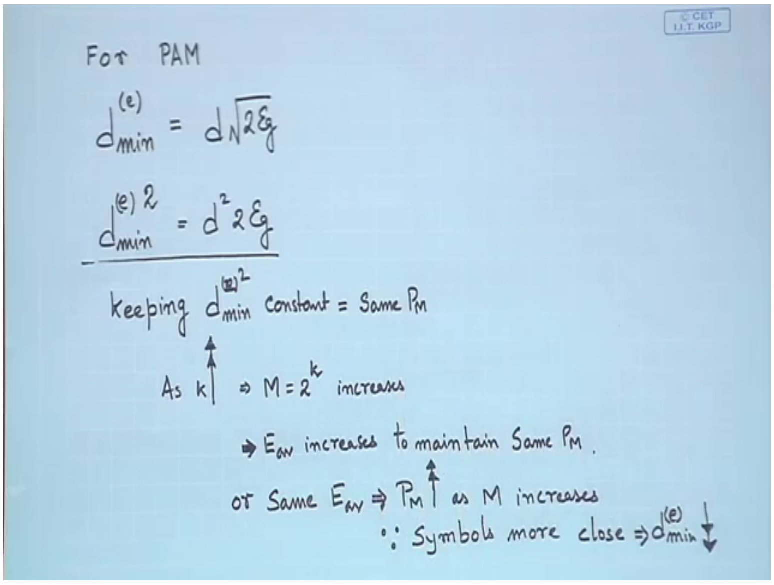
So, if this is my signal space x axis there is a 0, and you have the first situation where it is d and minus d there is a certain probability of error.

So now if I have to maintain the same probability of error; that means, I have to maintain the same minimum distance, I have to increase the signal energy. So, when I am going from M equals to 2 to M equals 4 by keeping the same minimum distance without changing the minimum distance, because we set error probability is dominated by the minimum distance if I have to keep the same minimum distance I have to increase the size of the constellation; that means, I have to increase the energy of the consolation.

The average energy goes up, So that means, if I have to maintain the same error probability; that means, keeping the same minimum distance average as energy required simply increases right. So, this is the first part. The other views if I want to keep the same average energy. So, let us take M equals to 4, I want to keep the same average energy M equals to 4; however, I want to use M equals to 8. So, if you want to use M equals to 8 in that case we have to accommodate mode constellation. So, it is not to scale right. So that means, I have to pack more symbols within that same average energy.

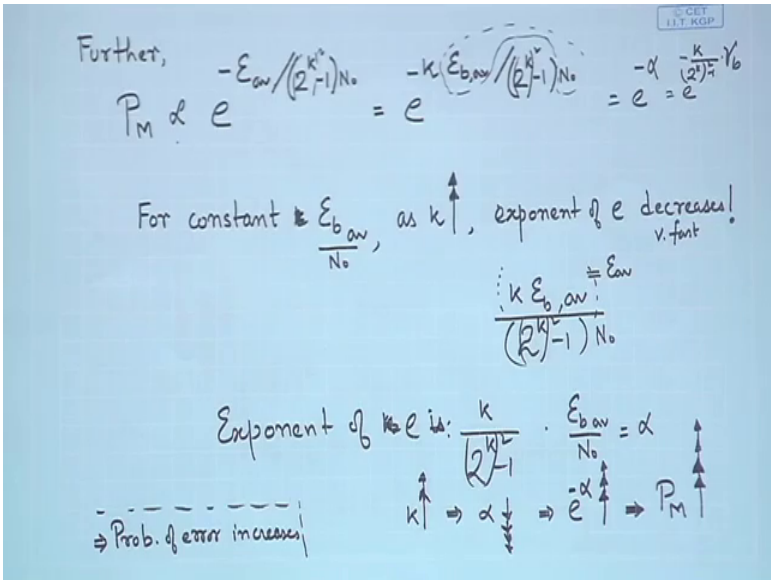
So that means, the PDF now N naught is unchanged. So, there is variance of noise is unchanged. So that means, if I maintain the same signal power the variance of noise as experienced would be the same, which is the spread of the signal right. So, if the variance of noise has remained the same the way you could keep the error probability the same is by maintaining the same minimum distance, because you are interested in calculating the area under the tail probability right. So, which will be remain the same if this distance is kept constant; that means, this area is remaining constant. Here what is happening is the constellation points are coming closer right compared to the previous case. So, you are one constellation point you have another constellation point, you have another constellation point, you have another constellation point. So that means, you are now integrating this area. Previously you are integrating area which is much smaller; that means, you had 3 d over hare. So, you have area under the curve was here. So that means, we could see a clearer picture in one case your area under the curve was here. But now if you pack more signals PDF is not changing, but the area under the curve is changing right; that means, you cover more area under the curve because PDF is unchanged the signal power is remain the same, the noise power is remain the same, the variance is remain the same only thing is our thresholds have come closer, right here wherever was one threshold if the threshold is much closer. So, the threshold is much closer area under the curve is much more So that means, the error probability increases and that is what we exactly have here, if I keep my same energy in that case if I increase M my probability of error increases. So, these other 2 views that you can get from this particular figure. So now, if we go ahead and try to understand that what could possibly making this happen that is intuitively what we have explain. So now, till now so, instead of explaining intuitively if you look at the expressions things will be a bit more clear.

