Modern Digital Communication Techniques
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Lecture – 45
Optimum Receivers for AWGN (Contd.)

Welcome to the lecture on modern digital communication techniques. So, we have seen particular realization of the detector, where we have started off with the posterior probabilities; that means, received vector r how would you find which particular signal to be said which particular signal was sent and we needed to calculate the posterior probabilities, we expanded it by the base rule and then we saw the denominator term has probability of the received signal r, which is in depended of a sm. And in the numerator we had probability of r given Sm times probability of Sm which is the prior probabilities.

And then we saw that if the pf Sm are all equiprobable; that means, all signals are equiprobable we have the m p receiver balling down to the maximum likelihood receiver, and we moved on to see how does the maximum received maximum likelihood receiver would look like when we expand the likely would function, we are taken the log of the likelihood function we expanded it and we saw the there is a distance metric that is inherent in it, an maximization of the log likelihood function is same as minimization of the distance metric of the received signal, with respect to a particular Sm, and they we will be saw that the distance metric is computed by considering the vectors.

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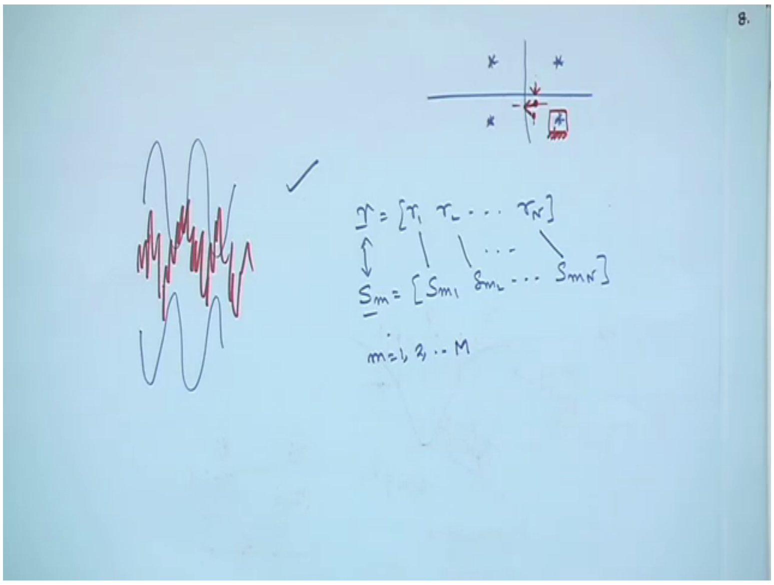

So, you could think off the vector r as r 1 r 2 r n produced by the output of the matched filter or the correlator and Sm as Sm 1. So, up to Sm n which is which are the components.

Of the signal which is already pre known. So, if you have to calculate the distance between these 2 take away with take the difference square it take the difference square it up to all the terms and add then together you get the distance of this vector from this vector and then you keep changing m equals 2 1 2 upto capital m, and then you choose that m which makes its distance minimum that is one possible way and then we look further and we saw that you do not need to calculate the length of the vector because when you are calculating distance you have the non square of the vector or non squared of the vector Sn a mth signal and then there is a correlation component.

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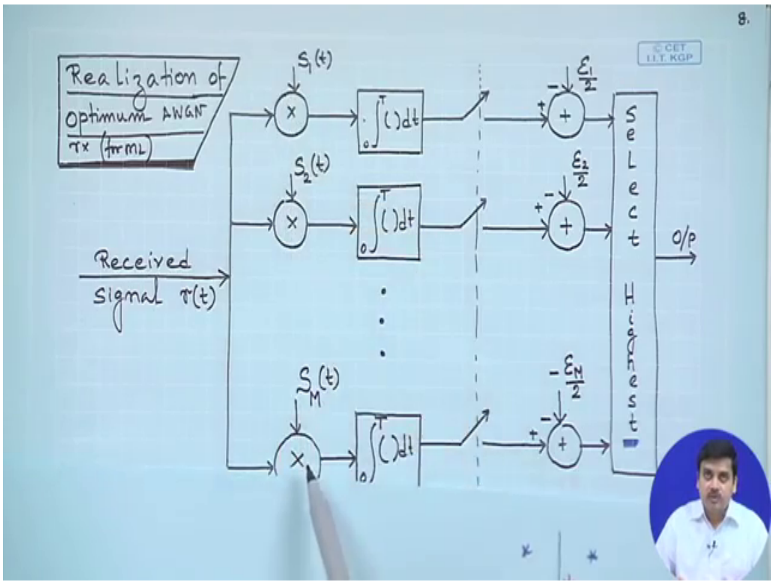

So, since a non squared of r is common for all Sm. So, we took that out and we saw that there is a correlation component as well as there is a correlation component and there is an energy component of the signal, which could also be looked at as the bias term. So, when you look at in the integral form we had the correlation of the received signal r with the different possible wave forms take away the bias term of each possible wave form and select the one which maximizes right.

