Modern Digital Communication Techniques
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Lecture – 46
Performance of Digital Modulation Techniques

Welcome to the course on modern digital communication techniques and as you are well prepared, now that we have discussed receivers; that means, we have seen the signal transmitted through the channel received and processed and finally, a decision has been made. So, now, it is important that we try to analyze the performance of the system that we have created.

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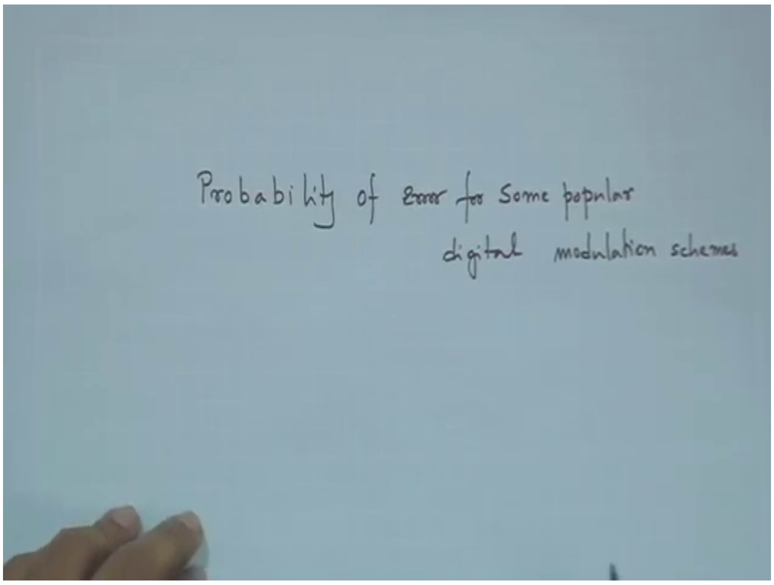

So, what we are going to discuss is the probability of error for some popular digital modulation schemes. So, we have already identified that why do we want to call it an error and just to summarize that since the detector chooses from one of the possible sets; that means, in a very Layman's term, we can think of as if there are 4 colored balls and some colored ball has been received from the other side that the detector has to make a choice between which of the possible color balls it would have been. So, that is mean when is it will finally, make a choice. So, if it makes a choice the choice can either we write or it can be wrong.

So, therefore, we want to calculate on an average; what is the chance? It makes the right decision or what is the chance that it makes a wrong decision and we wanted and we had designed a system. So, that the percentage of time or the chance that it makes an error is as low as possible. So, therefore, we get into study of probability of error and what we are going to look at it is some popular digital modulation schemes. The reason is there are so many different digital modulation schemes and we have studied quite a few it may not be possible to look at all possible modulation schemes.

So, we will take a few which are very important and which will serve as baseline and then you can developed based on whatever we discussed here.

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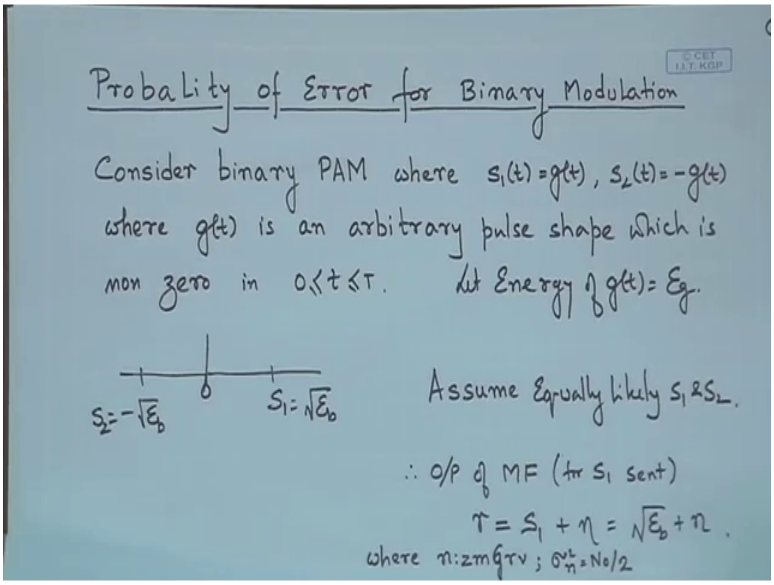

So, first we take a look at the binary modulation scheme. So, first we take a look at the binary modulation scheme and what we have is a PAM binary PAM signal where we have the signals S 1 t and S 2 t and S 1 t is G of t while S 2 t is minus G t. So, this is well know where G t is the arbitrary pulse shape.

So, we are not talking about a particular pulse shape and of course, whatever we have here is non 0 for the interval 0 to t. So, let the energy of G t b G E g.

So, these notations we have using throughout nothing much new for you. So, the signal space diagram is as over here. So, again this is standard notation; what we have been using and we will assume equally likely signals S 1 and S 2. So, what we have is the output of the matched filter for S 1 being sent.