(Refer Slide Time: 10:51)



So, what we do is let us take the palm whichever way were doing right now, that d minimum Euclidean distance is given by this which we know. And d min squares is of courses expression. So, if we keep d min square Constant we going to get this nearly the same P M that is what we explained. So, as k increases by this arrow I mean k increases in the upward direction it means it increases, M increases exponentially 2 to the power of k right it increases several times faster. E average increases to maintain the same probability of error that is what we have already explained. Because if you would keep the same minimum distance you would go from 2 to 4 your error probabilities is going to remain the same right. Or if you want to keep the same E average in that case your probability of error increases that what we have seen as M increases, this is because symbols and more closely packed. And if symbols are more closely packed the minimum distance decreases if minimum distance decreases probability of error increases this is what we have seen qualitatively.

(Refer Slide Time: 12:04)



Further you can also note that the probability of error is proportional to E to the power of minus E av upon 2 to the power of k square we have the term because we have to look at the expression of error probability which is here. So, from this you can say that P M is less than equal to 2 M minus 1 upon M E to the power of square of this that is minus 6 I am going to use k because log base M 2, I am going to use gamma b which is equal to gamma and this is equal to gamma b. So, this is gamma b divided by 2 to the power of k squared minus 1 right. So, that is what we have so that is what we said in this particular expression as well.

So, when we use this particular expression we have not considered this part, because as M goes this number almost becomes close to 1. So, as M become larger and larger this close to one this is a constant is not much effect not much variation with M over here compared to what we see over here. So, M is log M is 2 to the power of k there is not much variation with k. So now, if this is fine we move ahead if you are with this expression without the constant term in front of it is uses proportionality. So, what we have is this expression for constant E b by N naught as k increases. So, we are maintaining this ratio as constant, because this is constant. So, average bits constant.

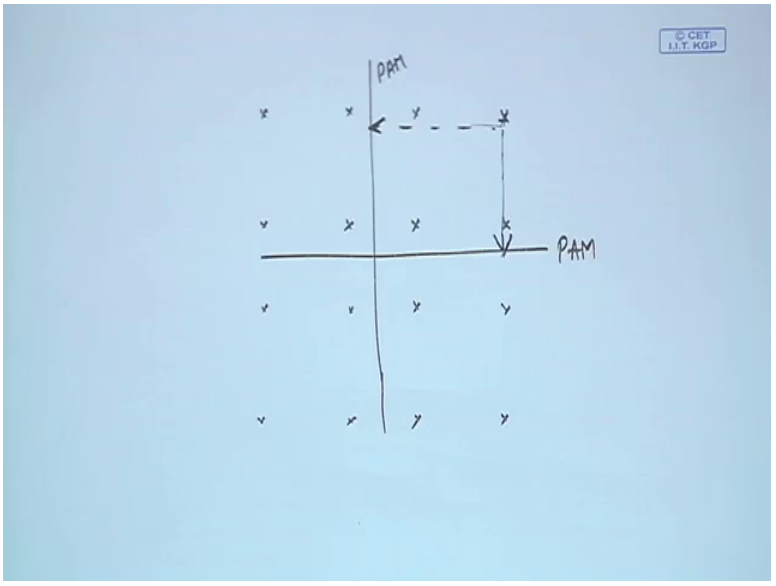
The exponent of E decreases very fast. So, let us see So, you could write this as minus alpha right which you could write E to the power of minus k upon 2 to the power k square minus 1 multiplied by gamma b that is what we have. So, the exponent is k E b a b upon 2 to the k square minus 1, upon N naught now clearly you can see that if k increases E average increases of course, the numerator increases, but the denominator increases at a much faster rate because it is 2 to the power of k squared right or 2 to the power of 2 k. So, going by this for k increasing the alpha decreases at a very fast rate and therefore, E to the power of minus alpha increases E to the power of minus alpha increases means probability of error increases. So, clearly from the expression also you can get a view of what happens as you increase k. So, in summary what we can say is that in this kind of constellations where we are having more bits to be sent; that means, you have to recall a previous disc discussion or will use this particular one for this k equals to 1 for this k equals to 2 and for this k was to 3; that means, 3 bits right.