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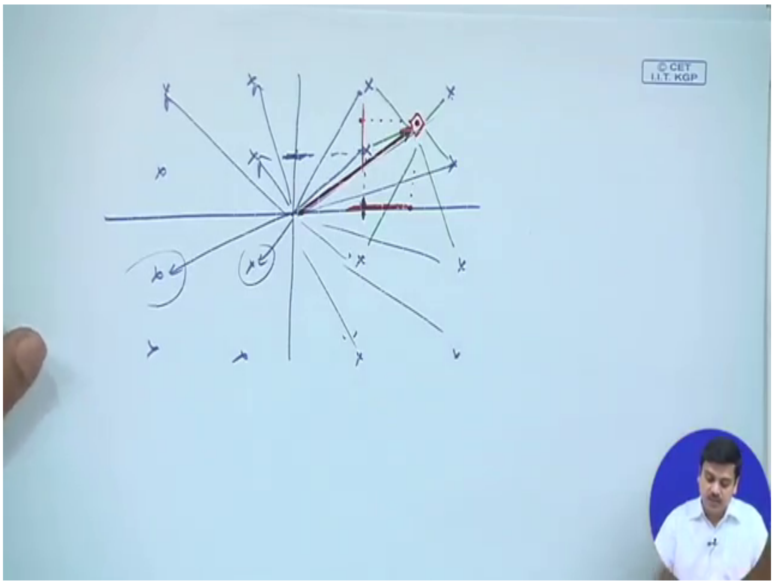

So, these are the 2 ways of looking at it so; that means, we have the minimum distance or maximum correlation. So, in terms of minimum distance we briefly stated that suppose you have these constellations that are to be sent. Now this can be thought of as having a component of the i and thought of the component on the q. So, as if on the i channel this was sent on the q channel this percent and because of noise this got shifted somewhere there and because of noise this got shifted somewhere here.

So; that means, the received signal would be at some location which is here, the minimum distance metric tells that you have to compare the distance with all possible vectors. So, these are the different vectors right and you have to see the distance of the difference which is the nearest distance to it tell the shortest distance we identified as the outcome, we also saw this could be calculated as the projection of these vector should rather this color. The projection of these vector on all possible vectors and take away the bias term of the signals. Now from this picture it should be clear why we talk about taking over the bias term because the energy of the signal is different from energy of this signal. So, you have to take away the bias term and take that output which is the maximum of correlation; that means projection of this on a vector and taken away the bias term is the one of the possible realization.

So, we have seen 2 possible realizations, but the end result is given a particular received we have to identify which particular signal was possibly send which is the possible waveform or which is the possible constellation point, which could have resulted in the received signal or the received vector.

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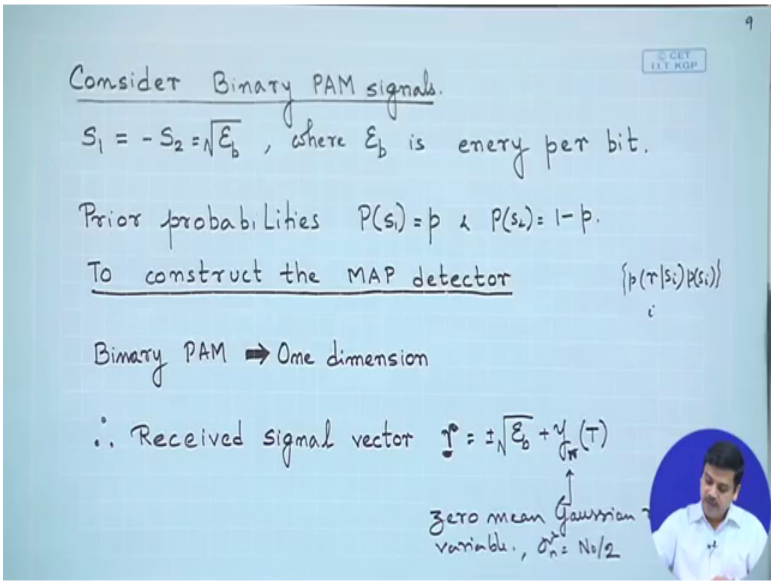

So, we moved on further with this, and we will consider binary PAM signal for simplicity let us move to a binary PAM signal. In the binary PAM signal you have 2 signals S 1 and S 2 this is well understood we have been discussing this we have the picture here s one and S 2 and where E b is the energy per bit.

