Modern Digital Communication Techniques
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Lecture – 41
Optimum Receivers for AWGN (Contd.)

Welcome to the lectures on Modern Digital Communication Techniques. So, I said in the previous lecture we have covered modulation and now we are into the receiver. So, we have started discussing about the AWGN channel which we have discussed in the previous lecture .Aand then we wrote the expression of the received signal at the receiver when it passes through an AWGN channel. And then we started discussing about the receiver structure for which we had to recall our understanding of the modulator of the symbol mapper, where we said that the symbol mapper essentially takes a bit stream and produces waveforms.

And the job of the receiver is to do the reverse; that means, it has to take the waveform and reverse map it to the bit stream.Aand therefore, you could define the criteria for designing a good receiver in the sense that you would like to make the minimum amount of error. And then we said that we want to design a receiver for AWGN channel which minimizes the probability of error.

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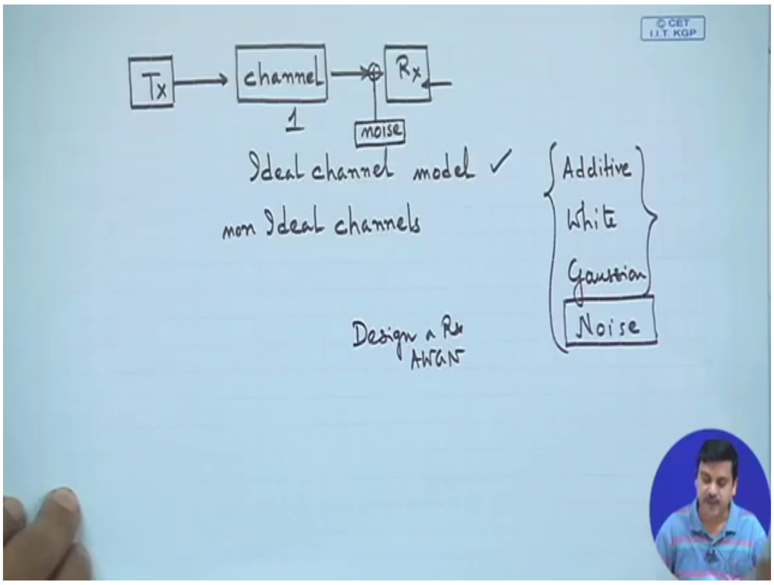

So, we will quickly look at some of the things which we discussed in the previous lecture so that we can continue with our discussion on the topic.

So, quickly let us see, we provided a model where there is a transmitter channel and the receiver and noise gets added at the receiver. We also said we will take a look at the ideal channel where channel gain is 1. And effectively it is also known as the additive white Gaussian noise and we want to design a receiver for AWGN channel.

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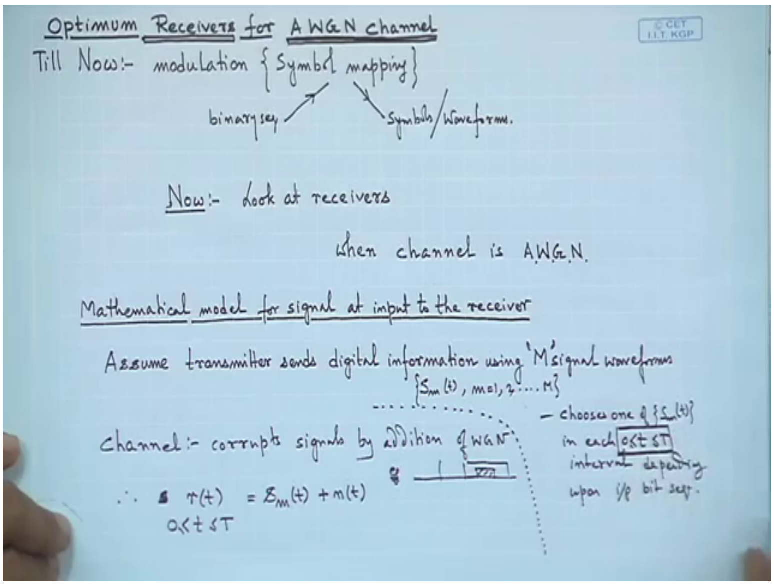

So, we said let us design an optimum receiver for AWGN channel. And since the modulation does a binary to symbol mapping or waveform the receiver does the reverse process. And we looked at the received signal r of t which was equal to the S m of t; where S m is the transmitted signal, where m ranges from 1 tp up to capital m; that means, selecting one of the possible waveforms. And there is noise getting added we said this is white because the receiver uses a filter which is much wider than that of the signal. And the noise is spectrally white in that portion. And the receiver sees this signal in the time interval 0 to capital T that means in each symbol interval.

