Modern Digital Communication Techniques
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Lecture - 43
Optimum Receivers for AWGN (Contd.)

Welcome to the lectures on Modern Digital Communication Techniques. So, we have covered a very very important aspect in the previous lecture.

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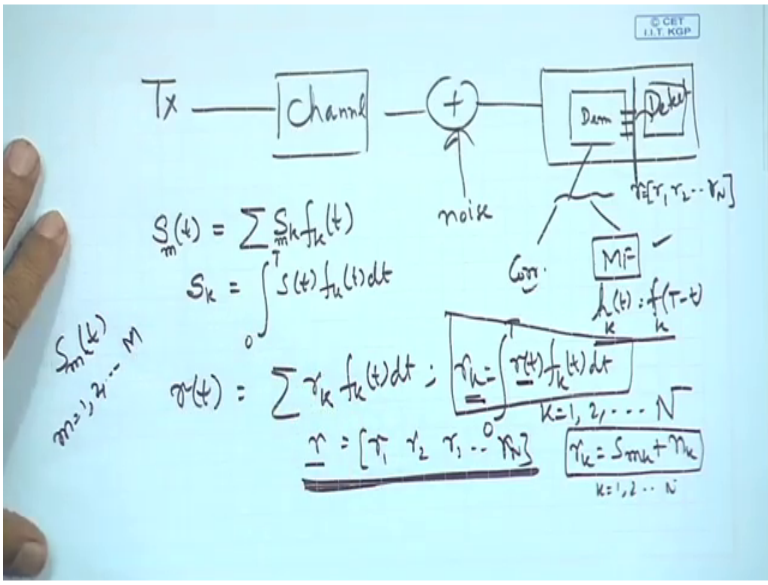

And summarily if you would like to see what we have done so far in a nice way we could briefly state that- we have started with the transmitter, we have seen one channel at least, where there is additive noise and we said that this is not completely theoretical it has practical usage. So, it is not simply your fancy that what we are doing here. And then not only that we have started looking into the receiver and at least we have covered a stage where we have reached the first part of the receiver which is the demodulator.

So, in a demodulator we have seen 2 different kinds the correlator and the matched filter and what we see the matched filter uses an H the impulse response which is the flipped version of the correlator function and rather it is H k that is the k-th filter is that of the k-th correlator or the basis function and we have seen one of the important characteristics of matched filter is that by choosing an impulse response which is matched to the signal in our case matched to the basis function what you get is the output of the matched filter at this point has maximum signal to noise ratio.

So, this is a very very important stage where we are. So, once we discuss the detector then at least we have travelled one path from the source encoding transmission of signal through modulation techniques passing through an AWGN channel. And the receiver procedure through which one should be able to realize a basic transceiver system what we have not covered is non ideal situations where there is channel estimation required channel equalization required and carrier synchronization and clock synchronization required which will do after we complete the receiver discussion as well as study the performance evaluation in terms of error probability once we complete the AWGN receiver.

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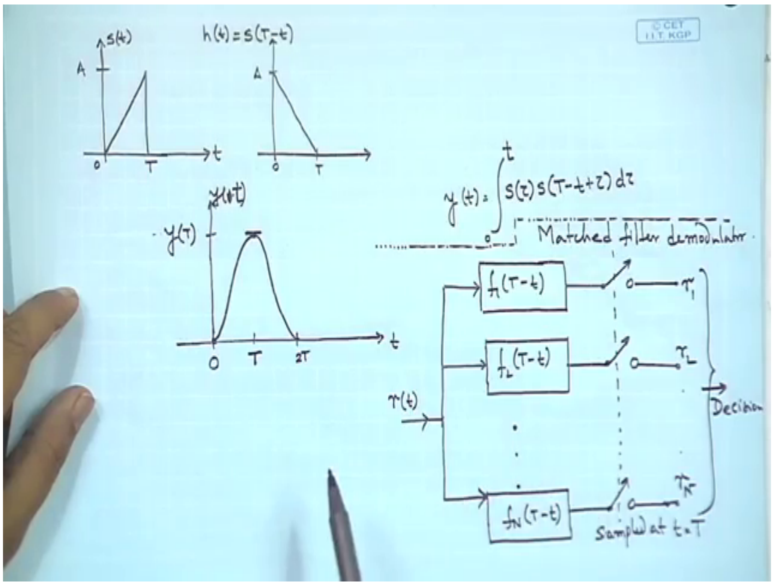

So, proceeding further this briefly summarizes what we have done earlier. So, it is the same picture for received signal send to a parallel bank of filters each filter uses the basis function in the flipped fashion as the impulse response when it goes through the filter there is convolution; convolution of the flipped function is as good as correlation. So, the output at the sampling time t which is the symbol period we get the component of r on this basis function. So, we get all the components that are desired.