So, we also have the prior probabilities P of S 1 being p and since there are only 2 symbols the probability of the other symbol being sent is definitely 1 minus p. Now we have to construct a map detector for this particular example. So, to construct the map detector we must remember that we are dealing with a binary PAM which is one dimension right so; that means, there is the matched filter output projects the signal on one of the dimensions and it gets a single component. So, the received vector received signal r is can be written as plus minus root over E b because your signals that we have here are root over E b n minus root over E b, which is also described in this particular statement here.

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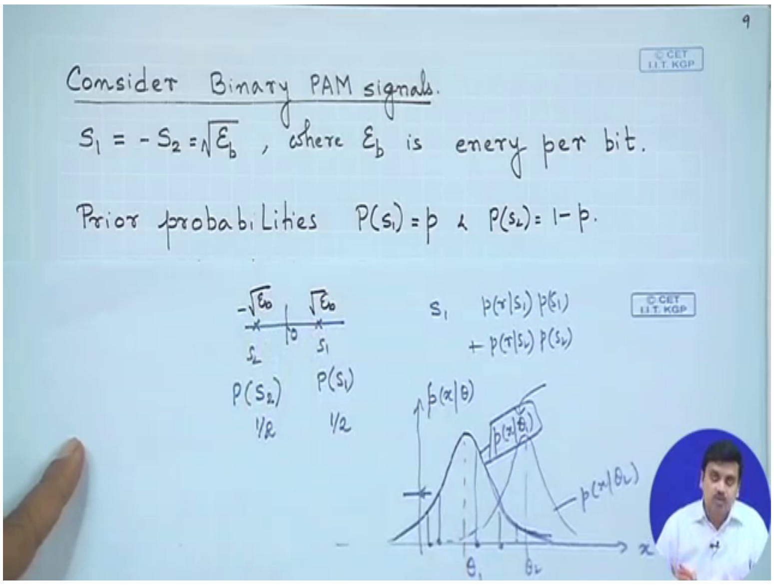

So, when we have the received signal has either plus or minus 2 t b; that means S 1 or S 2 and there is the noise component. So, this is the outcome of the matched filter at the end of the sampling time t. So, this is the received signal vector.

So, this part we have already seen it is 0 mean with the variance of n naught by 2 in our early discussions. So, when we have to construct the map detector remember we have to construct p of r given his i multiplied by p of S i this is what we had said and find that i which maximizes this right.

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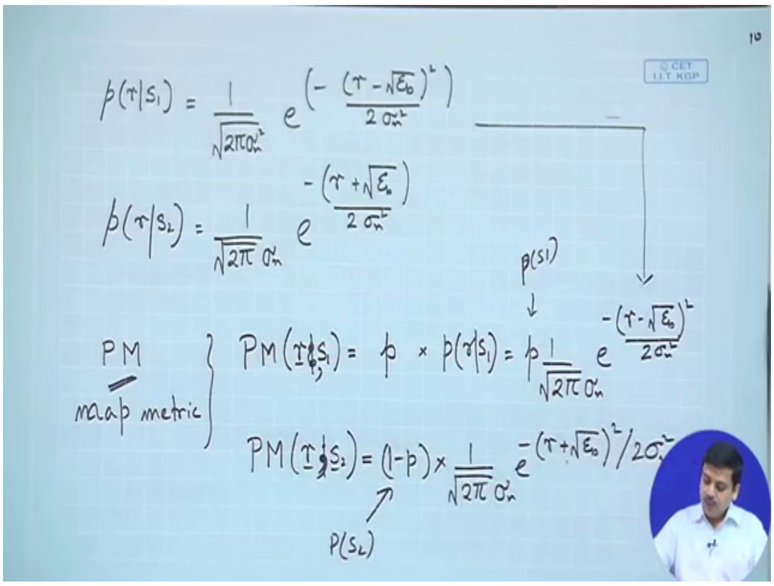

So, just reminding what we had discussed earlier. So, we need to calculate p of r given S 1 and p of r given S 1 is simply r minus root E b of because this is mean of the signal and this is distribution of white noise P of r given S 2 would be plus E b in one case you have minus E b because in one case the mean is E b the other case the mean is minus E b. So, r minus E b is plus E b that is what we have.