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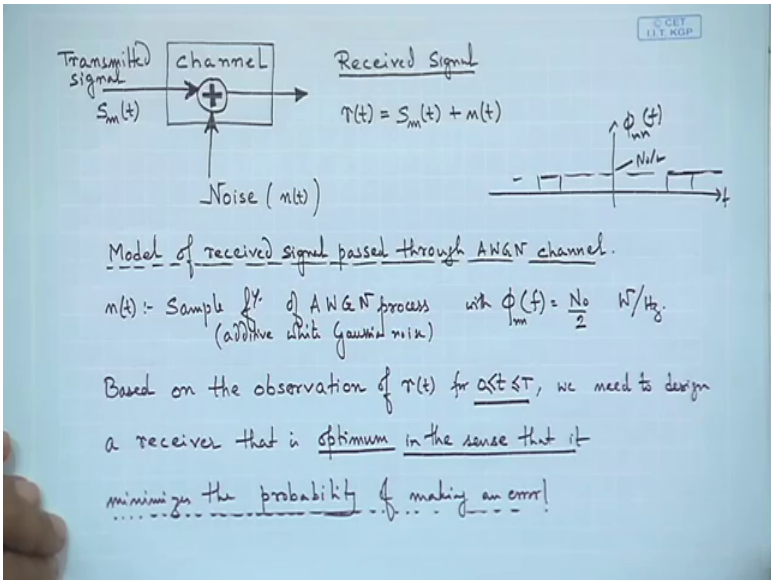

And we did discuss about this kind of a figurative representation this is the expression of the received signal this we all discussed in the previous lecture. And we described the noise with these power spectral densities as N naught by 2. And then we said based on the observation of r in the interval 0 to t that is a symbol in interval we would like to design a receiver that is optimum. And we described the meaning of optimum in the sense that it minimizes probability of making an error which we just described once again.

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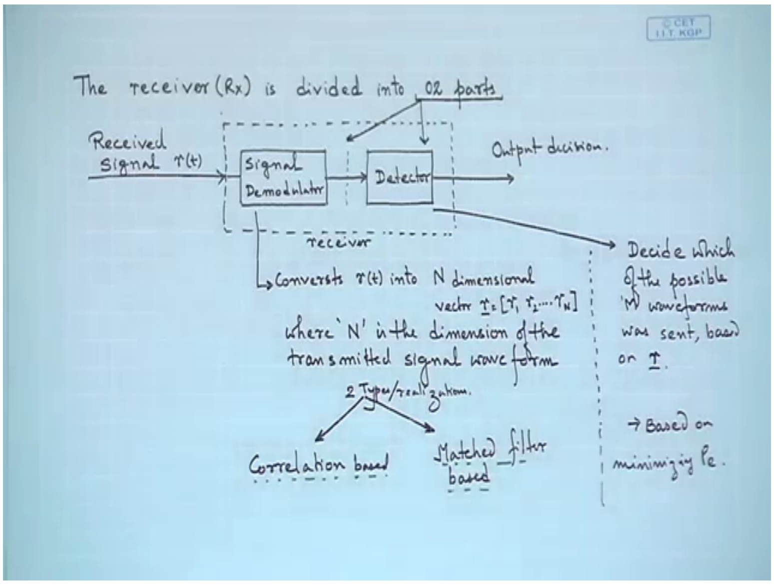

And then we looked at the receiver architecture where we said that the received signal is first demodulated. So, it goes through a demodulator and then it goes through the detector you have typically studied demodulator in analog communications and you will see some form of similarity in this as well.

So, together it forms the receiver it takes in received signal it produces output decision and this decision is about which of the possible waveforms was sent this we again revised a few minutes back. And the demodulator we were discussing about this it converts r t into N dimensional vector this is what we had stated. So that means, it is decomposes r into r 1 r 2 up to r N where N is the dimension of the transmitted signal.

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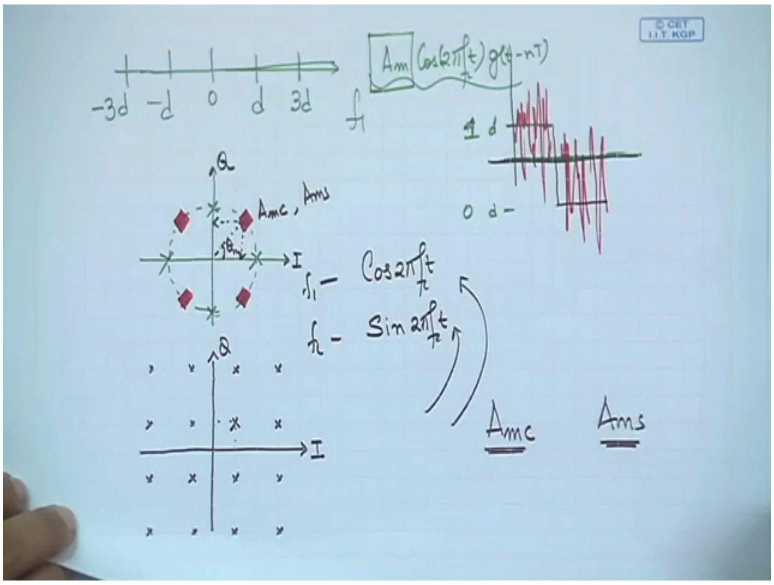

So, to explain the dimension of the transmitted signal we did take a look at the pulse amplitude modulation which is one dimensional; that means chooses only the amplitude of the basis function. So, this is the component of the signal on the basis function that is what we explained for phase shift keying we said you could decompose into I and a Q channel and you could have projections on the I as well as on the Q. And these are the components on one basis and the other basis then we moved on and discuss that well there could be a Q m constellation whereby you straightaway have this relationship. That means, there is an amplitude in the cosine carrier amplitude in the sin carrier and it goes as quadrature amplitude modulated signals.

