Modern Digital Communication Techniques
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Lecture – 23
Characterization of Signals and Systems (Contd.)

Welcome to the lectures on Modern Digital Communication Techniques. Till the previous lecture we have almost nearly completed describing the signals the systems as well as the narrowband noise process.

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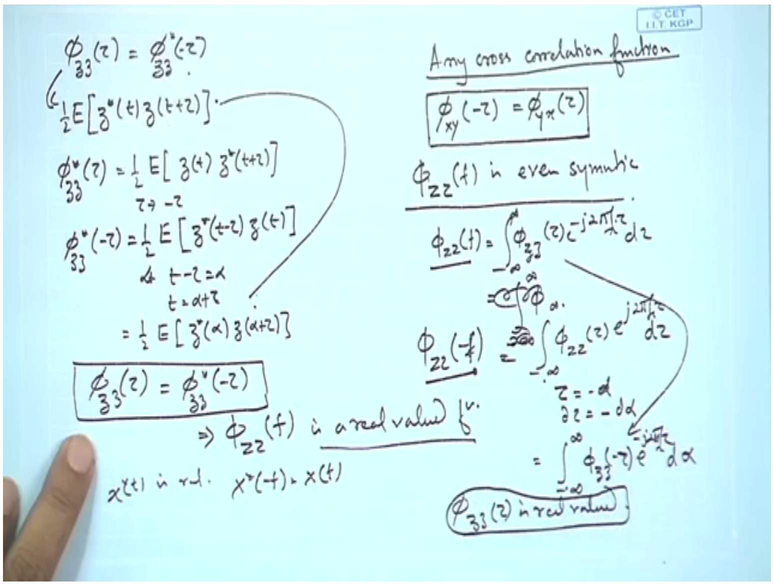

And we have related each of these from their passband to their equivalent low pass. Just a quick note on what we have been discussing in the previous lecture. So, that it helps you in, unless confused.

In this particular step that we did where we said it is even symmetric, we intuitively or we had assumed that phi zz tau is equal to phi zz of minus tau and whereas, we had actually this is basically comes from the set of assumptions that we have already set forward regarding that phi zz of this; however, what we yet had to do is that this is real valued. So, phi zz tau is real valued is the assumption that we basically use over here. And in that case it is phi zz tau is phi zz of minus tau and of course, this appears when we take this odd function so that means, this particular thing that we did is essentially for tau equals to 0 or for all cases.

So, whenever phi zz tau is real valued and this is real valued as we have seen for the situation, where this correlation term goes to 0. So, one option is when tau is 0 and the other thing we assume that let phi xy tau is 0. So, that is what we have taken over here. So, we said that if tau phi xy tau equals to 0. So, that is real.

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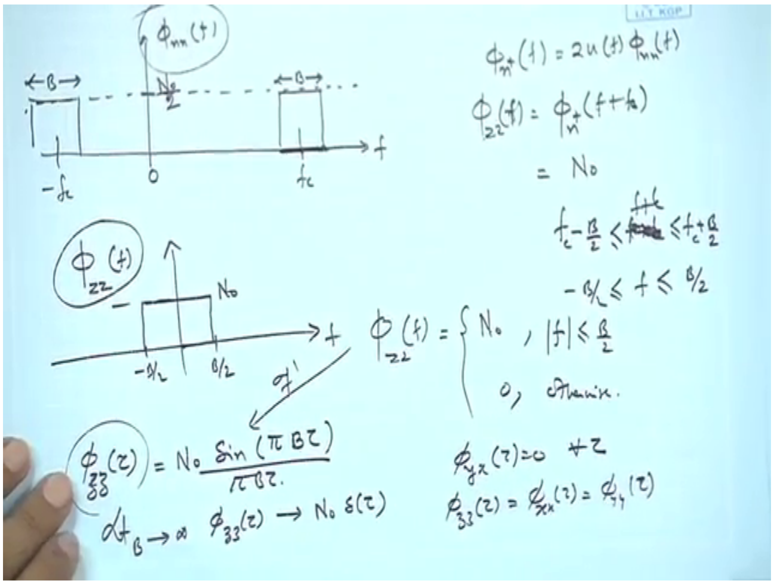

So, that is what we had used in this analysis. So, that we even symmetric and when we were calculating this stationary noise process and we arrived at this, we use these particular relationships. In fact, that is not directly needed as well.