So, we must be careful that we are talking about S 1 out of S 1 and S 2. So, in that case, the received signal would be S 1 plus noise and S 1 being root E b we have root over E b and of course, noise is described as 0 v and the variances n not by 2.

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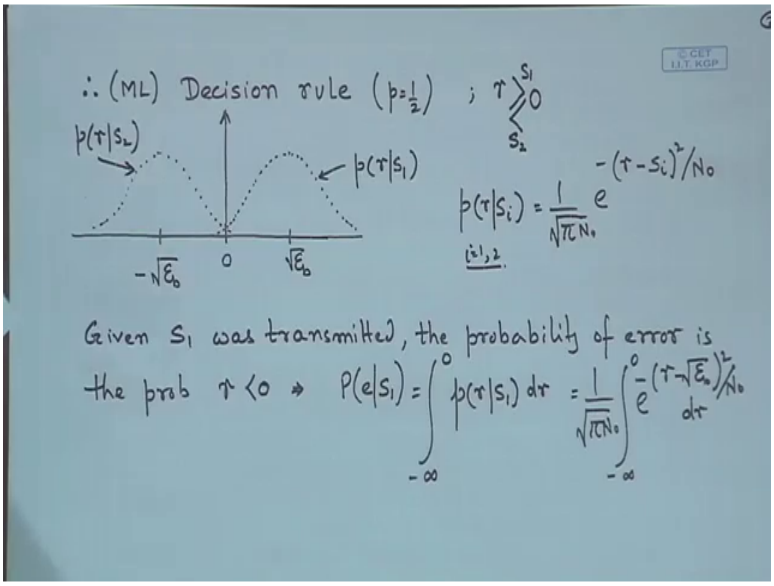

So, now we have to go to the decision rule and we have studied the decision rule where P is equal to half.

So; that means, we are taking the maximum likelihood receiver because equi probable and for such a situation you may remember we got the result that the received vector is to be compared with 0, if it is greater than 0. This is S 1, if it is less than 0, it is S 2. So, in other words, when I have this particular signal space, 0 is the threshold point and anything on this side will be decoded as S 1 anything on this side will be decoded as S 2. So, now, since we have sent the signal r, we have received the signal r, we have the signals space on this axis and would like to draw the conditional PDF of r given S 1 because that is how we made the receiver.

So, the conditional PDF of r given S 1 would be the Gaussian PDF with the mean of S 1 that is root E b and the variance of n naught by 2 that is the variance of noise if S 2 would have been sent your PDF would have been Gaussian with mean at minus root E b that is a signal value of S n sigma of n naught by 2. So, that is how you have p of r given S I being 1 or 2 denoted by this expression which is well known to us.

So, now given S 1 was transmitted the probability of error; that means, if S 1 is transmitted, we are trying to find that the decision is a wrong decision is the probability that the received vector r is not on this side, but on this side. So, when I am talking about probability of error, what I mean is suppose we have received several r values; that means, have received r, let us say 1000 times and what percentage of time suppose all the 1000 times I have sent S 1 and every time I sent S 1 so; that means, if this is S 1, I have sent it first time I received r is here second time it is here third time, it is here next time, it could be here, it could be here, it could go there.

So, the mean is here, but it will be scattered around this and it could go there. So, it could go everywhere. So, whenever it goes on the negative side if this is 0 we will identify it has S 2. So, identifying it as S 2 is error when I know that S 1 has been sent we are trying to calculate what fraction of time is it on the on the region of S 2.

So, that is what we have you want to calculate the probability that are is less than 0. So, that is probability of error conditioned on S 1; that means, we are saying S 1 is sent is integrate from minus infinity to 0 because you would like to find this part.

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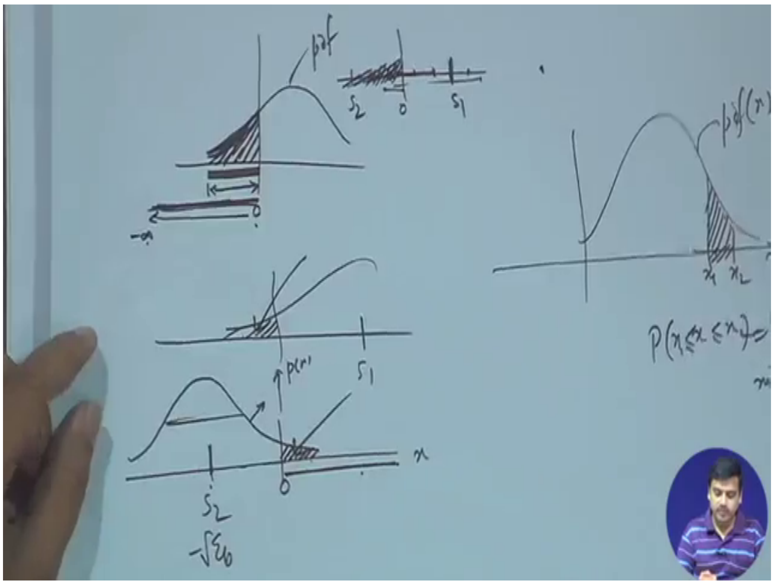

So, this PDF goes like this. So, if you have to make it a bigger picture like this suppose. So, we want to find the area under this curve if this is the PDF the probability is equal to the area under the curve under, this curve covering this range of values, right.