So, in these 2 cases there are 2 basis functions one along the I, that is this one you one along the Q which is along which is this one.

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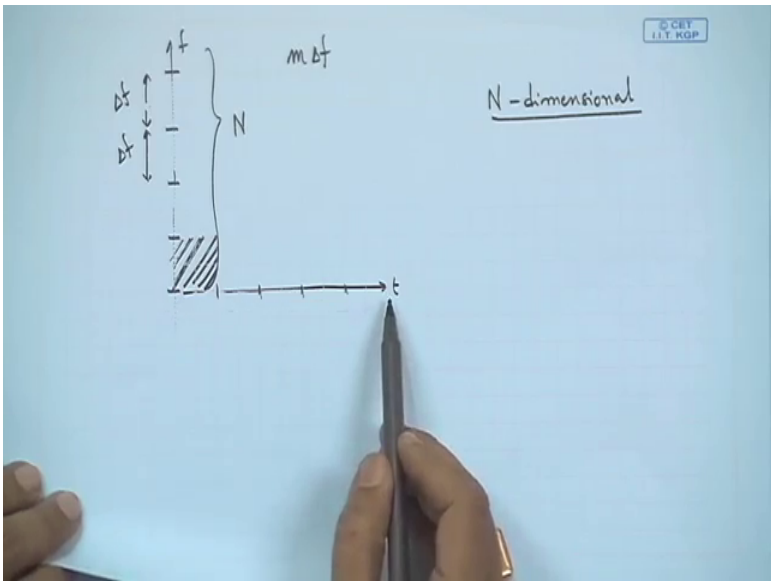

So, these are the 2 dimensions that we have looked at then we looked at the N dimensional scenario, where we also said that if there is t. And there is an N 1 and N 2 dimension you could have N 1 N 2 dimensions in the space.

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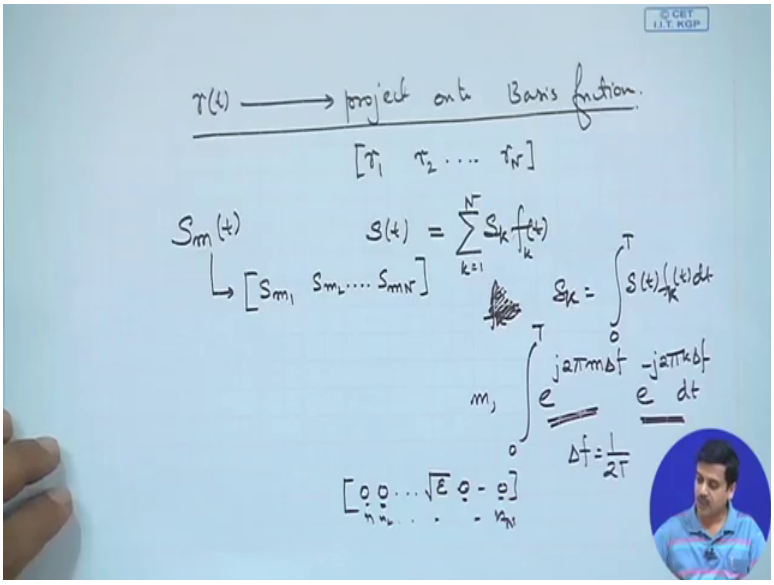

So, you have N dimensional signal. And then we explained finally that if you have r of t what we are basically stating when we say that r is broken into components. It is basically r of t is projected onto the basis functions and it produces these components of r on the basis functions.

So, we should recall that if there is a signal S t and we have f ks as basis functions which span the signal space; that means, the signal could be expressed as a linear combination of these basis functions then you could decompose or you could represent S t in a vectorial form. And this was in one of the particular lectures we did discuss the signal space representation and that the signal follows all properties of a vector when we break it down into this form then we can do vector operations on signals as well. And S k that is the coefficient or the component can be computed in a form that is projection of S t on f k. So, that you do by integration 0 to t S t f k t; where f k is one of the basis functions and we recall in case of emery fsk this was the transmitted signal and you had to correlate. So, it was basically projecting the signal onto different basis where you have this k as the different frequency components different frequency components different signals.

So, once you do this projection by virtue of orthogonality which we got delta f equals to one by 2 t then the projection could be 0s in case of ideal on all other signals there would be an energy in the desired component when m is equal to k and 0. Again on the others so; that means, projection of rt; that means, when you break r of t into these components for emery fsk ideal case you would get like this in case of noise you would be getting some noise components. So, the base signal would look like this in case of noise there would be some N 1 N 2 and. So, on noise components present along all of these.

