Modern Digital Communication Techniques
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Lecture - 42
Optimum Receivers for AWGN (Contd.)

Welcome to the lectures on Modern Digital Communication Techniques. In the previous lecture we have discussed about the receiver architecture, where we described that the receiver is can we think can be thought of as having two stages: one is the demodulator followed by the signal detector.

So, in the demodulated part we have said it could be realized using either a correlated receiver or a matched filter. We have developed the expressions or the models how does the correlator receiver work.

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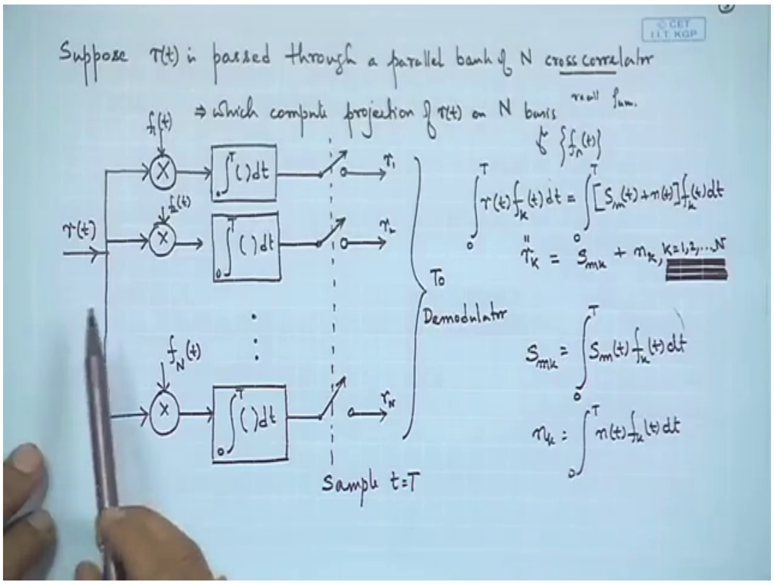

So, this is the structure that we had where we explained that the received signal is passed through a parallel bank of correlators, where the received signal was projected onto the basis functions. Projection means, you multiply with the basis function integrate from 0 to t and sample it at t equals to capital T the end result was the received signals components were received.

Now, these components we studied are made up of the component of the signal on the basis function as well as the component of noise on that basis function. So, each received component of the signal was having component of signal and component of noise.

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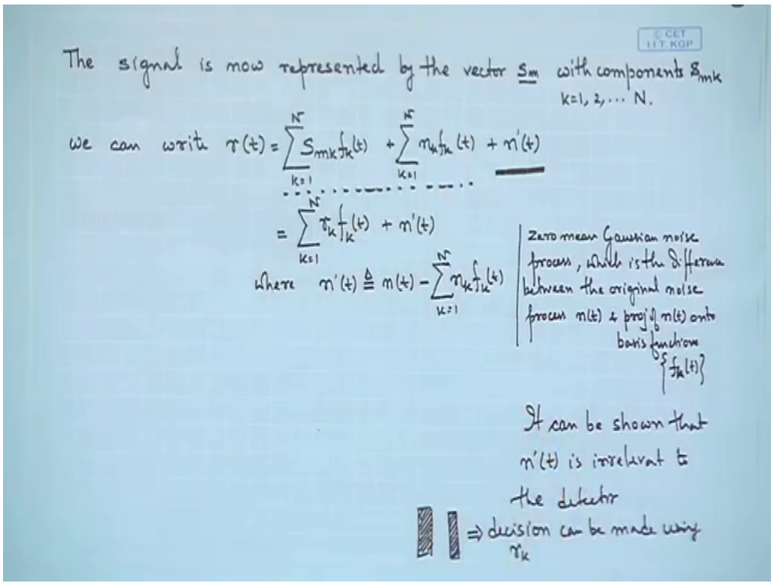

And then we said that this particular signal could now be written in terms of expansion of the signal using this basis functions, and the noise using basis functions plus noise which is lying outside the basis set right.