So, the range of value is from 0 to minus infinity, you rather minus infinity to 0 in this, for this PDF, right that is a standard thing. So, we have minus infinity to 0, the conditional PDF and over r r d r because r is a received a vector in case, it is one dimensional over here. So, p of r given is one with directly use over here and then we proceed further with expression that we have here is we make a substitution that r minus root E b over root n naught is x by root 2 or in other words, we want to replace this part in the new; in the integrator to E to the power of minus x square.

So, that we get well known expression in terms of E to the power of minus x squared and our evaluation terms of the easier.

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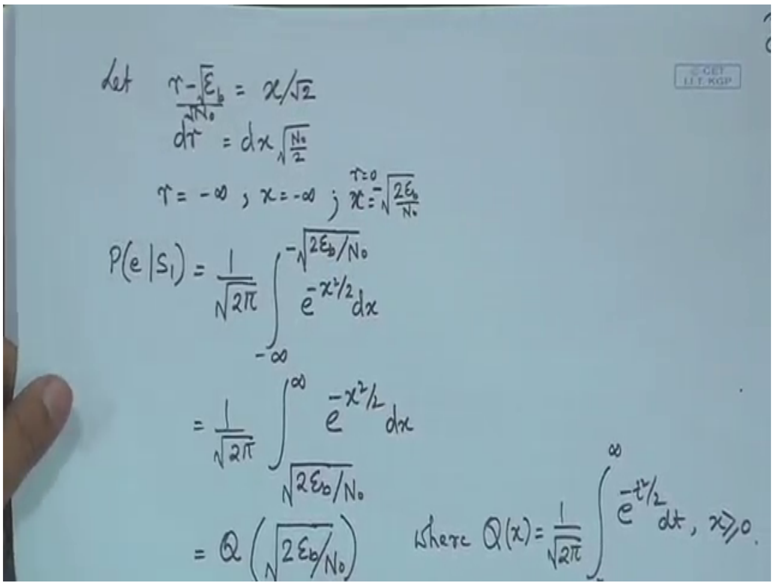

So, now if we let r if you make the substitution. So, d r equals to d x and some scaling factor for r equals minus infinity, x is equal to minus infinity, r equal to 0, x has a particulars change you can do all this change easily. So, this integration limits change the integration limits that you see over here gets changed to minus infinity to this number and you have the expression looking like this.

So, moving further on this; since there is a negative singned sign on both of them, you could swap the limits and you get an expression which appears here and this expression can be substituted with the Q function with the argument of this term where Q of x is 1 over root by integrate from x to infinity E to the power of minus t squared by 2. So, this is the standard function and which you can only find values by numerical evaluation.

So, we use it in terms of Q function because this looks a bit cumbersome. So, from now onwards, we will be using mostly Q functions to represent the error probabilities. So, in summary, we have started off with the binary PAM, for the binary PAM, we have considered equi probable transmission. For equi probable transmission, we have calculated the conditional PDF of a given signal for the received vector the received signal and then if I know as one was sent the probability that it is detected as S 2 is the area under the curve in the region of S 2 because as we said, if there is a PDF function and you want to find the probability that the random variable lies in a certain interval x 1 and x 2 all you have to do is find the area under the curve of the PDF of x in the interval x 1 to x 2 and this is going to give you probability that x lies between x one and x 2 right that is a standard thing.

So, now since we want to find the probability that it lies in this range, we have to integrate the PDF in this range and that is what we have done that is what we have done over here. So, when we do that integration; do some change of variables and what we end up with is an expression of the Q function where the Q function is described in this way.

So, now the Q function is something which is not very intuitive and you cannot make much out of this. So, there are some approximations of Q functions which are sometimes very useful which will refer to. So, now, the preceding from this point what we calculate its p of error given S 1, now what will be the probability of error given S 2; that means we considered S 1 was sent and we find area under the curve over here?

So, now if we say that S 2 is sent; what is the probability of error we have to first see the PDF of received signal given S 2 which will be this is x and this is P of x because the mean is that of S 2 root E b and it as a sigma of n naught by 2. So, it will be detected as S 2 as long as noise does not push it on the right hand side if this is. So, whenever it goes to the right hand side it is in error.

So, if this the PDF called then we want to find the probability that the received vector lies in the range 0 to infinity. So, that will be integration of this part of the curve because of symmetricity; this area is equal to this area, right symmetricity because of equal probability and it is centered around 0 equi distance from 0 and this PDF curve is also symmetric and we also have equi probable.