So, this is what we had discussed in the previous lecture. So, moving on we first look at the correlator receiver because we stated here that this demodulation can be realized in 2 ways one is the correlation based and the mass filter based and which we will look at in today’s lecture.

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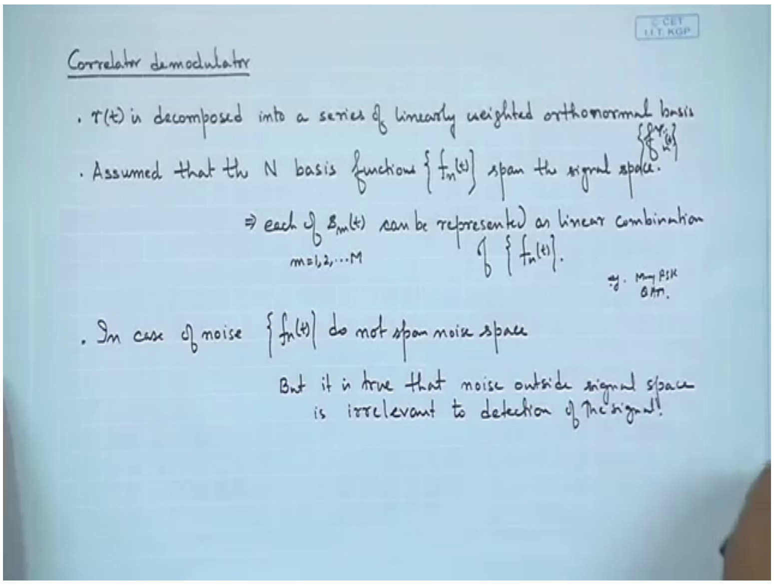

So, when we take a look at the correlation demodulator what we do is r t that is the received signal is decomposed into a series of linearly weighted orthonormal basis functions now this is not new for us because we have already discussed this thing that what is meant by decomposition. So, it is similar to S of t being represented as a weighted linear combination where the weights are the components of the signal on the basis functions.

So, now since S of t could be constructed by taking S k which are the coefficients along with f k as the basis here it is the reverse process; that means, finding the coefficients for the signal corresponding to the basis function. So, it is the reverse thing and that is basically projection which we have explained over here. So, this is the construction of the signal this is the decomposition of the signal. So, you are breaking down into components. So, instead of S t we now have will apply the same philosophy on r of t; that means, will apply the same philosophy over here.

So, we assumed that the N basis functions span the signal space. So, there is this is always there. That means, whenever we discuss communication systems or when we discuss signals and we talking about basis functions if your basis functions do not span the signal space. Then, what is going to happen when you expand the signal using whatever limited basis functions you have you are not reconstructing the signal in its entirety there would be some amount of error which is left. So, we do not want to do that over here. So, since it is within our control we can find the basis functions and in order to find them we had done gram Schmidt orthogonality by which given a set of signals if you have a whole set of signals from which you could construct orthogonal signals.

So, once you have the orthogonal signals the u I you can do the rest of the processing. So, we assume that there are N basis functions which span the signal space each of S m t. So, S m N equals to one to capital m indicates the different waveforms each of the waveforms can be represented as a linear combination of f N t. So, this is what we have explained; that means, each one of these waveform. That means, suppose I have a waveform let us say this or I have another waveform which is this or I have another waveform which is of a different phase altogether. So, right each of these waveforms could be represented as a linear combination of this basis function where the component of this on this basis would be used to represent it in the vectorial form.

So, in case of noise that is important f N t do not span the noise space that is clear because noise has a bandwidth which is infinity. So, if we think of bands which are orthogonal to each other.

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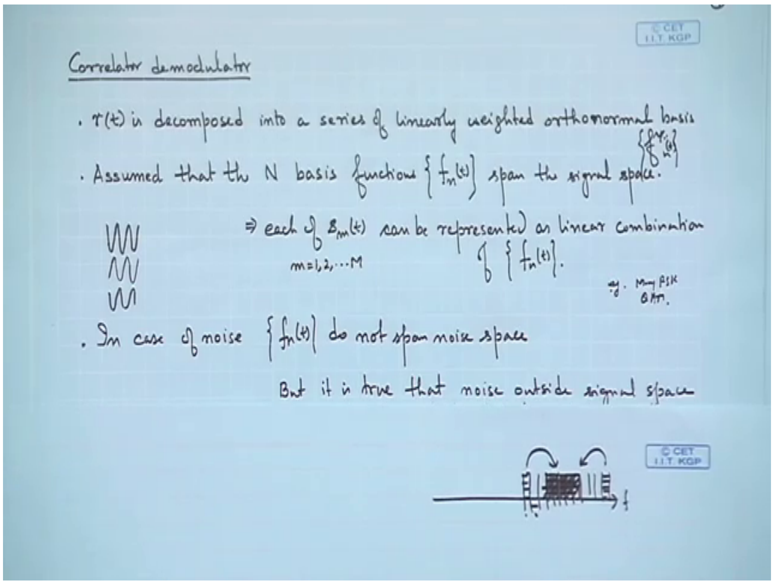

So, for instance if I simply take this is the f I divide it in 2 orthogonal bands this is not present here; that means, if I integrate this frequency with this frequency they are they will produce 0 because this spans this space this pans this space. So, if you take noise it spans the entire space. So, with the limited set of basis functions you cannot represent N t. So, that is all it says.