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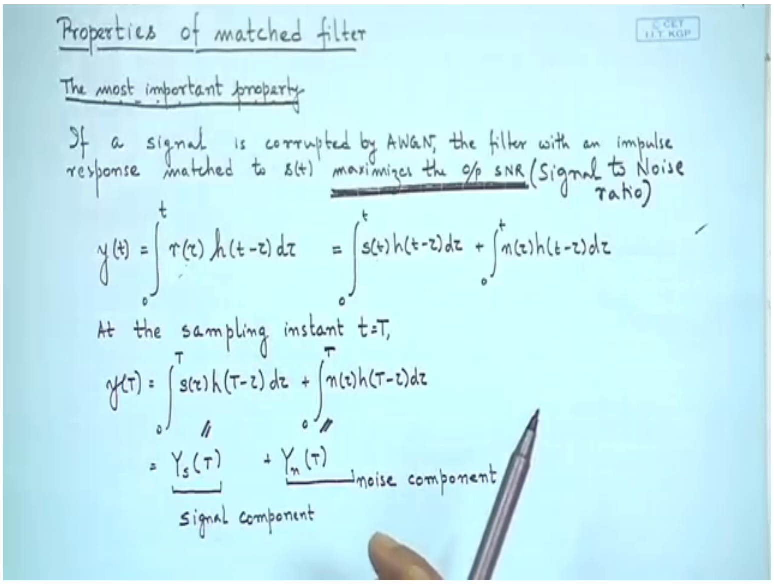

So, if this is the signal the matched filter would be the flipped version that is what we have here. And when we discuss the properties of matched filter we looked at the filtering operations is convolution. And then received signal is broken into two parts the desired signal and the noise component. And then be computed the signal to noise ratio of the signal component and the noise component where we define signal noise ratio as the power of the signal to power of the noise.

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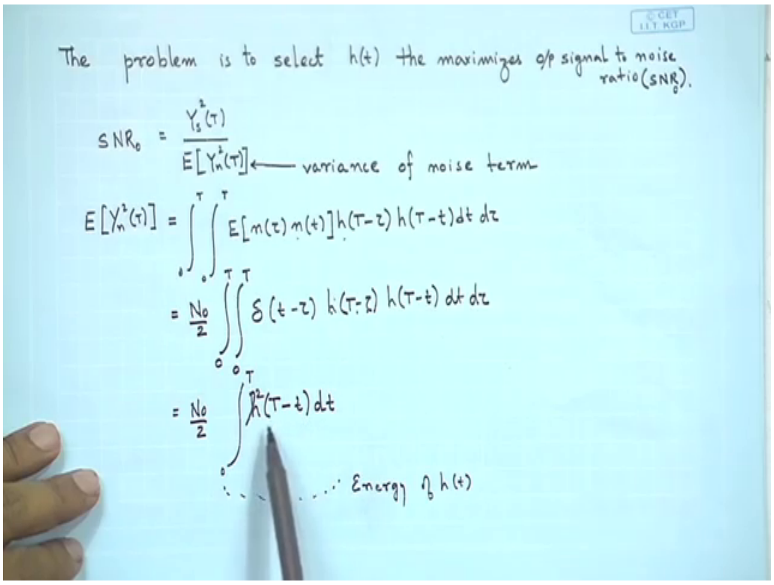

We computed the noise power which turned out to be n naught by 2 multiplied by the energy of the impulse response. So, as long as the energy is kept constant the denominator would be kept constant and our problem was to maximize signal to noise ratio at the output. So, that to maximize the signal to noise ratio at the output by choosing H of t, because signal is not in our control only H that is impulse response is on our control. And then we see the denominators if it is to be held constant then we are only left with the numerator.

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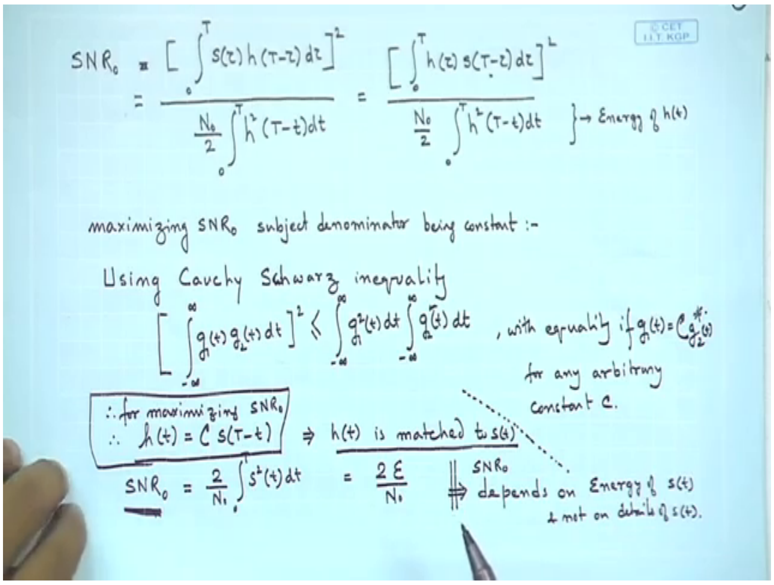