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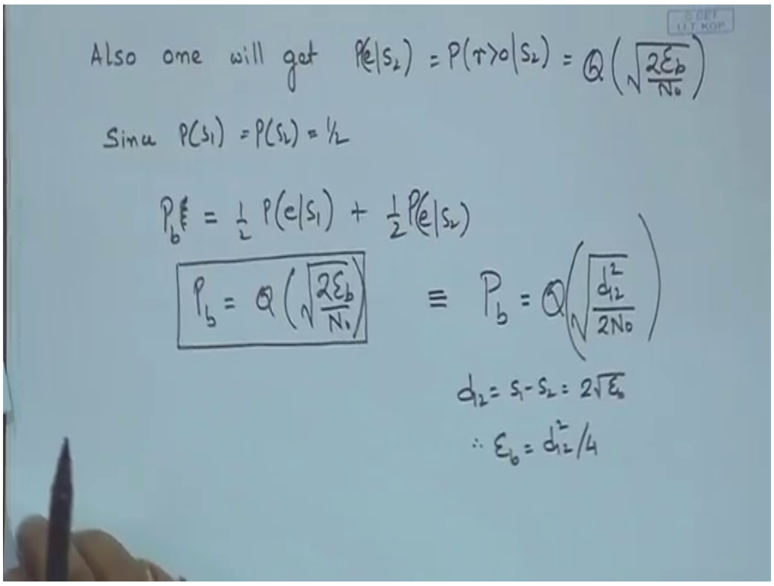

So, what we have is the probability of error given S 2 is what we just described probability of the deceased vector greater than 0 given S 2 is the same Q function that you are going to end up with and since probability of sending S 1 is equal to problem sending S 2 is equal to half probability of a bit error is equal to probability of sending S 1 multiplied by probability of error given S 1 plus probability of sending S 2 multiplied by probability of error given S 2 together P of S 1 is half of p of S 2 is half P of E given S 1 is a must P of given S 2.

So, therefore, P b is already equal to P given S 1 which is Q function root over 2 E b by n naught, right and since you know that d 1 to that me the distance between the first signal and the second signal is to root E b that is very straight forward that is 2 root E b that is very straight forward.

So, you have d 1 2 squared is equal to 4 times E b or E b is equal to d 1 2 squared up on four. So, we can replaced E b by d 1 2 squared by 4, you are going to get d 1 2 squared by 2 n naught. Now this is also important because the probability of error is dominated by the minimum distance among the constellation. So, I will briefly tell you that if you have mean here and you have constellations there.

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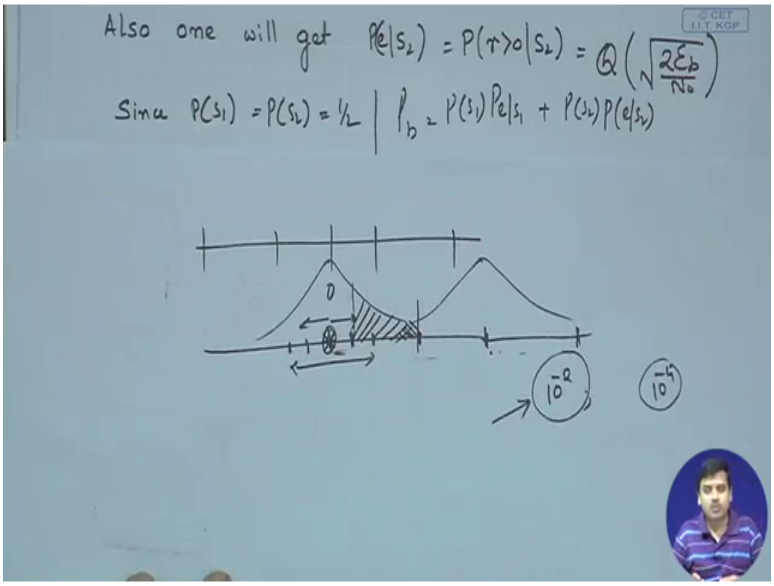

So, this is a symmetry case not much of a problem, but suppose I have constellations. So, my error probability will be dominated by this region compared to this region because if I have sent and constellation point here there is high chance of it going into error because region is narrow, right, where as if I send a symbol over here, there is less chance of it going into error because the PDF goes like this the PDF goes like this if you look at the area under the curve it is much more compared to the area under the curve over here, right.

So, and things will be clearer again, suppose I have a probability of error which is 10 to the power minus 2 and I have for this case probability of error 10 to the power of minus 4. So, when I am taking the average probability of error have to if we equi probable, I have to add this and this and divided by 2 if there are let us say 2 in that case you are clearly finding that this is much more having a much more weight compared to this and hence the probability of error will be dominated by the high probability of error terms which is due to the minimum distance between the constellation point this is a bit qualitative analysis, but of course, things can be made more clearer.

So, for this reason, it is important to also write the probability of an expression in terms of distance between the constellation points. So, now, as we said earlier we will be using the expansion of Q function. So, as we use the expansion of Q function we considered the Chernoff bound.