But there is an important thing it is true that noise outside the signal space is irrelevant to detection of the signal so; that means, if my signal is present in this band of frequencies all it says that the noise present in this band or noise present in this frequency will not influence the detection of the signal which is spanning this set of frequencies. So, this is the f axis right you can imagine it in this way. And if you think in terms of basis function it is an abstract way of representation not necessarily connects it to this, this is one particular realization. So, it also applies there equally.

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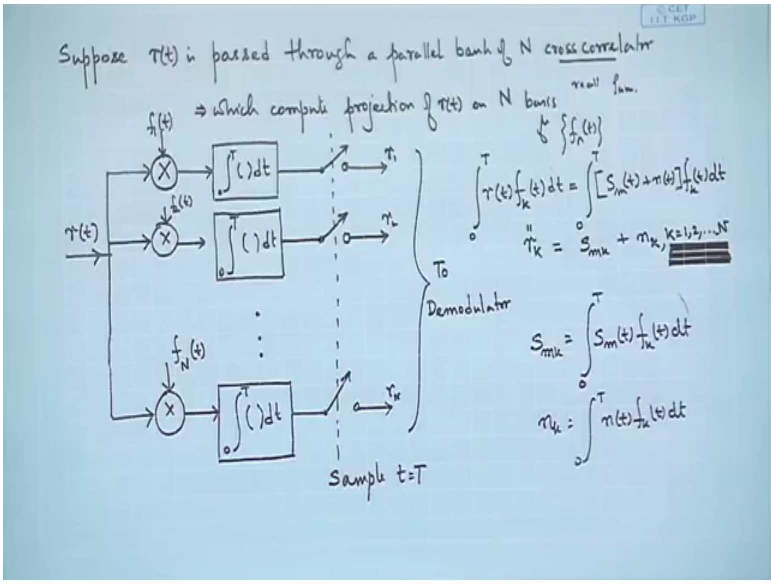

So, this is something what is important for us to remember. So, then what we talk of is suppose r of t; that means, the received signal it passes through a parallel bank of N cross correlators right why is it. So, simply because when we are projecting r of t onto the basis functions we are as if correlating this signal r of t with each of the basis functions the outcome would be the component of the signal on the basis function and that is what is of our interest that is the correlation value. So, this can be can be represented in a form that we have r of t r of t is now split and send to several different paths. So, it is being sent to several different paths in 1 of the paths r of t is multiplied by f 1 of t the first path it goes into the integrator 0 to capital T d t and then it is sampled at a time t equals to capital T.

So, let us see what happens. So, this effectively means that we have 0 to t integral r of t f k of t. So, k is one in this particular case d t so; that means, we are projecting r on f k; that means, we are finding r of k. So, here what we should get is r 1 of small t and at the sampling instant we are going to get r 1 of capital T. So, since this is r. In the next step you can expand r of t into S mt plus nt because N is the m-th waveform right. So, this is how the expression looks like we look at what this yields right. So, the left hand side could mean r of k. So, this one will produce r 1 of t this line where r of t gets multiplied by f 2 of t then integrated from 0 to t. So, by this expression k is equal to 2 and you get the value over here is integration of r t f kt over the period 0 to tdt and that is equal to r of 2 and that is read at the time t equals to capital T.

So, if you proceed in this manner you have to do the projection on the N basis function. So, we have to talk we have talked about this N basis functions. So, this particular realization is as if you are correlating r with the different correlators cross correlators and this cross correlator functions are the different basis functions. So, you will be getting r 1 that is this result known as r k r 2 up to r N that is what we had started off with this is what we had started off with. So, we wanted this and this is simply how you get it a pictorial representation of what we have been explaining verbally.

So, when you realize this in circuit the architecture could be realized in many different ways that depends upon the optimization of implementation, but what it basically tells us is that you receive a signal you need a multiplier right multipliers are there in demodulators. So, what you have here is let us say cos 2 pi fct this is cos 2 pi fc plus another frequency maybe or this may be sin if there are 2 dimensions cos and sin nothing else over here if it is a emery fsk it is e to the power of j 2 pi one delta f t this one is e to the power of j 2 pi times 2 delta f t and so on and so forth, right.

So, now if you look at this particular expression which we had just explained the left hand side is simply r k k equals to 1 2 3 4 up to N; N different dimensions and integration of this term S m t with f k t is projection of the m-th waveform on the k th basis. So, we write S mk. So, when we had written S s of t earlier we had described with S of t. So, here this is any waveform. So, m is the m-th waveform and projection on fk would produce the k-th component of the m-th waveform. So, that was represented in this structure. So, here remember m changes from one to capital m different wave forms and k changes from one to capital N the different basis dimensions.