So, when we look at the numerator that is this portion we use the Cauchy Schwarz inequality using the Cauchy Schwarz inequality we find that the maximum value of this is attained when one of the functions is a scalar multiple of the other function.

So; that means, H of t should be equal to some constant times S t of tau so; that means, to maximize you need to have the filter which is matched to the signal we already started with the premise that let H be matched to the signal. And now we are saying that if you want to maximize the output of the matched filter you must choose impulse response in this fashion. So, what we have here is if you choose it in a matched fashion your SNR is maximized and the maximum value of SNR is 2 E by n naught where E is the energy of the signal this is a very very important result in the study of digital communications which we should remember.

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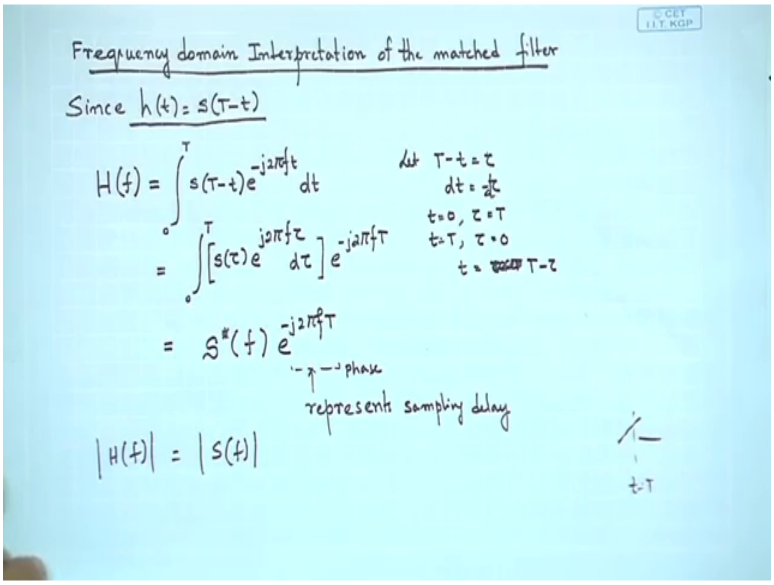

So, moving down further let us look at the frequency interpretation of the matched filter. So, what we have is since H of t is matched to the signal H of f which is obtained by Fourier transform of H of t. So, we have the Fourier transform of S t minus tau here and then by change of variables you could do a sequence of change of variables you would find you could write this had S of tau E to the power of j 2 pi f tau right and recall our discussion in the beginning where we stated some of the Fourier transform relationships. So, this is equivalent to the Fourier transform conjugate of S of f.

So, S of f is a Fourier transform of S tau. So, S conjugate of S f would be something like this. So, and there is a phase term of course, there is a phase term E to the power of j 2 pi f t you easily get to translation of variables change of variables. So, this phase term is because of the sampling delay because we sample the signal at the end of a sudden sampling duration which is t. So, you remember that there was a sampling gate which we say at at t equals to capital T you sample the signal and absolute value of H is the same as absolute value of f. So, there is no problem to it.

So, this is the frequency domain interpretation of the matched filter and you could calculate the signal to noise ratio even in the frequency domain the results would turn out to be the same.

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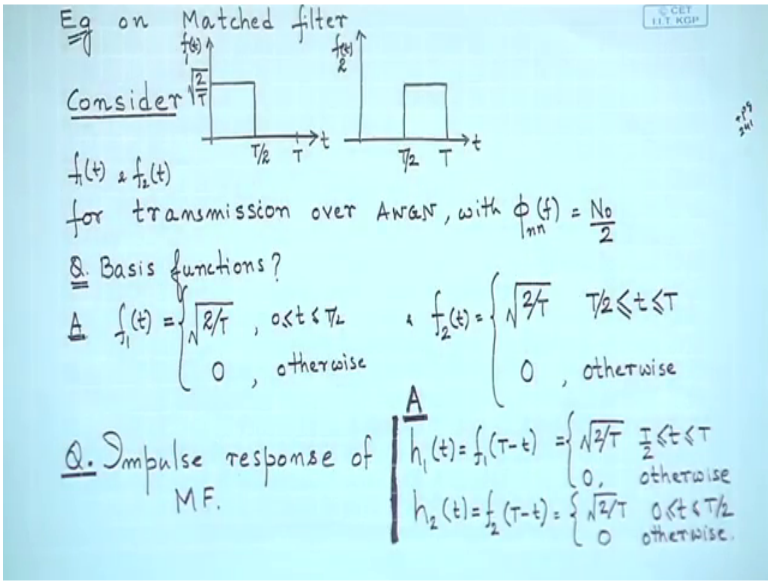