So, now the map metric would be PM of r with S m S 1 right e probability metric of r and S 1 would be P that is the probability of sending S 1 multiplied by probability of r given S 1 right. So, that is P because of this P multiplied by whatever we have over here. So, this comes there and this is P of S 1 right

And then the probability metric that out posterior metric that we have to calculate s p of r with S 2 is 1 minus p because this is P of S 2 applied probability if p is the probability of S 1 1 minus p is the probabilities of S 2 we already said that, and this we again have from p of r given s two.

So, now our job is 2 select S 1 or S 2 depending upon which particular metric gives the higher value.

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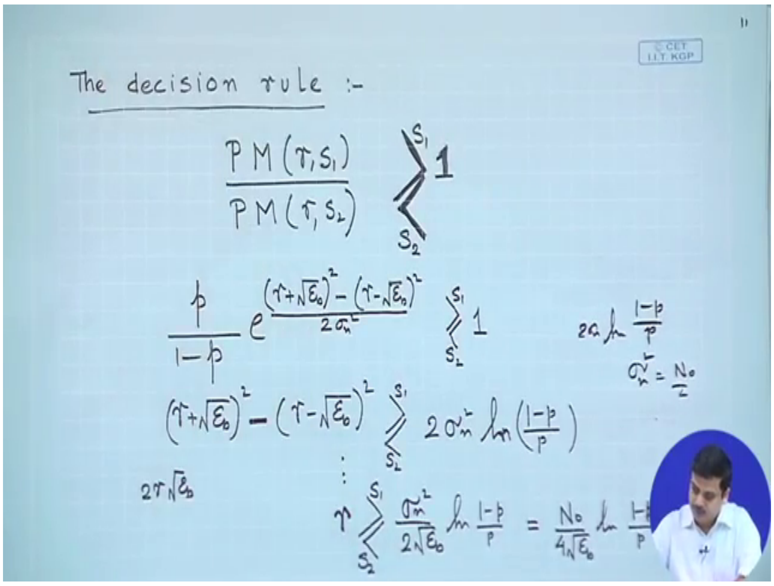

So, we could say that we have a decision rule in a way that we take the ratio of the metric of r on S 1 and r n S 2 and it is also sometimes called as the likelihood ratio test. So, these are the likelihood functions of r with S 1 and r with S 2 and we are taking the ratio of them. So, if this ratio is greater than one my decision would be S 1.

So, that is why we have if it is greater than one it is S 1 and if the numerator is less than the denominator; that means, if the denominator is greater then you have it less than one right or in other words if P of M r S 1 is less than P of M r S 2 then definitely S 2 was the transmitted signal and as per the decision rule. The decision rule tells we calculate these probability metrics which ever probability metrics gives the higher value you choose that particular signal.

So, now we have computed these probability metrics here we have already computed this probability metrics. So, we are saying you divide them take the ratio if you take the ratio.

You going to have p upon 1 minus p this and this would cancel out this particular term would cancel out with this term and in the numerator you going to have this term minus this term right. So, that is what we have the and since it is in the denominator it goes up. So, there is a plus sign of course,. So, you have p 1 minus p that is could be just stated e to the power of this term which has come from the denominator minus this term right.

And the right hand side remains the same; that means, if this is greater than one you would choose this to be S 1 if it is less than one we choose the signal to be transmitted as S 2 and that is the output of your decision device. So, what we have in the next step is 1 minus p upon p. So, 1 minus p upon p and since this is e. So, when it goes to the right hand side you have an ln. So, you have this term ln of 1 minus p upon p and this denominator to sigma n squared is getting multiplied. So, we have this term.

So, you left with only this distance on the left hand side on the right hand side. So, you have this. So, again reminding that if this is greater than this our decision is S 1 if this is less than this our decision is S 2. If you do a few steps algebraic operations you going to get r squared r squared terms which are going to cancel out, then you going to get 2 r root E b and they will be 2 root r E b. So, you are get r multiplied by E b and then you have E b squared you have sorry E b and E b and they will also cancel out. So, will 2 left with 2 times r E b multiplied by 2 and there is 1 factor of 2. So, 1of the 2 cancels out. So, you are left with 2 r root E b right.

And then you could bring the 2 E b 2 root E b on the right hand side you will be left with if r. So, what is r? R is the output of the matched filter in case of PAM if it is greater than the right hand side which is here then your decision is S 1 if it is less than the right hand side your decision is S 2. So, the right hand side again you could say that sigma squared n is n not by 2 we had chosen that sigma square n is n naught by 2.