So, we have S m k given by this which is exactly same as the expression that we have here right if you look at this expression is the same as this expression. So, it is the component of the wave form on the k-th basis plus there is this noise right which is the component of noise on the basis. So, noise is present all over, but we are correlating on a particular basis or we are looking at a particular basis. That means, we are trying to see; what is the component of noise on that direction along which I am taking the component of the signal. So, wherever I am taking the signal in whichever direction whichever basis I am also getting the component of noise on that direction.

Now, if there is no signal on a particular direction that is not of my interest whatever noise is there is not of my interest either because I am not using the dimension at all. So, I am not using the dimensions. So, intuitively you can think of if I am not using the direction where signal should not be present whatever noise is there should not influence me because I am not looking at that direction at all. So, that is why we said at some point that noise outside the domain of the signal does not influence the decision criteria right.

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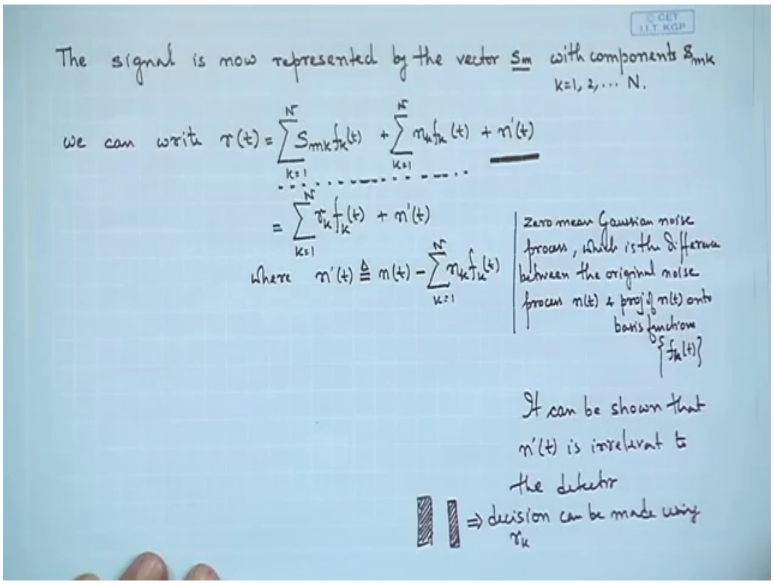

So, moving further we could say that the signal the original signal is represented by the vector S m with an underscore I use underscore as a notation simply because making all this bolds form bold fonts within the using hand notes is very difficult. So, I use underline to indicate vector in this particular discussion, and in different discussions, different courses you will find different notations some people use an arrow on top some will use a cap on top. So, will identify at each location what do we mean. So, as of now we will continue with this particular notation and with components S m k reminding again N dimensions.

Therefore we could write r of t the received signal is equal to. So, this is the received signal do not get confused with what we had discussed here. So, here we are discussed the k-th component, but now I am talking about the full signal. So, the full signal how would you generate you would some r k multiplied f k t right and that is what you have. So, rk is basically this. So, S m k and nk multiplied by fk. So, it is the same story. So, what we have over here r of t is the signal with the projection the coefficient and the signal added over linear combination over N dimensions same with noise; however, we have a very important thing over here.

Initially we said that nt is white Gaussian noise now since fk is limited it does not span the noise space there must be some amount of noise which is left which is outside the signal space that is N prime of t. So, N prime of t is beyond so; that means, whatever is nt take away the projection of noise on the signal space the remaining portion is N prime of t right. So, this particular part is rk fk because this is the signal and the noise what is received in the signal space and some amount of noise which is outside the signal space. So, N t N prime of t is defined as the extra noise which is not in the signal space. So, it is a 0 mean Gaussian noise process which is the difference between the original noise process and the projection of N t under the basis function. And we have already said that it can be shown that N prime of t is irrelevant to the detector decision. And decision can be made using r r of k only right this we will see we will show at an appropriate point.

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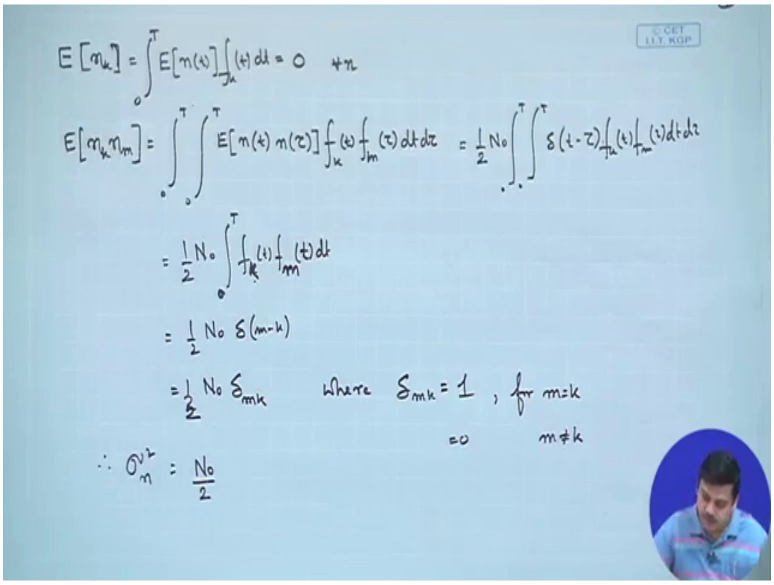