So, these are some of the important things we should remember. With this let us move forward and let us take an example on matched filter. So, to take the particular example what we do is we consider 2 functions f 1 and f 2. So, this is f 1 and this is f 2. So, f 1 is having a value which is root over 2 upon t; up to t by 2 and it is 0 for the period t by 2 to t f 2 is 0 in interval 0 2 t by 2 and it is nonzero or having the same value as f 1 in the period t by 2 to t.

So, looking at these 2 one of the important conclusions that you can directly make is that these are orthogonal to each other clearly because if I multiply f 1 with f 2 and integrate for the period 0 to capital T what will you get 0 there is a nonzero value, but there is a 0 value. So, up to this point this has nonzero value; however, this has 0 values. So, therefore, the product is 0 and from t by 2 to t this has nonzero value, but this has 0 combining you have a 0 value in the entire range. So, that clearly means it is orthogonal and otherwise it is visibly also available. So, this would also mean that one could choose this as the basis functions as well so, which is different realization of a basis function you have seen basis functions in the I n q domain; that means, one that modulates the cosine career one that modulates the sin career.

But now you are seeing that you could find basis functions in the time domain also. So, it is not new if you are very careful you will note that we discussed n dimensional signaling when we discussed n dimensional signaling.

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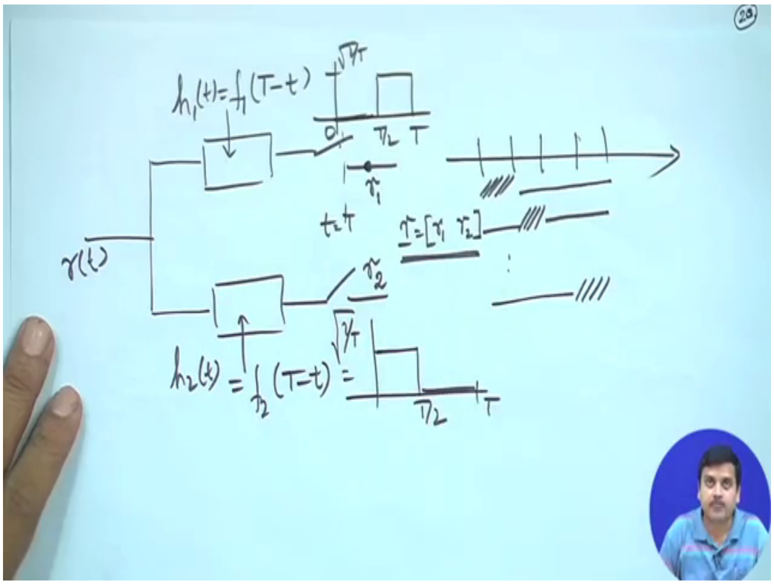

We said that certain duration of time could be divided into time slots and you could send a signal in one of those time slots do not send anything there you could send signals in one of the time slots and not send anything there and like that you could have it in this manner right. So, if you have it in this manner in that case these signals are orthogonal to each other and you have n signals. So, we have 2 signals which are orthogonal to each other and created in the time domain.

You could in this fashion create a longer duration of time and divided it you could also create n orthogonal signals. So, this is in the baseband form. So, f 1 and f 2 are used for transmission over AWGN; that means, to send signals over AWGN you could either choose f 1 or f 2 to indicate the presence of a 0 or presence of one. And of course, what we have is the power spectral density is n naught by 2 and if the question is what are the basis functions. So, clearly as discussed just now these are the basis functions already and represented mathematically it is f 1 of t is equal to this amplitude root over 2 by capital T it is a constant in the interval 0 to capital T by 2. So, that is it is a constant value in the interval 0 to t by 2 and 0 otherwise and f 2 is the basis function which is this constant amplitude root over 2 by t.