So, what you have is n naught by 4 and expression as it is. So, you could say whatever is the outcome of the matched filter if it is greater than this expression I would choose it as S 1 and I would choose this as S 2. So, what does it mean let us take a look at this right. So, as if you have this with thus now things will be bit better.

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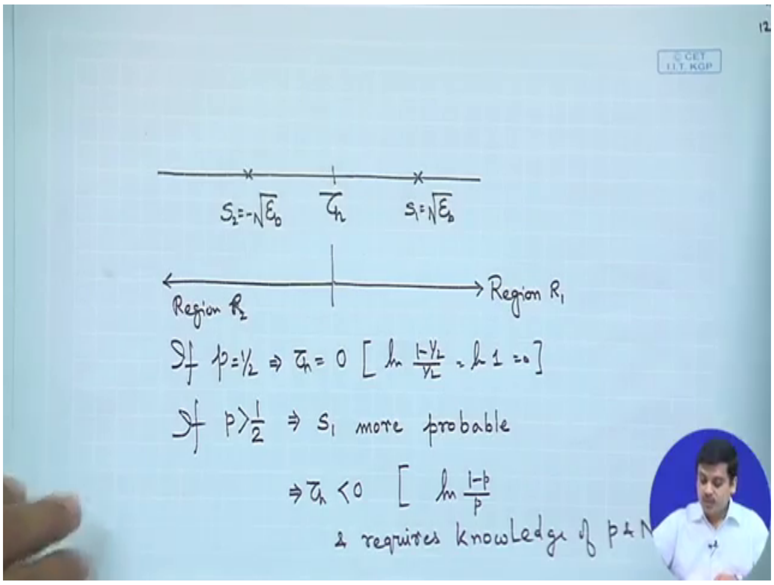

So, as if this is my single space and we said that one of the signal is root E b the other signal is minus root E b.

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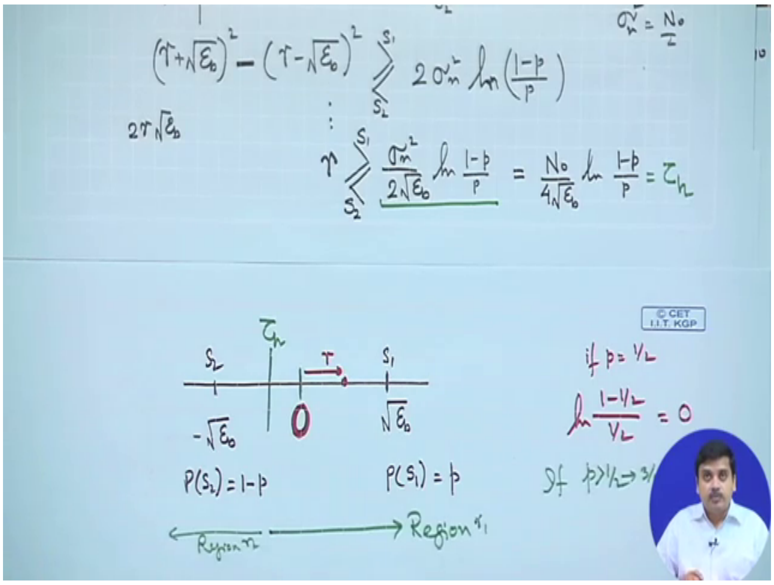

And now this has a probability of. So, this is S 1 this is S 2 probability of S 1 being sent as p and probability of S 2 being sent is definitely 1 minus p, and then the matched filter produces sum r. So, if you have let us say S 1 which has been sent suppose some particular signal has been sent noise will corrupted and it will be somewhere. If S 2 would have been sent noise would have got added and it would have been somewhere. So, now, suppose which one is sent we do not know, but we have received an r which is here. So, let this p r. So, what it tells is you compare r with this particular value right and if r is greater than this value then you choose to be S 1 and if r is less than this value you chose it to be S 2. So, now, let us take look at how this is important. So, if p is equal to half. So, if p is equal to half so; that means, you are going to get 1 minus half upon half and ln of that. So, which will be 0. So, 0 on the right hand side; that means, our decision r will be compared with this point of 0.