So, what happening on the other side? On the other side you are if you are bit duration T is constant right. So, then you are bits per second is k upon T bits by second bits per second. So, as k increases if this is your bit rate that is a b, b increases beats per second increases; however, what we see P M also increases; that means, probability of error increases. So, on one hand you have an improved throughput, but you have a poor error probability right. So, if you decrease bit rate you can you will decrease your bit rate; that means, you are decreasing number of bits per symbol. So, as I increase M we are sending more bits per symbol. So, if I am using binary PAM I would use only 2 levels if I am using quaternary PAM, PAM I am going to use 4 levels 1 2 3 and 4. So, in this case I am sending 2 bits per symbol in this case I am sending one bits per symbol. So, in this in the one bit per symbol case my bit rate is low, but my problem of error is good is also low.

When I send more bits per symbol I pack more bits per symbol without changing the total transmitted power in that case a bit rate of course, increased, but my error probably has also increased right. So, in order to maintain the same error probability I must use much more energy right will see an exact comparison later on. So, at least it tells you that by going from one constellation order to other that is going from binary PAM to emery PAM M equals to 4 8 16 you are definitely increasing your bit rate, but if you are keeping the same transmit power your error probability increases significantly.

So, when you are increasing bit rate in order to maintain the same error probability you must increase your transmit power as well as right. If you do not then error probability will increase significantly. So, this is something that we should remember. So, with this we can move on to the next very important thing or the related thing that we have studied is quadrature amplitude modulation. And at this point we can remember that when we discussed QAM we have said that QAM can be composed of 2 PAM signals one along the I axis one along the Q axis.

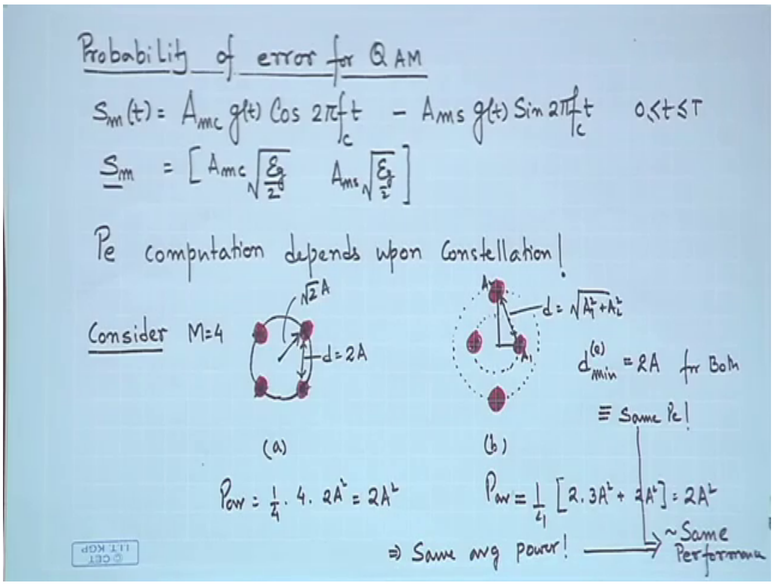
(Refer Slide Time: 18:14)



So, what we at said is that if there is PAM along this axis and there is PAM along this access right. So, then what you have is a QAM. So, any constellation point can be thought of as a PAM here and a PAM here right. So, if I receive a signal I mean project it is component on the I channel and I am going to project it on the Q channel and I can use PAM for decoding. So, whatever we have studied for PAM we can now extend it to the analysis of QAM when we are doing error probability analysis. So, let us look at the study of probability of error for QAM. So, again to summarize is what we have is the signal in this form which is not new to you. So, I do not need to spell it out and the signal vector is in the next line again since it is known and described thoroughly in all previous lectures again I leave it up to you to connect.