In the interval t by 2 to t and 0 otherwise, right; so, this is mathematically represented and then we would ask; what is the impulse response of the matched filter. So, the impulse response of the matched filter would be very interesting here. So, you have we have to get back to the matched filter structure. So, if there is a receiver we should have there are 2 basis functions. So, we should have 2 filters and this clarifies some of our doubt that this one should be f 1 of t minus t and they should have the impulse response f 2 of t minus t. So, this is H 1 of t so; that means, we are using impulse response as the basis function and flipped version right. So, that is exactly what we have that is H 1 of t is equal to f 1 of t minus t.

So, if you put t minus t over here. So, you have to put capital T minus t over here and that you will be getting as root over 2 by t for the interval t by 2 to t. So, if you are going to flip it around t. So, it is going to go there right. So, we can put the values and you will clearly get this thing for same for f 2 t the second impulse response would be f 2 t minus t. So, f 2 t minus t would take this value for the interval t minus t lying in this interval right small t minus capital T lying in sorry capital T minus small t lying in this interval. So, if you put that inside this you are going to get that for the interval small t in the interval 0 to t by 2 it is nonzero. That means, you want to get H 2 t as sorry you could you could draw it here.

So, you would get H 1 of t as this thing and H 2 t you are going to get it as this and here the value is root over 2 by 2 here it is root over t by 2 here it is 0. So, this part it is 0 look at this the matched filter is this one and for this one the matched filter is this one right it is 0 over there all right. So, you could construct matched filter in this fashion.

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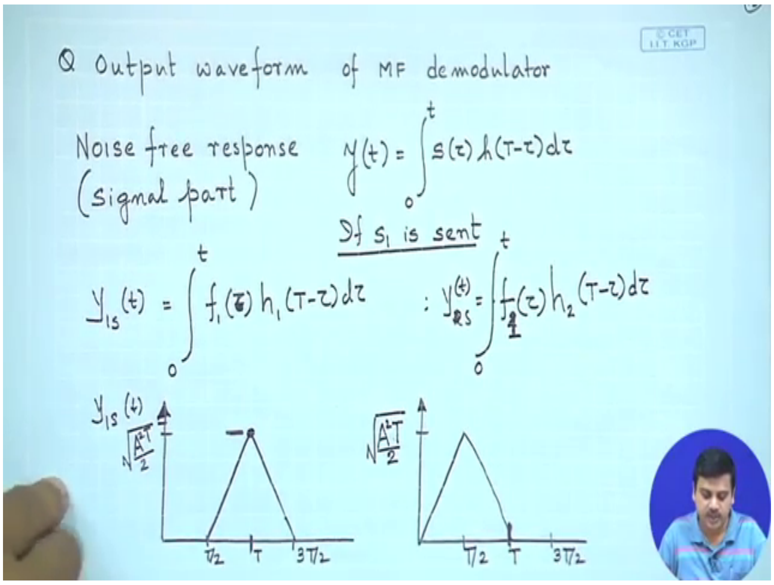

And then the output waveform of the matched filter if you have to see; so, there is noise of course one could see the noisy response, but what we are interested now is the noise free response; that means, only the response of the signal that is what we want to see over here. So, it is the outcome of the convolution from 0 to t S of tau H t minus tau so, because this is convolution.

So, if S 1 is sent Y 1 of S will be equal to; that means, output of the first matched filter will be f 1 of t right; that means, this is the signal that is sent convolved with this one and the output of the second matched filter would be the signal that is sent if for signal is sent convolved with the second matched filter right. So, what you are going to get if you solve this then you would if you do work it out the first matched filter will give you an output like this. And the second matched filter will give you an output like this which when sampled at t equals to capital T the first matched filter is going to give a maximum value which is a square root of t by 2 or square root of a square t by 2, whereas, the output of the second matched filter will be 0 and that again tells you that the second matched filter is orthogonal to the signal.

Because at small t equals to capital t; that means, at this point the signal value would be 0 whereas, this one the first matched filter is going to give the peak value. Remember we found that is going to if the highest signal to noise ratio. So, noise power is constant, but signal power is maximum only when the integration period is capital T.