So, if r is greater than 0 then you would choose it has S 1 and if r is less than 0 you would choose it as S 2. So, this also confirms to the nearest distance of the minimum distance criteria. So, we could say that this is the region for this particular case if instead of this situation you have suppose I would rather make this with this colour to indicate the same. Now if you have p to be greater than half right if p is greater than half; that means, S 1 is getting transmitted more often than S2. So, if S 1 is plus 1 S 2 is 0 that was we have a sequence were there are more ones than zeros right.

So, p is greater than half. So, p is greater than half you would look at this expression the numerator is less than the denominator right p is more than half numerator is less than half 1 minus something more than half denominator is half so; that means, log of some number which is less than one therefore, this is negative right therefore, the log of a number which lies between 0 and one is negative so; that means, in that case you have. So, this right hand side this thing we could mark it as tau h that is a threshold, in that case we could say this is let us say tau h

So, if r is greater than this you would select it as S 1 if r is less than this you would select it as S 2. So, now, we can see that if in this case we know the prior probabilities then the way we implement our receiver would be you simply get the output of the match filter, compare it with the threshold which requires the knowledge of n naught E b and p right p is the probability of transmitting a particular constellation. It requires n naught E b because accordingly things get scaled.

So, if p is some value greater than half let us say it is 3 by 4 say it is 3 by 4. So, this has a particular value and this is going to scale that. So, depending upon a certain E b by n naught your th is going to vary right. So, you need the ratio of E b with n naught and you need the value of p which is going to produce the value of t h you going to compare r with the th and then you are going to make a choice that which particular signal was sent; and we would call this region r 1 let say and we will call this region let us say region r two.

So, if r false in the region r 1 we would say S 1 sent if it falls on the reaching r 2 you would say S 2 has been sent. Now suppose you have certain prior probabilities, but instead of implementing the map receiver you start to implement ml receiver. So, if you implement the ml receiver; that means, you are going to choose p as half equal probable; that means, this term is going to be 0 th will be 0. So, now, whatever is the transmission probabilities you will take the received signal, and compare it with the midpoint mid distance between these 2 constellation points right. So, you are going to get a certain probability of error which will be different from the map receiver; only when the probability of transmitted signal is equal to half then only the ml receiver.

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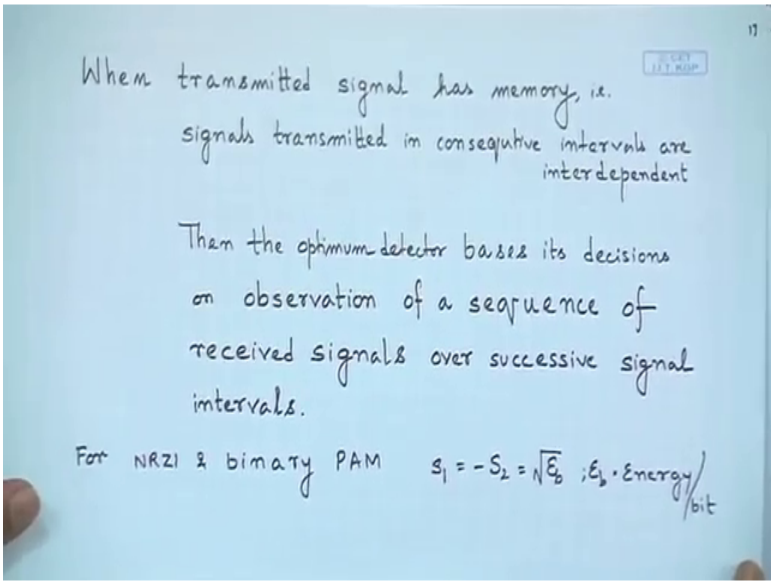

And the map receiver is going to produce the same outcome of the same performance in terms of error probabilities. So, with this we would like to move on to look at another interesting thing.

So, and look at how the receiver structure would be like. So, now, we would like to take a look at the situation, when the transmitted signal has memory. So, when the transmitted signal has memory the signals transmitted in the consecutive intervals are interdependent right definitely. The output now is dependent on the output before plus the incoming information sequence in that case the optimum detector which we have just discussed should base its decision, on observation of the sequence of received signals over successive interval.