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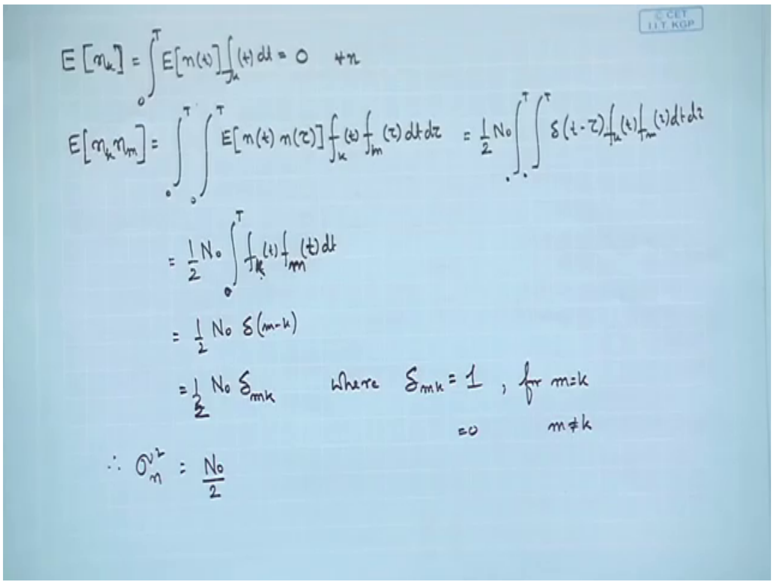

And then we looked at a few properties of the component of noise which we found it to be 0 mean the correlation was half N naught delta m k which effectively means that they are uncorrelated for different values of m and k.

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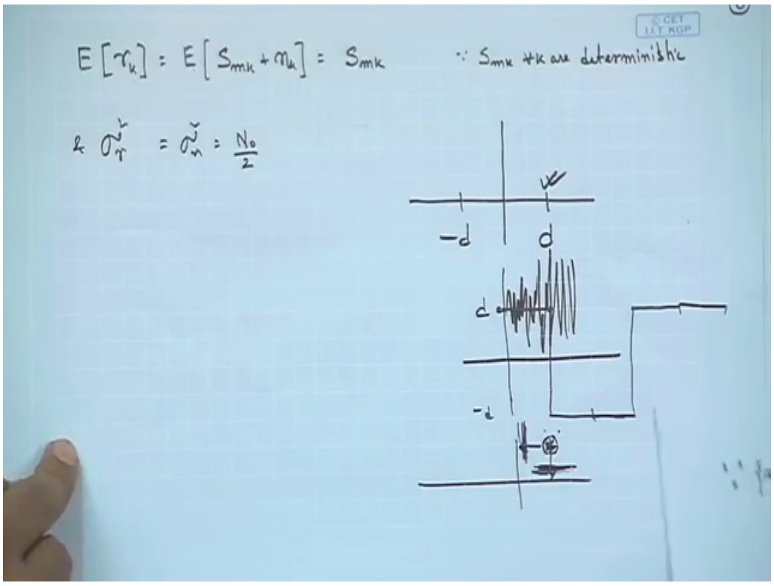

And then we looked at expected value of the component of the signal which turned out to be the component of signal itself the transmitted signal itself and variance is of course the variance of the noise component which is N naught by 2.

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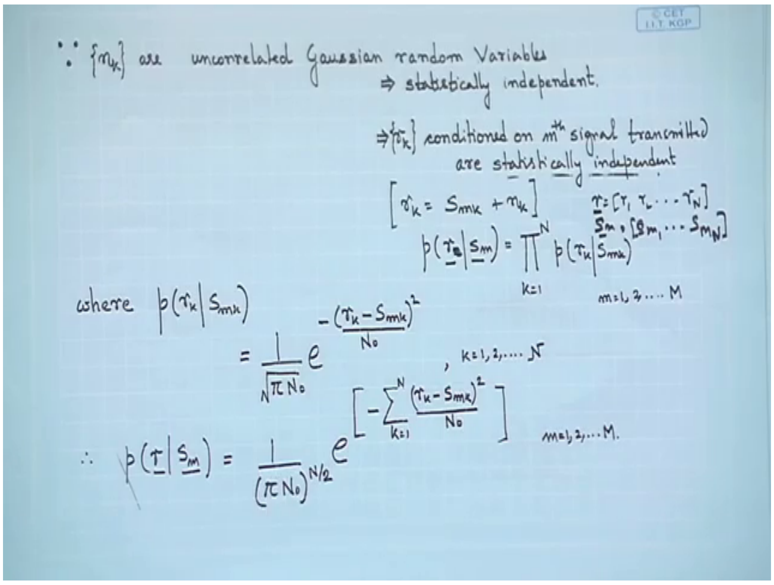

And then we further looked at the distribution of the received component given a transmitted signal and we also looked at the distribution of the joint the joint distribution of the components of r given a particular transmitted signal which was in this form.

So, in this part of the lecture that we described; now we will be looking at the next receiver structure that is the matched filter realization. And once we are done with the matched filter realization and the correlator receiver will take advantage of this particular expression that we have arrived at which gives us the pdf of the received signal components given the transmitted signal because at the initial part of our discussion.