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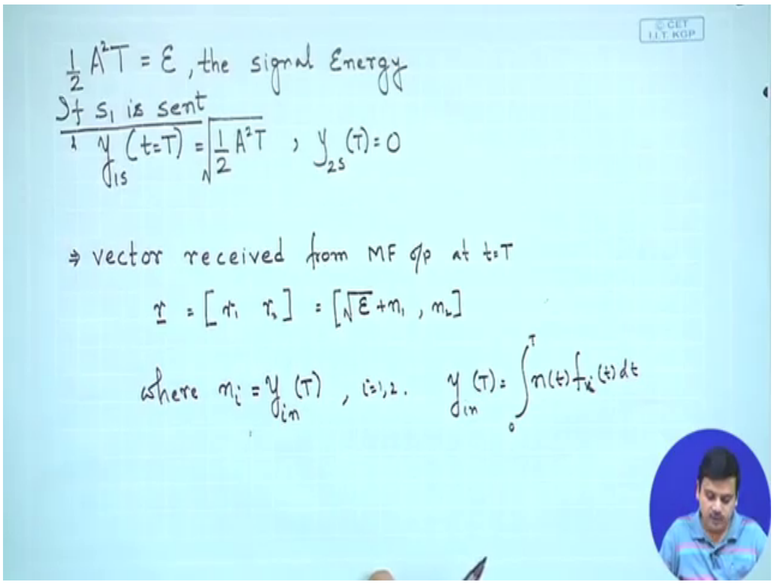

So, these are the outputs that you want to get from the matched filter and of course, in the previous page we had this in the Y axis which is the energy. So, you could calculate that and this clearly we have shown in the graph in the previous page we have clearly shown this in the figure here right. So, that should help Y one S at t equals to capital T is equal to this value and Y 2 S at t equals to capital T is 0 that is what you have over here it is having a 0 value, right.

So, now in the vector form this is important. So, the vector received from the matched filter remember we are trying to draw r of t right we are we are having r of t being passed through one matched filter to another filter sampled at t equals to capital T. So, whatever value you have here is r on the first dimension and r on the second dimension and together you get the vector r which is r 1 r 2.

So, this is the vector dimension that you get. So, this is the outcome of the matched filter. So, that is what we have over here the vector received from the matched filter output is r 1 and r 2 r 1 is the output of the first matched filter which will be this is the signal part that is what we said right and there will be the noise part also we did not do the noise calculation in our exercise here.

We did only the noise free part and what will be the noise part the noise part will be ten tau H t minus tau d tau and that would be the component of noise on that basis dimension and that we write it as n 1 and the outcome of the second one. Again back here it will be noise with H 2 and will write it as n 2. That means, the outcome of the first matched filter will be this signal plus there will be noise.

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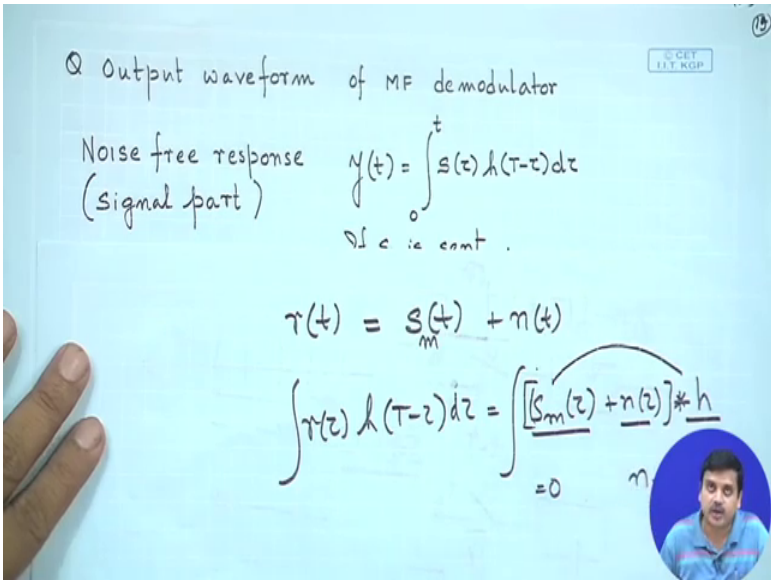

And in the second matched filter the outcome will be 0 from the signal component 0 from the signal component plus noise. So, it should not be although I do not have it here, but we should be remembering these things from that r t are equal to; so, S m t plus n t.

So, whenever we are convolving r tau H of t minus tau d tau whenever we are doing this in that case we are basically doing integral S m tau plus n tau I will write in short convolution with H right. That means, the first part of the integral is already shown over here. That means, S convolved with h t. So, you also have noise convolved with n t and in the in the second matched filter this convolution integral results in 0 while the noise component we mark it as n 2. So, that is what we have. So, we have n 2. Now, we could write that n I is whatever we have over here n 1 or n 2 is Y I n I taking the value of one or 2 at time instant t.

So, this is basically the component of noise on that basis dimension write this what we receive. Now if you we would have sent the second signal. That means, now we are talking about sending S 1 if we send S 2 that is the second function second waveform.