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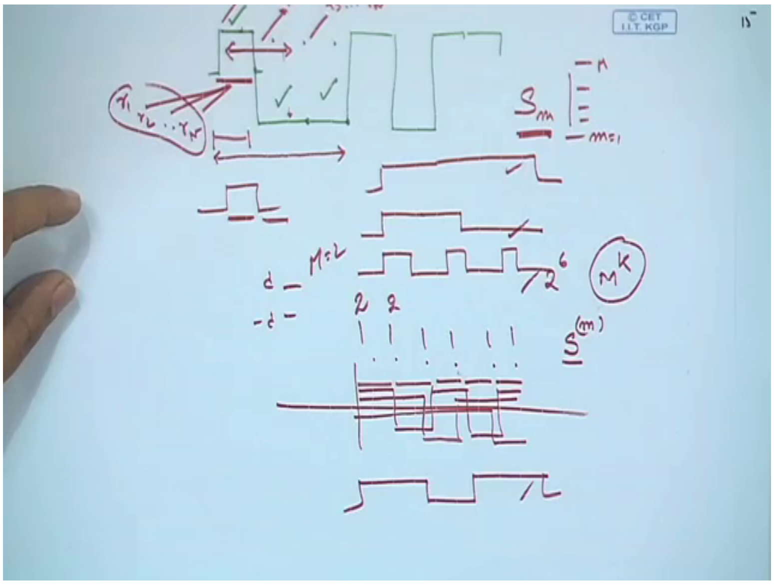

So, what it means is that suppose I have a signal which is present here followed by a signal there and then maybe the next signal is this one, in the previous case we said that this signal is independent of the signal which is independent of the signal and so on so forth. But now we say that this signal is dependent on the signal. So, there is some kind of an x or operation or some other thing for example, we use NRZI and things like that where you use the prior output which influences this output.

So, the original information sequence is actually spread across more than 2 symbols. In that case all it says that when we design the receiver we should not base our decision on only one observation interval, we should base our outcome by looking at several observation intervals in a sequence. So, like we have to do the similar thing that we have done before, but we are going to do for multiple signals multiple signal intervals.

So, this is example of NRZI were you have this x or operation and let us take binary PAM modulation S 1 and S 2 as. So, the NRZI signal is the information bearing signal when there is an x or operation and the outcome is still a 0 and one which is used to modulate the PAM right.

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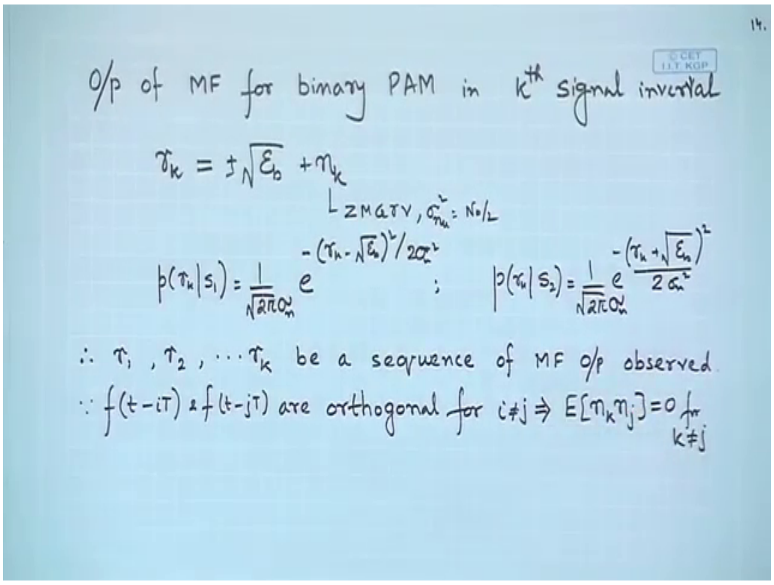

So, using that the output of the matched filter in the k th interval to please not we are changing the notation slightly. So, now, k is not indicating the n components it is just the k th signal interval because we are using so many variables it is very difficult to have unique notations we are reusing one of the notations.

So, r k the receive signal in the n th interval not the k th component is equal to plus minus root E b plus n k which is same as what we had indicated before and description of n k is the standard description that we have.

Now if S 1 was sent probability of r given S 1 is what we had seen earlier, and probability of r given S 2 is this which is here is a minus sign here is a plus sign we had seen it just a few moments before.

So, now r 1 to r k be a sequence of matched filter outputs observed right. So, I would like to remind once again. So, r 1 to r k would be the output of the matched filter at this point r 1 this point is r 2 at this point is r 3 and you will have r k right.

So, these are not the components of the received signal, earlier we had the received signal in this interval and this was decomposed into its components r 1 r 2 up to r n. So, the notation might be confusing. So, please do not get confused with the components we are talking about sequence of matched filter output observed; and f that is the basis functions are orthogonal for 2 different time intervals; that means, if i and j are not equal they are orthogonal right and that is clear because.