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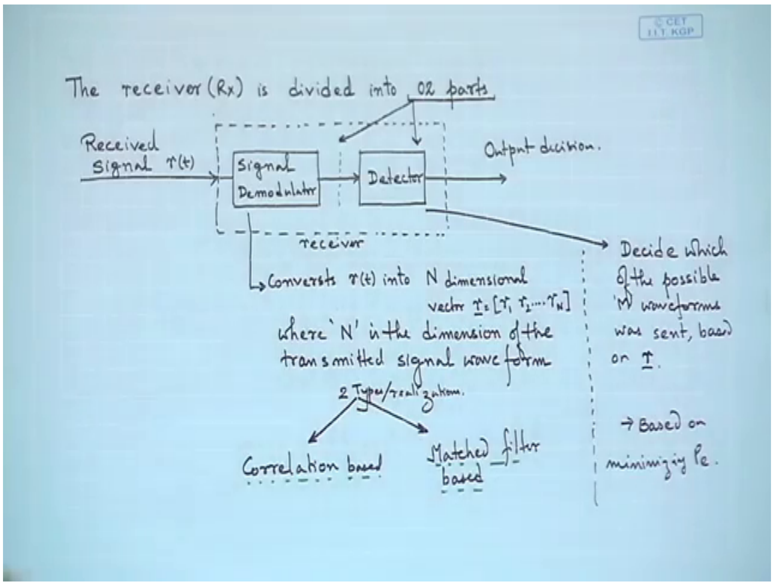

We stated that the receiver will produce components R t is decomposed into these vectors. So, what we did in the previous lecture was we saw how this decomposition could be done and what are the properties of these components as well as what is the joint distribution of these components given a particular signal.

Now, once we have studied this and we have generated this we will also see; how does these similar things come from the matched filter in this current lecture. Once we are done with these two we will proceed to use these components. So, in the detector where we say decide of which of the possible waveforms was sent based on the vector R. So, for that we will be requiring the expression for the joint distribution of these components given a particular transmitted signal. And thereby where we will be minimizing the designing the receiver based on minimizing the probability of error criteria.

So, with this we move ahead with our discussion on the different the different received different receiver architecture that is the match filter, but before proceeding on to the matched filter receiver let us briefly look at N prime of t which we are described in the previous lecture and said that it is that component of noise which is falling outside the basis set for the signal or outside the signal space.

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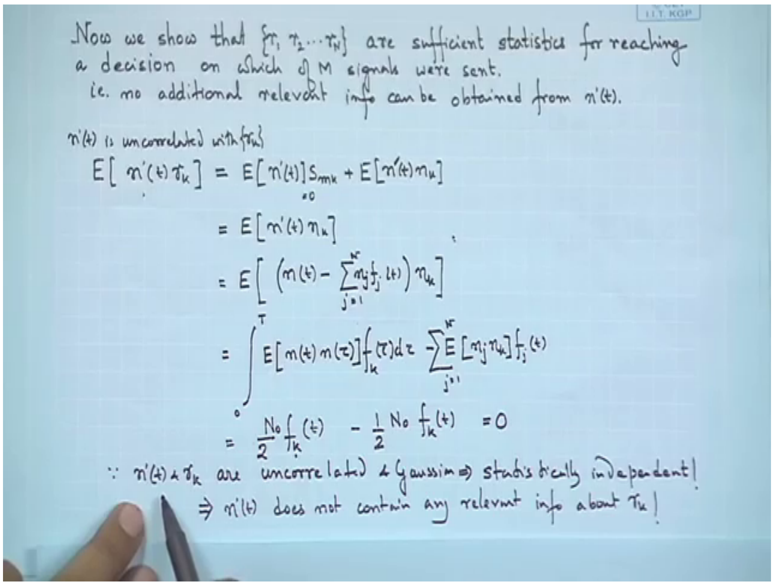

So, let us quickly look at that and then proceed to the matched filter. So, what we will do is we have to show that r 1 to r N are sufficient statistics for reaching a decision on m right. So, that is what we do. So, what we do is we take the correlation of N prime on r of k. So, we are trying to see if N prime of k is correlated if it is correlated to r k see we have actually constructed r fully from r k or S m fully from S m k.