The probability of error computation that we will do is now going to be shown will depend upon the constellation. So, what we are trying to say is we have qualitatively said that the d min square is affecting the probability of error we going to slowly get into the impact of such a thing in this particular discussion right. So, let us consider and just for the second I would like to remind you that when we studied QAM we said that the choice of the constellation would depend upon several factors. So this so, here we see one of the factors at least one of the factors we would influence the choice of the constellation or the design of the constellation. So, when we compute the probability of error let us take one situation where we have 2 possible constellation.

(Refer Slide Time: 20:55)



So, in these 2 constellations, these are the symbol locations. So, in both the case M is equal to 4. So, there is no problem on M, where 4 constellation points; that means, k that is number of bits is equal to 2 and we have maintained that the minimum distance between 2 constellations is 2A.

So now we are saying that both the constellations have the same minimum distance. So, minimum distance is 2A you can rest assured that there is not much impact on the probability of error right it will remain the same. And if you would calculate the average power; that means, here it is 2A squared and you have 4 2A squared and divide by 4 because of course, we are doing equiprobable you are going to left with 2A square does the average power because this is 2 root to A.

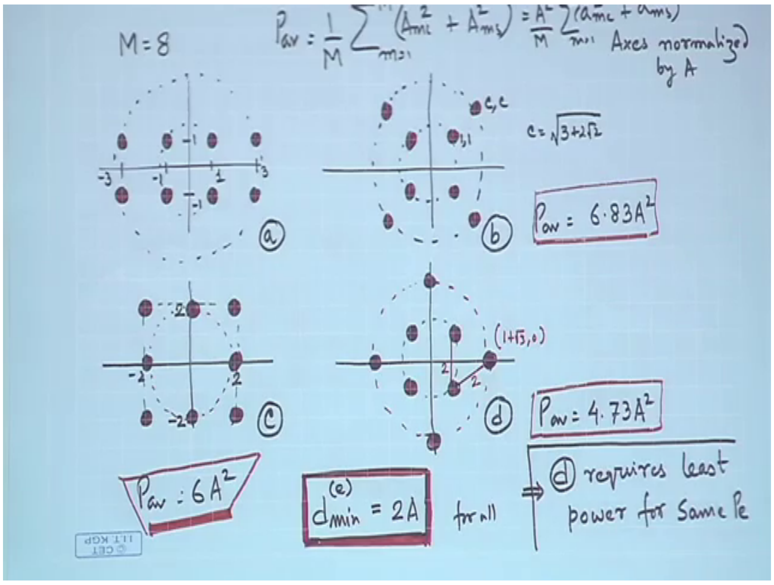
If you look at this constellation there is a particular description that is given again the average power would turn out to be 2A square, the since the minimum distance is same and average power is the same there should be no difference in terms of error performance. So, if you plot the probability of error verses signal to noise ratio for these 2 they will turn out to be the same right as you increase their SNR there error probably will decrease in the same fashion.

However, you mean observed in this case that all have the same energy, which is not true in this case right. The energy of this is less than the energy of this; however, we have maintained same minimum distance and the same average power right. So, this has lower power and this is higher power than the average and average would be same as this right. So, these are constellation which are closer these are contemporary. So, overall it has been maintained.

So, that clearly means that peak power to average power for this is much smaller compared to peak power to average power. So, the peak power in this case will be higher the peak power than the peak power in this case. So now, if you have a requirement where the peak to average power ratio which is the important parameters for signals passing through non-linear power amplifiers and you have a certain ratio which constraints your requirement, then you are going to definitely go for this particular constellation which has much lower peak to average power ratio compared to this. So, although these are the same error probabilities this have lower pa pr peak to average power ratio and hence I would prefer to go with this particular constellation.