So, let us look at. So, we will try to look at nk, and then finally we will develop the receiver further. So, the first property that we look about for nk is the mean value of N k. So, for mean value of n k we use the expectation operator on N k. So, expectation operator of n k n k is N t multiplied by f k integral. Since, expectation operator is there we bring in expectation operation here now e does not apply on f k because it is a deterministic function. So, e will apply of nt e of nt is 0 you all know. So, therefore, this whole thing becomes 0.

Next we look at the correlation property of N n k and N m indicating the k-th component of noise and the m-th component of noise right. So, we want to find if there is any correlation between these 2 components. So, for that we use the standard expression 0 to t 0 to t because we have 2 different noise sources here these 2 integrated. So, each integral having the variable d t and d tau and n k is nt multiplied fk and nm is N tau multiplied by fm expectation operator expectation operator it does not apply on this it applies on this right.

So, now we know that e of N t and tau is delta t minus tau from our earlier discussions; that means, noise at 2 different instants of time is uncorrelated for white Gaussian noise that is what we know and not only that it is equal to N naught by 2 right; that means, for t equals to tau this is equal to N n naught this; this term is equal to 1; that means, the correlation is N naught by 2 and t naught equal to tau this thing is equal to 0 right. So, it means this integral is valid only for t is equal to tau. So, we have N naught by 2 and we set t is equal to tau where it is delta and one of the integral goes away. So, we have N naught by 2 right and delta is equal to 1 for t is equal to tau.

So, we are left with integral 0 to t fk t f m t now you can easily recall that since these 2 are designed to be orthonormal basis functions this integral will be equal to 0 for k not equal to m and will be equal to 1 for k equal to m so; that means, this can be represented as half N naught that is here and delta m minus k which is equal to 1 for m equals to k and 0 for m not equal to k. So, what we have is the correlation between the 2 noise is equal to half delta m k which means that these 2 noise components are uncorrelated with each other and hence are independent, because we have taken Gaussian noise right. So, only for delta equals to k only for delta only for m equals to k we have the situation where this value will be equal to N naught by 2.

So; that means, for e of n k squared you have N naught upon two; that means, the variance is N naught upon 2 and the mean is 0. So, again we have a white Gaussian noise we have Gaussian noise with mean of 0 and variance of N naught by 2. So, therefore, we can proceed and state that the expected value of r of k we did expectation of this we did correlation of this.

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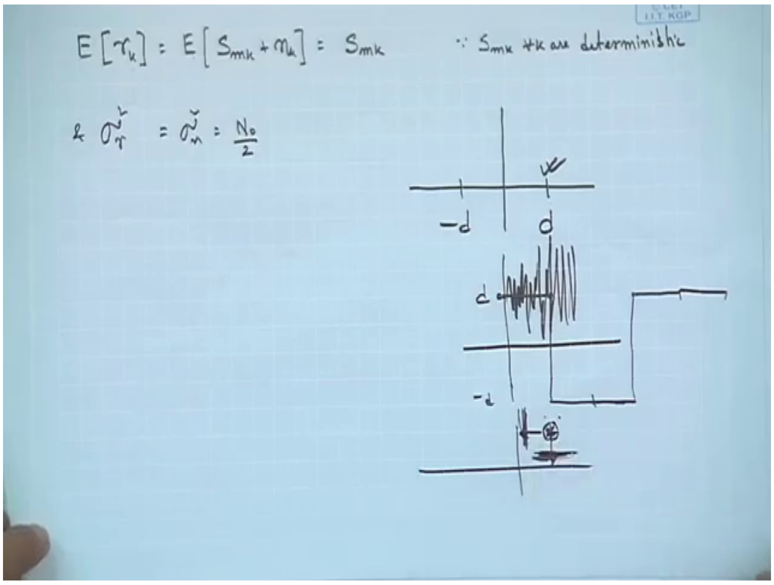

Now we move to find expected value of rk expected value of r k would be expectation of rk expanded over here right. So, k-th component m-th signal now this is a deterministic value we know the fixed value of it, it is a coefficient. So, expectation operator when applies on this is the value itself and e of n k you have seen to be 0. So, therefore, e of rk is that of S m k what does it mean? It means that the expected value of the k-th component of the received signal is the k component of the transmitted signal.