So, now we have to see whether this which is left out has at all any correlation if there is correlation; that means, decoding r requires information about this. So, we take expectation of N prime with r k. So, r k can be easily broken into S m k and N k so; that means, we have E of N prime times S mk E of N prime times N k right. So, expectation of N prime is equal to 0 and we have S mk is of course, S mk. So, now, we have expectation of N prime and N k right. So, when we do that we have to expand N prime N prime is N t minus the expansion of the noise. So, we are talking about expansion of the noise in terms of the basis set and N k is over here. So, moving further; so, we have expectation of N t multiplied by N tau times fk t as one of the components in the expectation.

So, whenever you do expectation of nt and nk. So, nk is basically N tau times f k tau right. So, so basically you have N of t and nk is basically this whole expansion and expectation of that minus you have the expectation which goes inside over here. So, you have fj already nj times nk already that is present over here. So, this expectation is N naught by 2 delta t minus tau so; that means, it is basically f k of t that is what we are left with and you have from this it is again N naught by 2 delta j k so; that means, for j equals to k it is valid. So, what your left over here is N naught by 2 fk of t minus N naught by 2 f k of t again which goes to 0.

So; that means, this going to 0 indicates that N prime and rk are statistically independent and hence you do not you do not infer any information about rk from N prime of t. So, this is something just for the sake for sake for you to know, but this is of course, important, but you can you may also skip this particular part if you find this not very relevant for your activity. So, then we move on to the matched filter demodulator.

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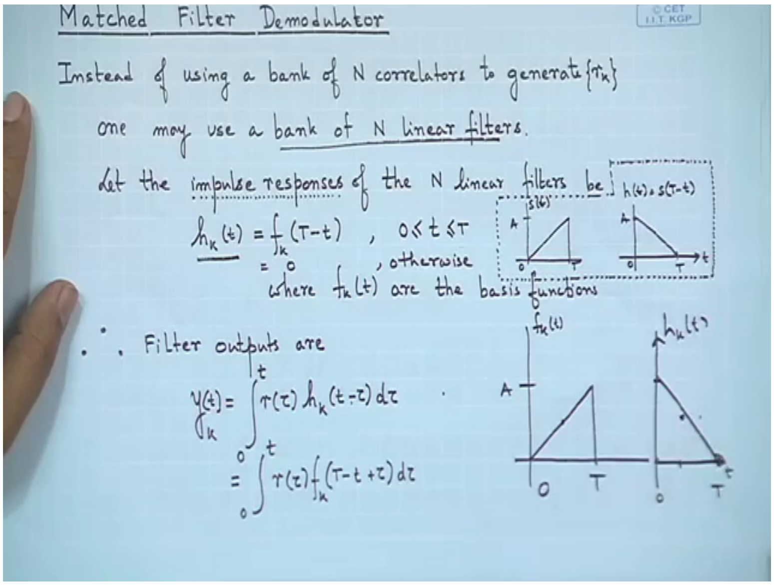

So, now in the matched filter demodulator instead of using a bank of N correlators to generate rk what we do one may use a bank of N linear filters. So, how do we do that? So, to do that we use the N linear filters with an impulse response that is given by f k of t. So, we are just taking a different view of whatever we had done before. So, what we do we choose filters with impulse response driven by h k k, because we have the different basis functions. So, h of t is equal to fk capital T minus t this means that you are flipping the basis function and you are assigning it as the impulse response.

So, this impulse response is the filter which has to be now used replacing the bank of N correlators this is just a different view of it. So, of course, when you do this we have f k of t as the basis function. So, we use the same basis function as we had done before and this filter coefficients or this filter impulse response is valid for the same interval 0 to capital T. Now, our filter output y k of t, because you are sending the signal to a bank of N linear filters each filter has this impulse response. So, your output of each of the filter would be integrate from 0 to t r of tau that is the input signal the input signal and convolved with the impulse response right that is the output of a filter convolution operation.

So, hk of t minus tau, so, we replace hk of t minus tau with fk by definition. So, we have fk t minus t plus tau. So, you simply replace t by t minus tau you will get t minus t plus tau because of this minus sign. So, this is the received signal. So, how is hk related to fk simply who draw a bigger picture here? So, suppose my fk is like this. So, when we draw hk 0 to t going by this definition; so h k of 0 so at time 0; so, this is time index t this is hk of t you have the value of f k t equals to 0; that means, f k of capital T fk of capital T is a.

So, you have the value over here and at small t equals to t by 2; that means, hk of t by 2 hk of t by 2 at this point would be equal to fk capital T minus t by 2; that means, somewhere here. So, which we could write it here and at small t equals to capital t; that means, when we are here we have f k of 0 f k of 0 is there so; that means, you going to get things like this; so, which is a flipped version of f k.