Moving ahead let us take the situation for M equals to 8. So, in case of M equals to 8 we have Examples where this particular one is one constellation configuration right.

(Refer Slide Time: 23:57)



Another constellation configuration is the one and I am marking with red as the constellation points right, and there been labeled. So, you can look at the exact labels and do the calculations yourself, I am not spending time in doing the calculations here. Because that is not a very complicated job, you should be able to manage that.

So, we are mainly focusing on the message that is being conveyed. So, we have the 4th constellation which I am filling up here. The forth constellation is here. So, if you look at all of this constellations they all have M equals to 8. And the average power calculation is simply represented here one upon M m is 8 some over M equals to 1 to 8 ; that means, I will take the power of this power of this is A mc squared plus A ms squared which is written over here. Then I will go to M equals to 2 I will calculate the power which is the non square of this vector, then I will calculate the non square of this vector and so on and so forth. Value to calculate the sum of the power of all divided by the total number is going to give me the average power right.

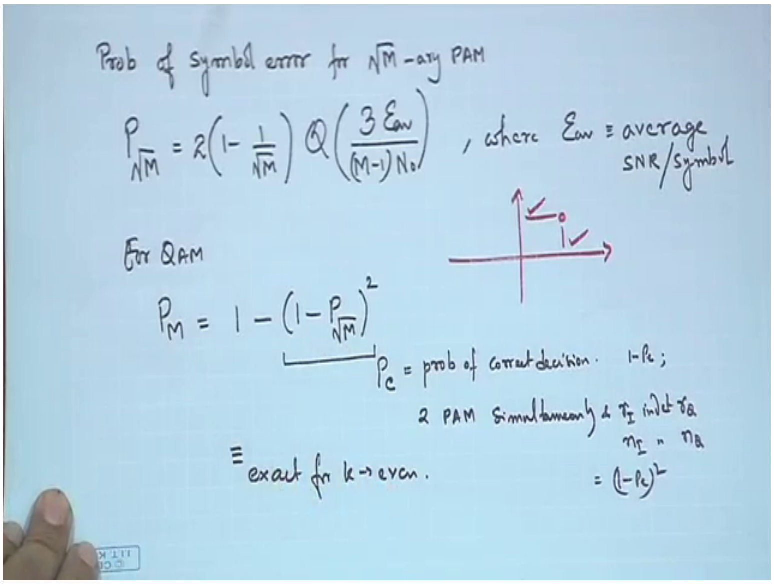
So, that is that way I can calculate the average power of all these and what we have noted down here the average power of these 2 constellations is 6A squared. We have also maintained the d min as 2A just like the previous constellation for M equals to 4, So that all the constellations have the same probability of error right. So, these constellations have a certain average power. These constellations you can see has an average per requirement as given by this. And this constellation has an requirement which is given by this expression. So, what we find is that although all of them will enjoy the same probability of error, but that is arrived at by spending different amount of power. And clearly what you can say is the this particular configuration here required the least amount of power. And hence this constellation is possibly one of the choices which can be used because by spending less power it achieves almost the same error probability as any other constellation.

So, through this at least we can explain that when we going to QAM the choice of constellation is not a very straightforward solution. So, we need to look at least one step deeper in order to select the constellation for QAM. If we are taken M greater than or equal to 16, then rectangular QAM requires only slightly higher power then circular QAM or any other thing. So, what we mean is that if we choose another constellation you could make many possible constellations. So, what we trying to say that if using rectangular qa QAM the extra amount of power compared to the best constellation in terms of minimum power would will slightly more. And we are willing to pay this extra price; that means, slightly more increase in power because we have a huge advantage in terms of ease of demodulation, because if demodulation would also require some power right from processing power. So, we would like to reduce on that front and make algorithms much simpler.

So, the for rectangle QAM you have M equals to 2 to the power of k where k is even your rectangular QAM because even means your I axis your k axis. So, you have on 2Access you would be like to choose k even. So, that you will be able to divide the bits evenly amongst both of them. So, QAM is basically 2 PAM on 2 quadratures with PAM each having route M signal constellation this is clear and we can we have always drawn that several times. So, this I do not need to explain.