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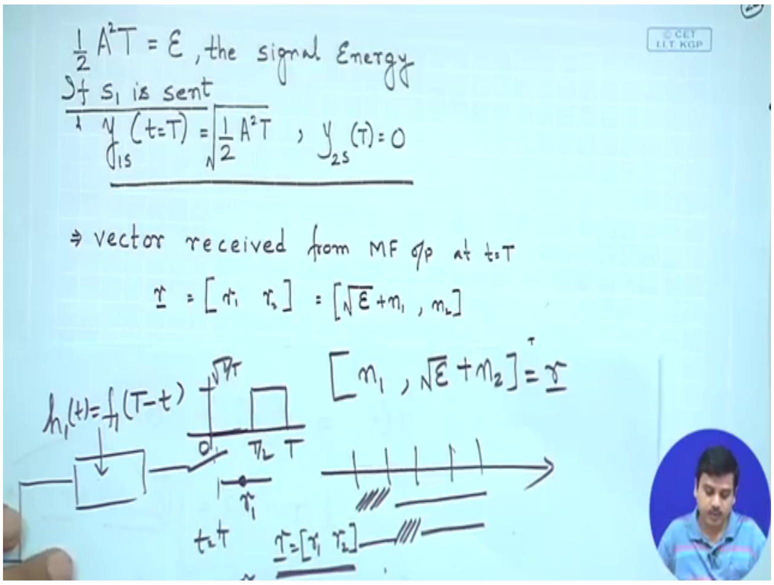

What shall we have received we would have received not this, but the first instead of this instead of this one we would have received n 1 from the first matched filter, because we have sent second symbol which is orthogonal to the first basis dimension. And we would have received signal power plus noise in the second thing. So, this would be my vector r if I would have sent symbol 2. So, this is just for clarification.

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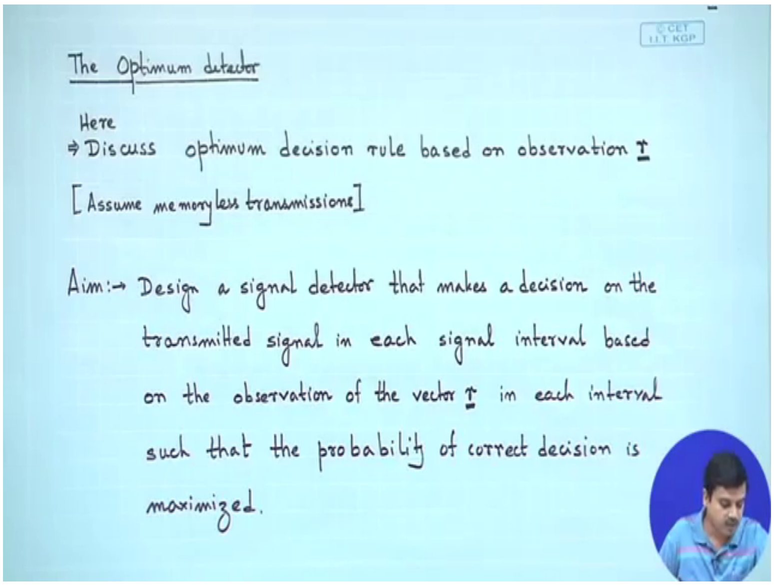

So, now since we have got the receiver since we have got the received vector we are now at a stage to discuss the receiver. That means, we said already we have the diagram here.

So, we have completed the de modulation correlation matched filter we have seen what kind of output the matched filter gives.

So, it gives a vector r we said it gives a vector r the detector uses these components to decide which waveform it has produced it has it has it produce this waveforms. So, just to summarize look at this S of t can be represented as sum of S k f k t correct where S k are the components of S t of S on f f k. So, S k is equal to integral 0 to t S t f k t d t right [FL].

Similarly r t can be represented as sum of r k f k t d t where r k are the components of r t on f k t d t. So, this is what we know and this is what we have achieved over here. So, r k means the component on the k-th dimension and k is equal to 1 to up to capital n. So, there are. So, r of t is composed of these components along with those basis dimensions. So, what the matched filter does is decomposes r has the components along the dimensions.

So, what it produces is the vector r made up of component in the first dimension on the second dimension up to n components. Now, since we have produced this components and just a side note the way it produces the component maximizes the signal to the noise power right the next part is now ready with us. So, since we have got these components here it goes to the detector the job of detect will be using these components to reconstruct the symbol please note it has these components. So, from components you can find r t right and r t is made up of S m t and noise. So, r k is made up of this we have seen already is made up of noise.

So; that means, if I have r k I also have S m k if I have S m k; that means, the k-th component of the m-th signal for k equals to 1 2 up to n; that means, for all k and for a particular m. So, if I have all these components then I can construct S t right. So, what I mean to say is if I have these components I know which waveform was sent right. So, I could modify this expression m put a sum m and put a sub m. So, if I have all the components I know which particular m-th waveform has been sent; however, the problem at the receiver is my information is noisy. So, from my noisy information I have to find out which particular waveform was sent. So, this is the problem that is available at hand.