So, this is the impulse response you are filtering with this impulse response; that means, you are doing the convolution of the received signal with that and this is the expression that you have graphically the filter graphically the basis function and the filter impulse response would look in this look like this.

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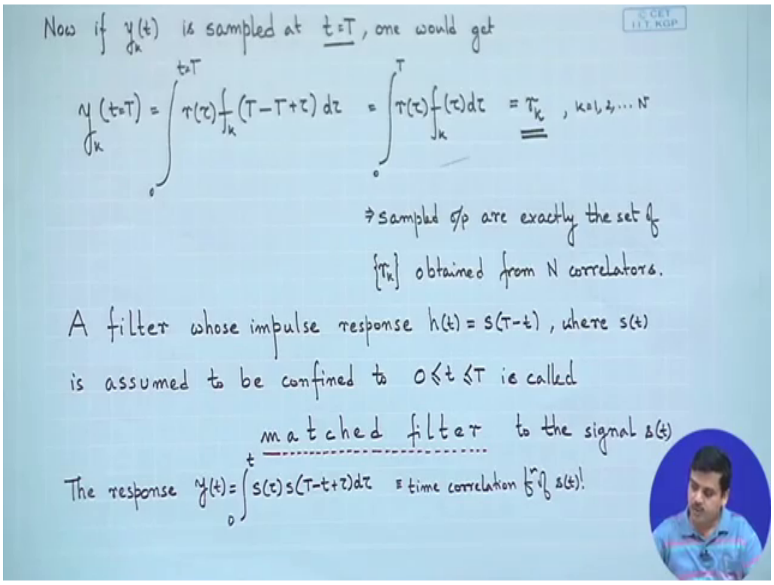

So, now, if you sample yk of p at t equals to capital T because at the correlator receiver you remember we had talked about sampling it at small t equals to capital T the expression you would get would be something like this that is yk will use this expression that we had seen before this is yk you would replace t equals to capital T. So, you have y k t equals to capital T integration limit from 0 to t equals to capital T. So, only small t gets replaced by capital T r of tau r of tau f k t small t becomes capital T tau d tau.

So, t and t cancels out you are left with integral 0 to t r of tau fk tau d tau which is the same r of k r k that is the kth component of the received signal as we had got in the correlator receiver. Now this is not very surprising at all simply because what we have done here is we had assigned hk as the flipped version of fk. And if you think it as a filter then filter requires convolution operation convolution operation there is again another flipping. So, because of this double flipping it turns out to be the same as the correlation with the basis function we flip it once here and again when you are doing the convolution operation that is here and you are sampling it at t equals to capital T.

So, overall what you get as the response output of the filter is same as that of the correlator, finally by this particular expression. So, when it is sampled are exactly the sets of r k component components obtained by the N correlators. So, what we end up with is again decomposition of r of t. That means the received signal into its k come or N components. So, whether it is the correlator realization that you want to look at or the mass filter realization in a double g N conditions where the filter impulse response is basically the flipped version of the basis function in that case we are going to get the components as the same components.

So, now we describe that a filter whose impulse response h of t is equal to S of t minus t which is the flipped version of the signal itself where S of t is assumed to be confined within this interval. So, you are flipping within that interval around that t is called a matched filter to the signal S of t right. So, this is by definition what is called a matched filter.

So, you have the input signal and the coefficient is just the flipped version within the signal interval for that you would call it the matched filter and the response is simply S of tau convolved with S of t minus t plus tau. So, this is if this is again if you look at it is the time correlation function of S t right that is what. So, basically it is the correlation of S with itself with of course, the time gap.

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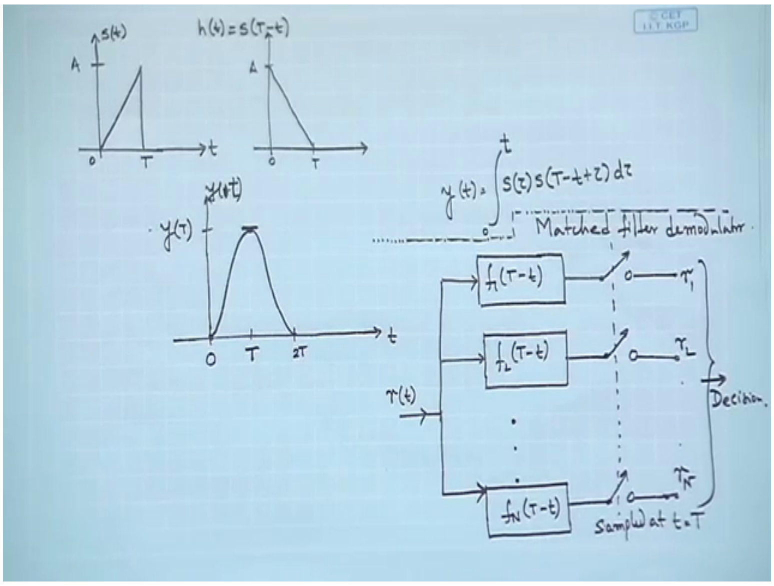