If I have 1 2 3 4 and I have 1 2 3 4. So, 4 PAM and 4 PAM would make 16 QAM, this is well known right, we do not need to describe this much. So, in order to calculate the probability of error.

(Refer Slide Time: 28:53)

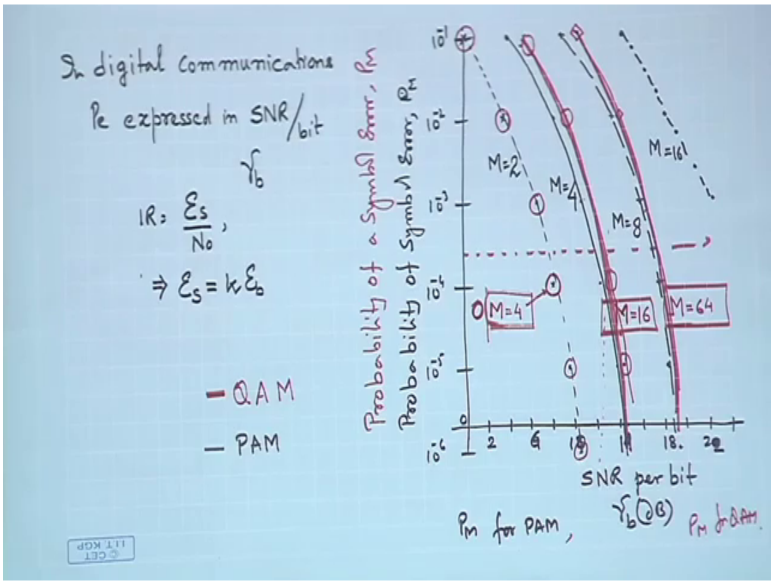


We look at the probability of symbol error for root M ary PAM where E average is the average SNR for symbol for QAM. So, there we had 6, but here we have a 3 because of translation of variables. So, the average problem of error for route M PAM looks like this. So, for calculating the probability of error for QAM would have to calculate the probability of the symbols being correct on both the directions. So, if root P M is a probability of error one minus P root M is the probability of it being correct. So, put the symbols have to be correct. So, you have this dimensions. So, it has to be correct here it is to be correct here.

So, when both are correct. So, probability of correct in one direction multiplied by probability of correct being the other direction. Because these are orthogonal directions and we have Gaussian noise we considered this to be un correlated which leads to independent. So, based on which we have this 2 PDF getting multiplied probabilities getting multiplied we have the square term. Now one minus probability of being correct is the probability of error. So, you can substitute this expression inside this and you want to get with probability of symbol error for QAM. And then once you do that you are property of a error expression, you can fill all this expressions over here. And then you can make the approximation that as M increases instead of having this one since one over root M decreases, you can simply say that P root M is P M has which is less than or equal to this expression here. And it can further be stated then it is less than 4 Q the squared this one cancels out you would have 4 Q squared and you are going to have 4 q So, Q square is much less. So, you can make an approximation again less than 4 Q. And for non rectangular QAM you are going to get an expression which is represented over here.

So, up to this step one should be able to easily arrive at by using the expression for QAM. And we would also like to remind you at this point about the possible expression for M ary PSK, which we are not deriving here because we have many more things to do, So that can be derived using the procedure which we have followed. So, you can break it into components and you can still derive. And we will look at briefly the error probability curves and we will continue to see it in the next lecture.

(Refer Slide Time: 31:25)



But very briefly what we find is I have marked in red the probability of error of M ary QAM and in black is the probability for M ary PAM, and these lines basically almost merge with the black line to just to make it separate we have another line as probability of error increases oh sorry, decreases these lines match with that of the PAM.

So, what we see is that as M increases. So, this is M equals to 4 this is M equals to 16 and this is M equals to 64. The error probability behavior behaves exactly same as that of PAM, as M increases your SNR required to maintain the same error probability increases. And for a constant SNR as M increases. Your error probability increase and the reason need not be explained any further because we have already explained the reason why it happens for M ary PAM and QAM is constituted of 2 such M ary pams one along the I channel one along the Q channel. So, all the descriptions and explanations at we have given for such kind of behavior for M ary PAM would apply in exactly the same way for M ary Q a.

So, we stop this particular lecture here and we continue with these discussions further the in the next lecture.

Thank you.