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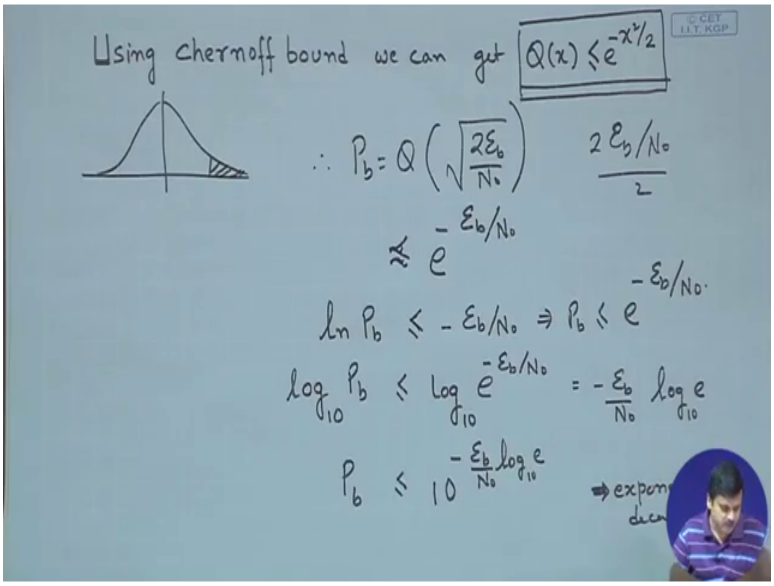

So, in using the Chernoff bound, you have the limit that Q of x is upper bounded by E to the power of minus x squared by 2. This is not a very tight upper bound, but you can use this approximation. So, P b which is equal to that of Q function of root 2 E b by n naught, it can be said P b is less than or equal to E to the power of square of this which is 2 E b by n naught divided by 2. So, it is minus E b by n naught. So, you have 2 E b by n naught upon 2. So, which is minus E b by n naught.

So far we have this expression and what might be of interest to you at a certain point is how does this particular function look like.

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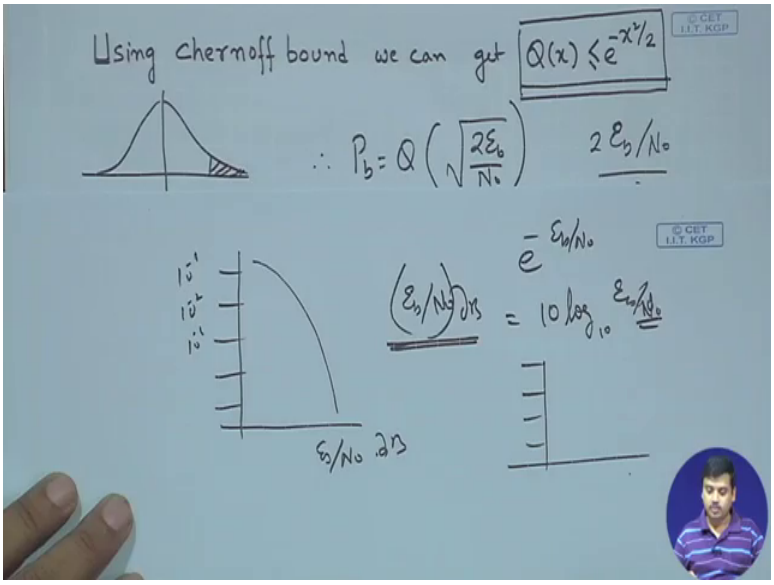

So, what you can do is you can actually take log of this. So, instead of plotting E to the power minus E b by n naught sometimes we are interested in E b by n naught given in d b scale.

So, E b by n naught when given in db scale is calculated as 10 log base 10 E b by n naught, there is whatever we have over here and what we can see at least from here directly there the probability of error, it decreases exponentially with E b by n naught in AWGN channel for binary pulse amplitude modulation BPSK.

So, if you were to plot E b by n naught, if your plot E b by n naught in db scale the figure that you might get in terms of error probability would look something like this where this axis we have SNR per bit.

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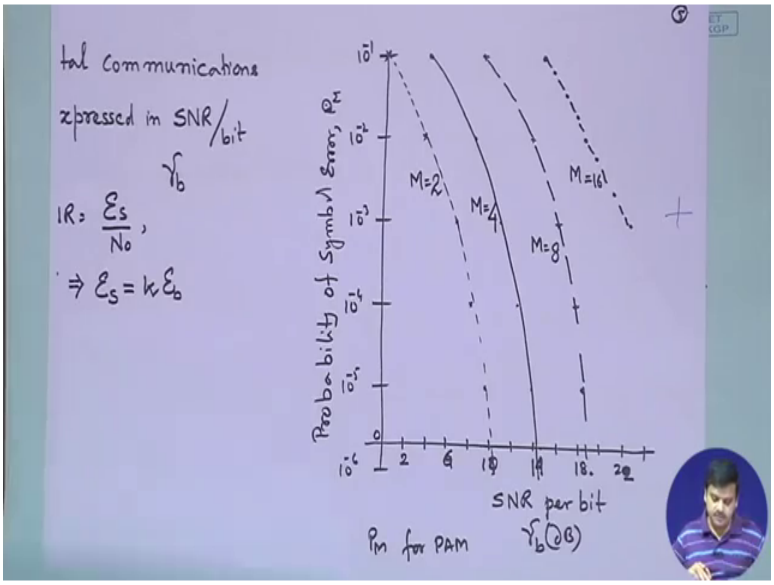