So, what we have over here is the typical output that you would get; that means, if you have a signal S of t represented in this form and h of h of t would be the flipped version. So, when you do the filtering operation with this you are going to get an outcome which looks like this. So, you have to just solve this you will get it in this form which has a peak value at small t equals to capital T. So, this is the peak value of the signal right and you can think of the matched filter demodulator as r of t broken into parallel to a bank of

N filters where each of the filters you had earlier f 1 of t. Now you have f 1 t minus tau earlier it was the correlator where you had a multiplier followed by an integrator and here since it is already flipped you have to do a convolution. So, this is a filter which does convolution and N result is the same as what we have seen and then it is sampled at t equals to capital T. So, what you get is r 1 r 2 r rn and these goes to the decision device right.

So, what we have seen is in case of a WGN where the channel has a gain of one and the received signal is broken is correlated against its against its basis function you get the components of the receive signal and you could also think of it in terms of passing the signal through different filters where each filter impulse response would be the flipped version of the basis functions. In that case you are going to get the same output as that of the correlator receivers. So, in both the cases it decomposes the received signal on the N basis dimensions and we get the coefficients which will be used in the decision making device.

Now, before we proceed to the decision making this device we will look at some of the important properties of matched filter and why is it. So, interesting to look at and why is it so interesting and very famous. So, this is this match filter receiver is one of the very important conclusions or understanding or take away from a course on digital communications, because whenever you receive whenever you are building a receiver. Typically, one would talk about realizing a matched filter receiver generally most of the cases there are of course, other receivers which are possible depending upon different situations, but a very basic and very important step in evaluating performance is to find the performance of a matched filter receiver.

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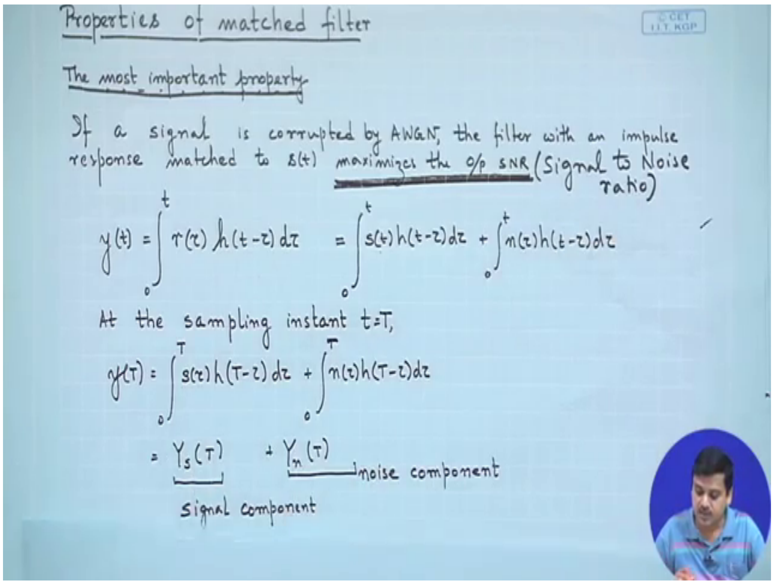

So, let us look at the matched filter. So, one of the most important properties of the matched filter is that that if a signal is corrupted by AWGN additive white gaussian noise what we are considering the filter with an impulse response matched to S t what you have already seen maximizes the output SNR this is very very important. So, that is why we go for a matched filter receiver the received signal y of t is given by convolution of the received signal along with the impulse response of the filter which is h of t.

So, there you see the convolution operation r of r of tau or r is broken into components of S and n. So, you have the signal component and the noise component at the sampling instant t equals to capital T you simply replace t by capital T you have convolution S tau h t minus tau. So, this is the expression. So, you replace small t by capital T this is what you get and you could identify that this part of the signal of y of t that is what is being processed the output of the matched filter is the signal component and this part is the noise component.

So, we have a signal component we have a noise component. So, when we discuss the signal to noise ratio SNR is the signal to noise ratio we need to operate on the signal component and the noise component we will find the energy of the signal and that of the noise or the ratio of signal power to noise power to calculate signal to noise ratio and we will see whether the selection what is the selection of h given S that maximizes signal to noise ratio.