That is what we are exploring. So, that is job of the detector. So, at the detector here we discuss the optimum decision rule based on the observation r. So, this should be very clear now. So, we have generated r. So, if you look at what we have generated in this page we have simply got components of r components of r on the basis dimension this is what we have got.

And now, since we have got all these components we want to find out which S m, because these components contain the signal, but it is noisy. So, you want to find this out. So, there is a decision making process, because we have several S m S m of t that m is equal to 1 2 up to capital M you have this problem.

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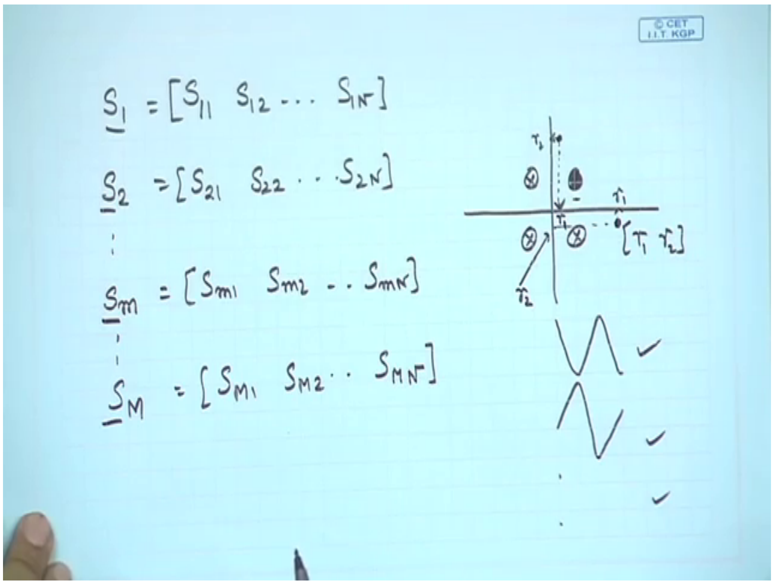

So, given this set you have to choose; what is the value of m that was sent. So, I would probably do it once again a little bit better way we have S 1 1 we have S 1 2 up to S 1 n this is the vector S 1 that is the signal S 1. So, I can construct S 1 of t by a linear weighted combination of this components along with this basis function I have S 2 vector as S 2 first component S 2 second component up to S 2 nth component right like that I can have S m as S m first component S m second component up to S m nth component and I can go to S m different waveforms k bits are required log of m base 2.

Look at this m th signals nth component. So, now, what we have stated here is we have these components for a signal, but these are noisy the moment is a noisy what is going to happen one signal might look more close to another right the moment is a noisy I actually have send this, but some of the components might be similar to another waveform and then I have to choose which of these waveforms was transmitted based on this noisy. So, there is a decision there is a choice that has to be made you have to choose amongst these right whereas, in the first part that we did here in the de modulation there is no choice there is simply a straight forward computation I have r t I simply projected whatever I get I get I do not make a decision.

So, now we have changed the problem and we have a decision making issue coming to you have to choose between one of these options right you cannot choose something which is somewhere between S 1 and S 2 it has to be any one of these choices. For example, if I have let us say q p S k. So, there could be a waveform like this there could be a waveform like this and there could be four other waveforms the detector has to choose any one of these waveforms and it has to identify probably this was sent it cannot do anything else it cannot say oh well something of a waveform which is none of these four has been sent in terms of baseband constellation it has to select one of these options it cannot say that oh something in between has come.

Which is actually done by the first part if you look at the first part first part actually projects the signal on the, I axis and on the q axis right on the basis dimension. So, whatever suppose I have sent this suppose I have sent this and noise gets added. So, noise will get added in the, I dimension as well as in the q dimension. So, suppose the received signal is here. So, what the matched filter does it projects this signal here as well as it projects this here and it tells that the vector is composed of the value r 1 and r 2 now using this r 1 and r 2 the receiver has to decode right. So, the other situation can be this was sent, but noise got added and the received signal has in the vertical direction shifted here and in the horizontal direction it is shifted here.

So, suppose the received signal has come here. So, it will produce r 1 over here and it is going to produce r 2 over there. So, this has a vector r 1 r 2 the job of the detector would be having given this value to find out or to tell which of these four was sent in the previous example if this was produced the job of the detector would be to choose any one of these four given that this is received or is output of the matched filter.

So, with this distinction we would like to stop this particular lecture here. And we will discuss the optimum detector in the optimum detector that is over here in the next lecture.

Thank you.