So, here I would make a small change and I would like to suggest that let us go for binary PAM and then we will discuss this.

So, what we can clearly see over here is that it falls exponentially. So, if you would have had a linear scale and this axis was in case of sorry if you have db scale over here and this axis is in terms of log; that means, 10 to the power of minus 1; 10 to the power minus 2 minus S 1 you want to get the curve which look like this we will see more of very soon.

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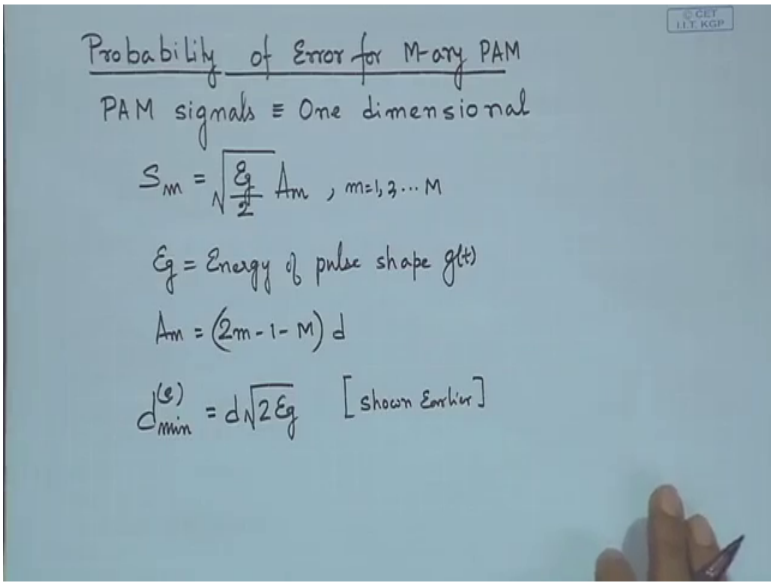

So, now we next move in to the probability of M-ary PAM signals. So, when you have M-ary PAM signals PAM signals are one dimensional that we clearly know and S m the modulating signal is root E g upon 2 A m were E g is the energy of the pulse shape and a m is 2 m minus 1 minus capital M into d and these we have covered before and minimum distance is d times root over 2 E g these things also we have considered earlier.

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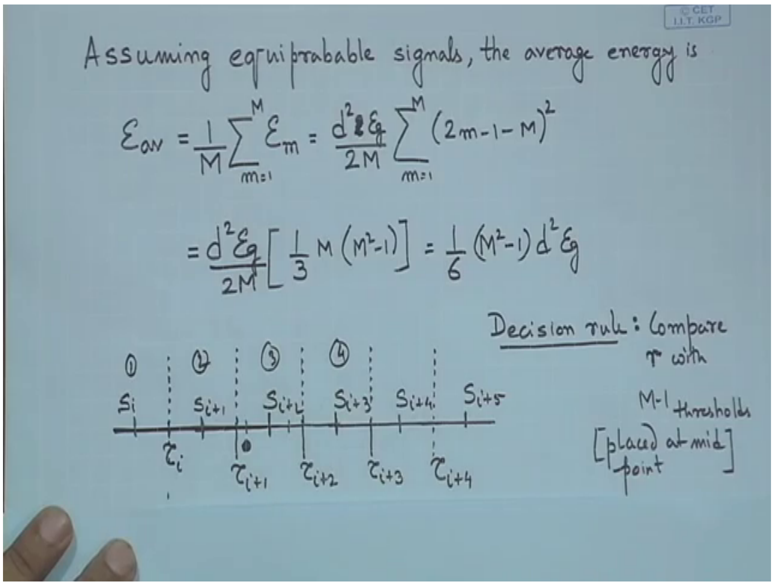

Now, if we considered equi probable transmission in that case the average energy of the received signal can be calculated as one upon M. This is because of equi probable E m is the energy of the M th signal which is equal to d squared E g by 2 M. Now this is straight forward from here, you can calculate this from A m and you have this expression which can be reduced to I mean you can do it is an algebraic reduction from which you can get this expression where capital m is the size of the constellation.