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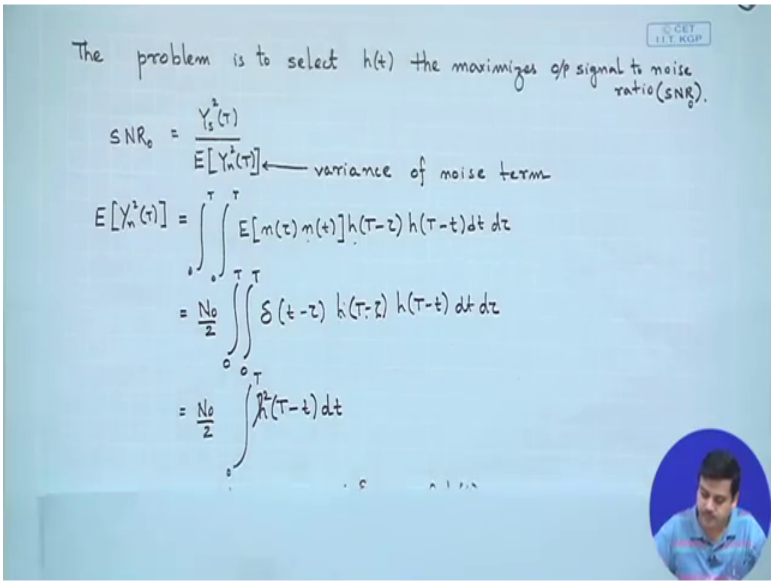

So, the problem can be stated as to select h of t as we just explained here. So, you have to find h of t to find h t from this which maximizes the output SNR an output SNR is defined in this form.

So, together what you have is we want to find ht which maximizes the signal component power to the noise component power where the signal component power is the signal convolved with the basis and the noise is the noise convolved with the basis which is due to this. So, just to connect these things we will be passing the signal through a matched filter and we would like to see; what is the ratio of the signal power to the noise power. So, the received signal has a signal component and the noise component. So, that is represented here signal component and noise component.

So, now by definition we have to find h which would maximize the ratio of this power to this power just a trivial note this power cannot be 0. So, there is a fixed value to this. So, we have to maximize this knowing this particular expression. So, SNR at the output is given by the square of the received signal upon the expected value of the noise power variance of the noise term. So, we try to evaluate the noise power y N squared t and expectation of that so again a standard procedure as we have been doing double integration E of. So, this comes straight ahead by looking at this expression.

So, since we have y N of t as this and y S of t as this. So, E of this would be expectation of this multiplied by the same term of course, the tau gets replaced with t that is exactly what we have over here y of this y square is this multiplied by that an expectation operator. So, the double integral from this we have 2 ns and we have 2 h values right and E will not apply on h it will apply of N and you can guess the result N t N tau will be delta t minus tau. So, which will leave us with and that is what we have it will give us delta t minus tau and delta t minus tau would mean it is nonzero only for t is equal to tau. So, if it is non0 for only t equals to tau. So, you have t equals to tau and this is equal to one and you have N naught by 2 this is a standard result which we have been using. So, t equals to tau would keep us with h square of t minus tau dt.

So, this is this portion within the integral is energy of the filter impulse response scaled by a certain thing. So, what we see that the denominator term in this is the energy term which does not depend upon the structure of h, but on the total energy contained by h right. So, the denominator term if you maintain constant energy on h; this h is not the modulation index that we had used in fsk. So, do not get confused. So, this h is the impulse response right. So, if we are talking about the constant impulse response. So, denominator is constant. So, if we have to maximize this you have to maximize the numerator based on the condition that this is held constant.

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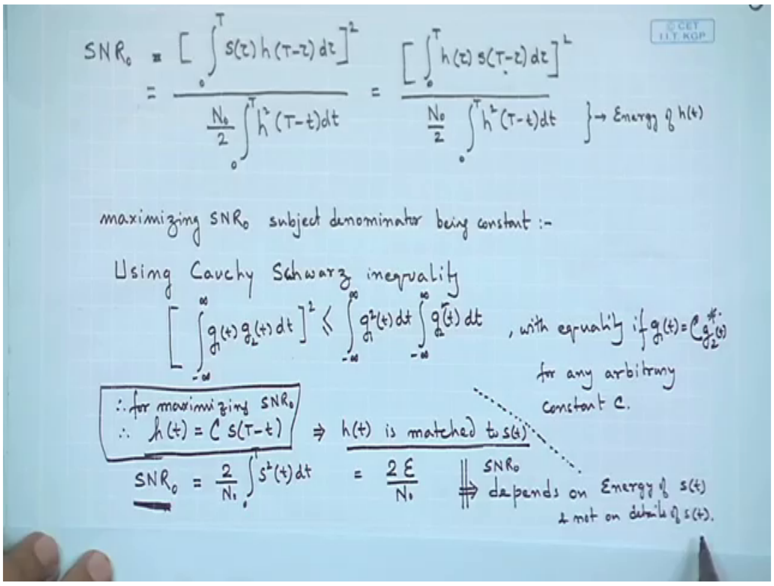