The basis function is here the basis function is there. So, the basis functions are like this. So, this is orthogonal in this interval right that is what simply says and the noise in these 2 intervals the noise component here and the noise component here are again uncorrelated and hence independent right that is what we have.

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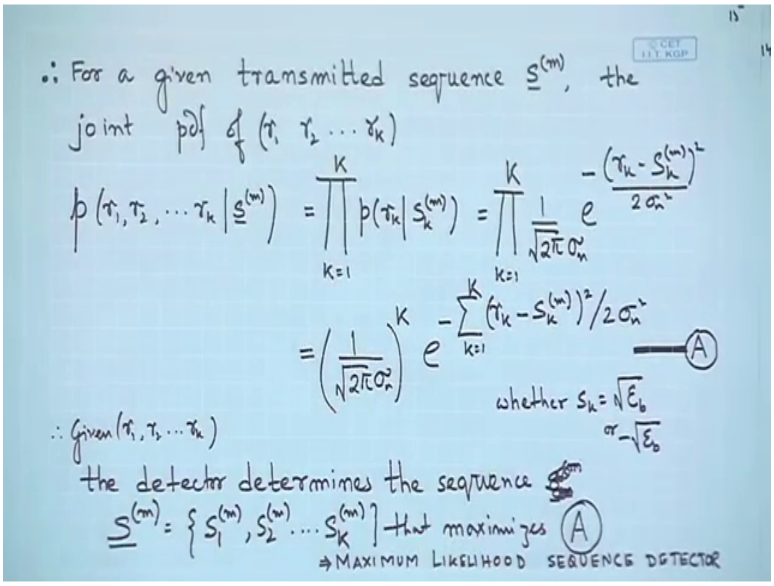

So, now if we considered a given sequence Sm, so we now have m in the power to indicate it is a sequence earlier we had written sm. So, please note this notation is not the same here you have m possible different waveforms, but here this is a sequence; that means, one signal another signal another signal another signal. So, we are sending this signals out then the joint pdf of these received signals in these k observe interval since we said them to be independent noise is independent and this basis functions are orthogonal similar conclusions as we had for the components the logic is a seminar and the condition the similar. So, the conditional distribution would become the sorry the joint distribution would become the product of the individual distributions.

So, now similarly as we had before the expression looks quite similar except the meaning of these terms are different than what we had used before. So, now, we have given the sequence of observations r 1 to r k the detector determines the sequence S 1 S 2 up to Sm

k. So, look at this we have these different signals in different intervals that have been sent right and this combination is the sequence Sm right.

So, if I have let us say a 2 symbols and I have 2 possible symbols lets a plus d minus d and I observed over one 2 3 4 5 6 possibles possible intervals. So, here there will be 2 possible sequence 2 possible options 2 possible options to you have 2 to the power 6 possible waveforms was 2 to the power 6 possible sequences. One sequence is all plus d another sequence is probably a plus d a minus d a plus d a minus d and so on and so forth. The other sequence could be then another sequence could be. So, these are all the different possible sequences and each of the sequence is indicated by this. So, note in this case our capital m is two; however, we have 2 to the power of. So, this is M to the power of k possible such combinations. Similar thing we are similar logic we had used in source coding. So, I mean should not be confusing at this stage.

So, now if you look at it what is trying to do it is trying to find we have the similar expression as we had for mp receiver or a ml receive. Now what we have over here is the distance of the received sequence from a particular sequence we are not talking about a particular symbol, but we have a particular sequence this is the distance right detector determines with sequence which one of these sequence. So, one sequence could be all plus ones one sequence could be like this another sequence could be.

So, the detector another sequence could be. The detector is trying to find out which of these sequences ah could have been sent and hence this kind of a receiver is known as the maximum likelihood sequence detector. So, it is not just the likelihood detector it is a maximum likelihood sequence detector. If you had to use the prior probabilities then you had to multiply over hear the probability of each possible sequence not generally again we are assuming here if all sequences equi probable then we need to just look at the particular possible sequence that could have generated this particular outcome.

So, this particular part that we have discussed in the last part of this particular lecture is just for your information sake because this kind of maximum like of detectors are used in complex receivers, where you have some kind of (Refer Time: 29:55) structure or memory that gets involved in the transmission or maybe when it passes through a channel and you would combine and you would like to decode the signal simultaneously with respect to channel equalization. So, this particular part is for your advanced knowledge state knowledge, but you may not use it in a very simplistic receiver and there could be other ways of extracting the benefits as shown by this kind of receivers.

Thank you.