So, now when you are building the decoder the decision rule that we have at this point is to compare the received signal r with M minus 1 threshold placed at the midpoint. So, what we see over here is S I to S I plus 5 at the signal points where as tau I tau I plus 1 at the threshold points. So, clearly what you can see is if I have 1, 2, 3, 4, consolation points, we have one threshold between first and second constellation point the second threshold between second and third and third threshold between third and fourth so; that means, if there are 4 constellation points I have received some vector I will compare with this threshold.

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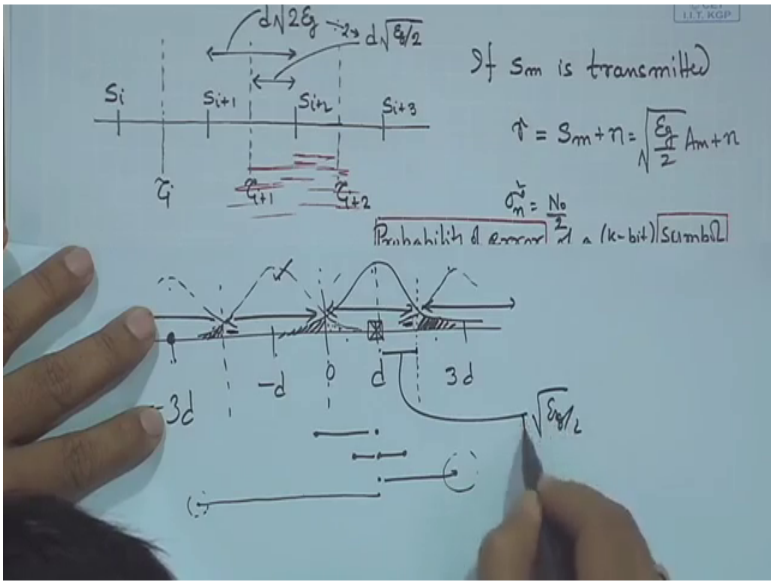

So, just to explain that briefly, suppose this is 0 this is d, this is 3 d, this is minus d, this is minus 3 d. So, if d is sent, I am going to, receive d plus noise if it is positive noise if the received vector goes there if; I have sent this; if it is negative noise, this we will go on this side.

So, now the received vector could be anywhere in any location. So, how does the receiver work? The receiver should work in a way that if it is maximum likelihood receiver and all the points are equi probable. We can extends our results from what we have seen before is the decision point will be mean to weigh between the 2 constellation points, right. So, whenever I have received the signal I will try to find that; which is the nearest constellation.

By the minimum distance criteria and it would be reduced to compare it with the threshold and find the region in which the received vector lies. So, if the received vector lies in this region, I would decide that most likely transmitted signal is the consolation point minus d without assuring what actually has got transmitted. So, if d was transmitted and noise as shifted it here, I would decided it as minus d, if d was transmitted and noise was shifted it here, I will decided at d, I will decided at d, if it is received over here that is huge amount noise been added, I will decided it has 3 d and if huge negative noise is added and it goes there, I will be decode it as minus 3 d.

So; that means, all these cases are of course, errors, but what I am simply doing is comparing with the threshold which lies midway right. So, that is what is mentioned over here.

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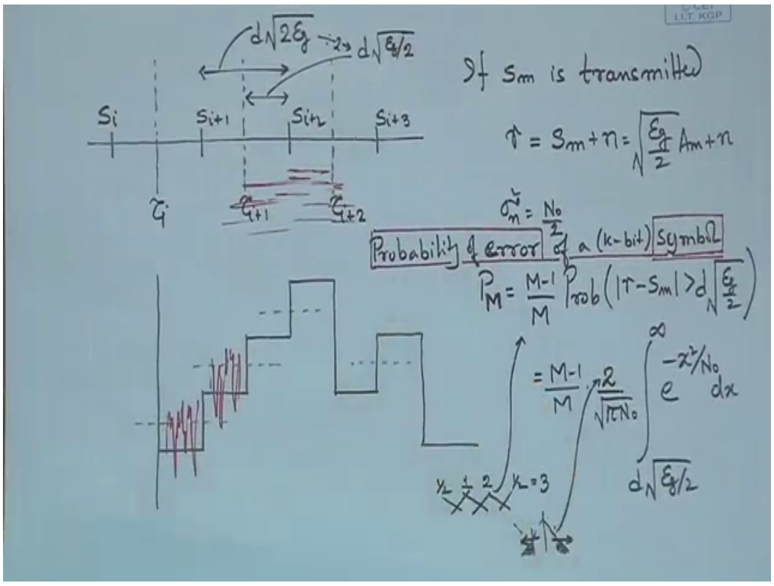

So, if there are m levels there are m minus one thresholds that is what we have and if this distance is d root to d root over 2 E g half the distance is the d root over E g by 2, simply you divide this by 2 and you are going to get this particular number. So, if S m is a signal that is transmitted the receive signal is of course, S m plus noise and S m as we have seen is root E g by 2 times A m.