So, that is what we have with us. So, SNR output; so, we all we of course, have to refer to this expression and we may need. So, if we can look at these few expressions we almost have it with us. So, we had defined received signal we had defined received signal r of t broken into S and N. And then we had broken these 2 integrals the first integral we called it the signal part the noise part and then we described what is SNR which is the signal power to noise power. And once we have done that we want to maximize this ratio and first we looked at this term. So, when we looked at this term we found that it is dependent upon the energy of h. So, suppose energy of h is held constant which is a pretty valid assumption we need to find how you can maximize the numerator.

So, this is the SNR term. So, whatever we had done a few minutes back is given in the denominator and what we have over here is the energy in the denominator. So, we have to find method to maximize the numerator right. So, that is what we are left with. So, if you have to maximize the numerator what we have to do is we will we will use the Cauchy Schwarz inequality which we had studied in one of the earlier lectures. So, I would strongly recommend you revise that when we discussed the signal space when we discussed the vector representation then the vector properties. So, there we had represented Cauchy Schwarz inequality. So, according to Cauchy Schwarz inequality if g 1 and g 2 are 2 functions or 2 signals this inequality satisfied. That means, integral g 1 g 2 square is less than or equal to g 1 square integrated times g 2 square integrated.

So; that means, and this holds with equality if one of the functions is some scalar multiplied by another function and in case of complex there has to be a conjugate of that for any arbitrary constant c. That means, what we have by looking at this expression and the numerator this expression is less than or equal to the integration of squared of h and squared of g. So, the best value that we can get out of this is when it is equal. So, for maximizing SNR we have h of t is equal to some constant times this thing so; that means, this expression will be at most equal to the maximum value will be obtained when g 1 is equal to some constant time dg 2 and that maximum value means that h of t is equal to some constant times S of t minus tau.

So, this expression attains the maximum value this expression attains the maximum value when one of the function is equal to the other function right so; that means, ht is equal to c times t minus t. So, there will be this c that will remain as the constant. So, what we will find is that c square terms appears in the denominator and c squared appears in the numerator as well c will be here square term. So, they will cancel out each other. So, it will not affect the S nr. So, this if you see h of t is some constant multiplied by flipped version of the signal itself which we had defined initially to be the matched signal or the matched filter to S of t. So, here we have ht is matched to S of t.

So; that means, we could see that the condition to maximize the SNR is reached when we have the filter impulse response that is matched to the signal itself. So, how does it translate to our case? So, when we are using the filter impulse response for the bank of parallel matched filters each of the filter impulse response is matched to the component of the signal that we are talking about.

So, f of k is the signal and the matched filter is fk of capital T minus t which is the flipped version of the signal itself. So, we are matching it to the basis function. So, that we should remember we should not get confused with S of t S of t in this case is the situation for which ht is there. So, when S of t has several components each filter will be matched to the component of the signal on that particular direction.

So, now if you use this expression you will you should be able to calculate SNR at the output is 2 upon N naught. So, from here 2 upon N naught this and this one will cancel out you will be left with S squared of t S squared of t is the energy of the signal. So, what you have is 2 multiplied by energy of the signal upon N naught. So, what it tells is that the output SNR or the maximum value of output SNR is equal to twice the energy of the signal upon the noise power spectral density. So, that is that is it depends upon energy of S t and not on details of S t why it does not depend on S t simply because we have matched the filter with that of S t.

So, whatever is S t h t will be matched to that so; that means, we do not need to know details of S t any further while calculating is not output. So, while calculating SNR output we already assumed given any S t; S t will be match to that and therefore, we can easily calculate the SNR output. So, in summary we can say that the matched filter receiver that we have implemented has a very important property that it gives us the maximum value of signal to noise ratio for a particular situation.

So, in summary we could state that we have arrived at a point or demodulator which gives us the maximum possible SNR condition that is ever feasible under AWGN condition and that can be realized through a matched filter receiver.

Thank you.