So; that means, if I have sent d suppose I have d N minus d and suppose I have sent d. So, the expected value would be d even though it is corrupted by noise. So, let us have a look at it. So, our transmitted signal would look like this. So, this is d this is minus t right and suppose we have this. So, it says that suppose I am sending only this signal right only d noise would be added on top of it say noise is 0 mean right expectation would be same as d. So, as simple as that right and if you think in terms of QAM there is a constellation there are 2 projections on these 2 axis.

So, noise will affect the I axis it will affect the Q axis. So, this received signal could go in any space, but on an average if I am sending this constellation for a very large amount of time then I am going to get the average value of the received signal as this particular point the mean value. So, there will be fluctuations in this direction there will fluctuation in this direction, but the mean value will be the same as this right.

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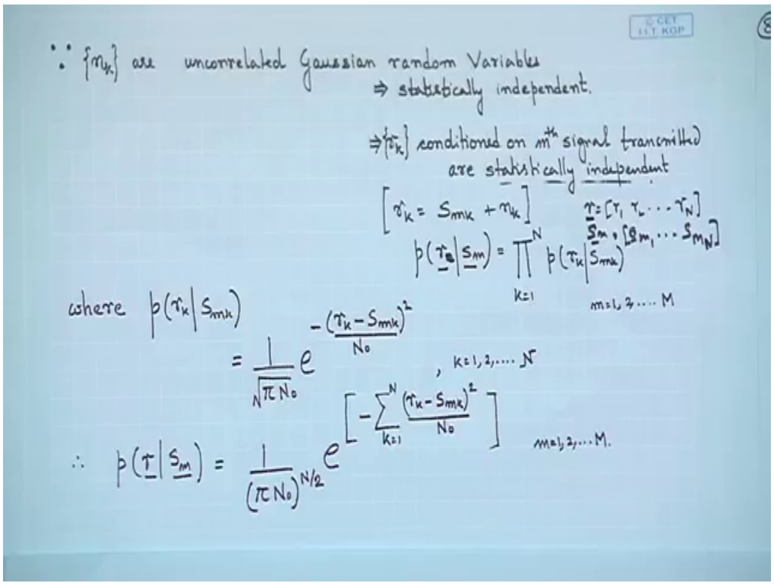

So, what we have is nk we have nk are uncorrelated Gaussian random variables and therefore, we have already stated that they are statistically independent we have already stated that.

So, r k conditioned on the m-th signal transmitted; that means, if we have a particular ms m signal that is transmitted would also be statistically independent because r of k is equal to S mk plus nk which has a mean m k and it has noise added to it this has 0 mean this is a fixed value so; that means, rk is a random variable which is distributed according to nk, but has a mean defined by this and that. Since it is distributed according to nk it will also have the properties of nk. So, it is also these are also statistically independent right and conditioned on the m-th signal means given that you know the particular signal if you take this as random then of course, results would be different.

So, we have r of k being equal to S m k plus nk if this is given; that means, the mean is known then rk is also a Gaussian random variable with mean of S m k and variance of N naught by 2 as per this and then we could write that the probability density function of the received signal vector r which is made up of k components k equals N components k equals to one to capital N given a particular signal is the product of the individual or the marginal densities p of rk there is a k-th component given S m. So, S m means S mk. So, what we have over here is that since these are statistically independent the joint distribution of these components of r turns out to be the product of the marginal distribution.

So, we have used that if we did not if we could not state that these are statistically independent then we would have had to find the joint distribution of the k components of r right. So, r is made up of r 1 r 2 up to r n. So, when we say pdf of r given S m, so, S m is also having components S m 1 up to S m capital N. So, this vector conditioned on this vector we should have a joint distribution, but since it is statistically independent it is a product of the marginal distribution. That means, you can simply work with each of the component independent of the other component that is what it says.

So, where the p of rk given S m k since we know from this that this has the pdf of n k with a mean of S m k what we have is the Gaussian distribution one over root pi N naught because there is 2 pi sigma squared N sigma squared N is N naught by 2 two and 2 cancels. So, we have pi N naught e to the power of minus x or rk minus mu or the mean is m S m k square upon 2 sigma squared sigma square is N naught by 2 2 and 2 cancels you are left with this.

So, if you substitute this back in this expression you are going to get this capital N times because there are N times multiplication. So, this is raised to the power of N there is a square root that comes as half e to the power of this multiplied pi times; that means, in the exponent we are going to multiplied N times. So, in the exponent you are going to add them up capital N times. So, minus summation r k minus S mk squared upon N naught now this is a very important expression that we have arrived at. So, we should be carefully noting this, because in the next discussion we will take advantage of this particular expression that what we have arrived over here to derive our receiver which is still to be developed.

So, please note that the way we are deriving the receiver is by continuing by starting with the signal model adding the effect of channel then the projection of the signal on the different components trying to understand the statistical properties about this correlation about these components and then given the received signal conditioned on a particular transmitted signal. So, we are looking at the joint probability of these components which interestingly came out to be statistically independent. And we have arrived at that particular expression of probability.

So, we will be using this expression to derive some kind of a receiver which will be clear very soon.

Thank you.