So, therefore, the received vector is this. So, this is the constellation point and n is the noise. So, because of noise the received signal could be in this fashion anywhere the moment it crosses this threshold, it will be decoded as the next constellation which is an error if it crosses the threshold, it will be decoded as this constellation which is the error if the perturbed signal lies in this range, it will be decide it will be detected as the correct signal.

So, now to decide the error probability, we need to look at the conditional PDF. So, if d was sent, the conditional PDF would be like this and if minus d was sent the conditional PDF would go like this right and so on and so forth. So, now, for this the conditional PDF goes like this. So, d the error probability given d has been transmitted is area under this curve going up to infinity area under this curve going up to infinity right and the area under this curve is same as area under the curve. So, we can say 2 times area under this curve for minus d, it is also same story. So, area under this curve plus area under this curve till infinity where as for these consolation points 3 d, there is only one side that we see over here for minus 3 d it is also one side, right. So, so what you have is one partial PDF one conditional PDF tail 2 times of that one and two. So, that makes one complete conditional PDF this is another complete conditional PDF, this has one side over here one side over here. So, together if we considered we have another PDF with 2 tails.

So, therefore, you basically have m minus 1 because there are 1, 2, 3, 4, such conditional PDFs, but form from which you have 2 PDFs which you are going to take on both the sides and to PDFs which will take on one side. So, you have 3 PDFs for which we are going to take on both the tales therefore, you have m minus 1 or 3 number of PDF to be con considered 3 number of tails to be considered.

So, to calculate probability of error, we have m minus 1 upon M probability that the distance of received from S m is greater than the half the distance that is half the distance; that means, this r minus S m if you look at this r minus S m r minus S m is noise. So, the probability that noise has a amplitude which is greater than d root E g by 2.

So, if noise is an amplitude which is more than this or more than this there is an error therefore, you have taken the modulus sign indicate on both the directions for which you have a 2 and this is the PDF of 0 mean white Gaussian noise and M minus 1 upon m is what we have just now I explained that there are n minus one double sided tails that have to be considered.

So, and the integration limit is of course, if you take one of the tails let us take one of the tails so; that means, this is the distance which is root E g by 2 times d. So, up to infinity and this is the result. So, this again you can identify it has the Q function of this particular expression.

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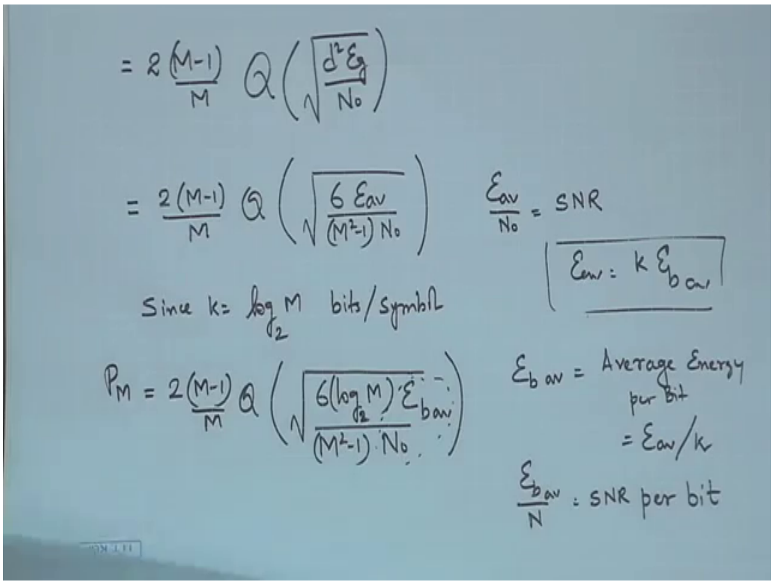

So, what you have is end result is 2 times M minus 1 into Q of this expression. Now again evaluating with respect to the average signal energy which we have calculated here. So, d squared E G that is what we have like this. So, you can calculate the in terms of average energy which looks like this right the average energy if you convert d squared E g from this expression to the average energy you are going to get an expression which looks like this.

So, you have this d squared E g and you have E a v. So, your d squared E g equals to 6 multiply it by. So, from this you have d squared E g equals to 6 E a v upon M squared minus 1 that is what you have d squared E g is equal to 6 E a v upon M squared minus 1 and n naught is of course, there and E a v upon n naught is SNR. Now since k is log base 2 of m, right. So, we can write that E b average bit energy is equal to average symbol energy upon k or you could write E a v equals to k times E b a v, right, we could we could say that; right.

So, in that case, if I write in terms of average bit energy we are going to have due to multiply log base 2 m which is nothing, but the K and then you have E b by n naught you have E b by naught. So, you would like to find the probability of error in terms of E b by naught or as a function of E b by naught. This is an important; not over here that for analyzing digital communication systems, we would refer to take the x axis as E b by n naught and not signal to noise ratio which makes it which makes the comparison different from analogue communications. So, we stop this particular lecture here and we continue further with the performance analysis in the next lecture.

Thank you.