So, anyway now back to this point that generally there is a confusion regarding this power spectral density. So, you should remember that in the pass band it is n naught by 2 whereas, you are in the low pass equivalent it is n naught. So, that you should correct it, also there is sometimes a reference to that whether you are considering one sided, or you are considering 2 sided. So, if you are considering one sided noise power spectral density it will be n naught if you are considering 2 sided it is n naught by 2. So, accordingly you should set all your expressions proper.

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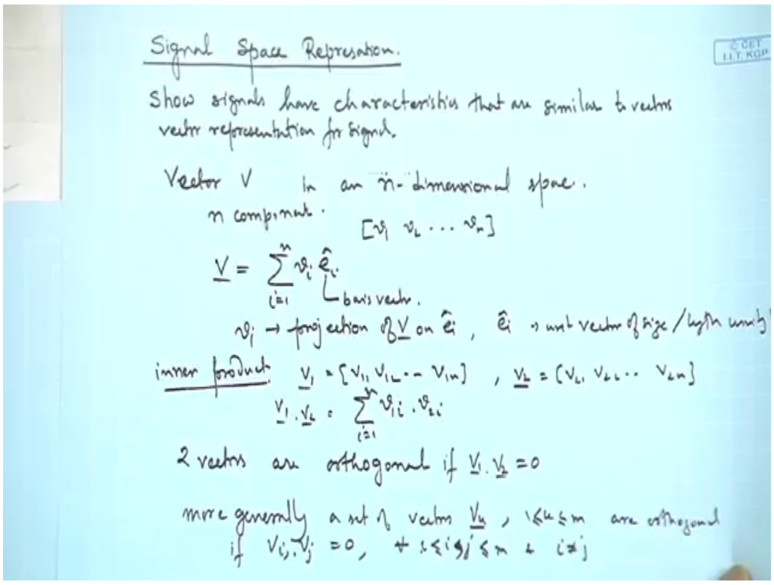

So, the next thing that we are interested in is the signal space representation. Now the these things that we are doing we could straight ahead use these things without actually describing, but these things are sometimes needed to visualize or to write the expressions as we have already declared in the beginning on an ambiguous statement. And when we talk about the signal space represent representation it gives us a pictorial image or a view of the signal in the vector space.

So, that is what we are interested to do so that means, we wanted to show that signals have characteristics that are similar to vectors, right. And we have a vector representation for signal, right.

So, this is our objective and before we do that we would like to summarize some things with the vector. So, let there be a vector v in an n dimensional space. And it is characterized by n components. So, you would have the n components like v one, v 2 up to v n, and it is also represented as a linear combination of the vectors. So, basically this vector you could write it as summation i equals 1 to n v i e i, where e i are the unit vectors or the basis vectors, right. And you could also say that v i is the projection of v on e i. And e i is the unit vector of size or length unity. Then you could define the inner product vectors v 1 which is defined by v 1 1, v 1 two, to v 1 n. And v 2 as v 2 1 v 2 2 to v 2 n as v 1 dot v 2 this is pretty standard; however, we are doing is just for revision v 1 I times v 2 I.

So, this is how would how you represent the inner product and why this is necessary because you would say that 2 vectors are orthogonal if v 1 dot v 2 is equal to 0. So, you can also say that a more generally a set of vectors v k with k ranging from m to 1 are orthogonal. If v i v j dot product is equal to 0 for all i, i j in this range one to m and i not equal to j. So that means, if v i dot v j is equal to 0 where i is not equal to j they are orthogonal this is well known for vectors. And we are going to use this for signals as well very soon.

So, once we have that then we also have the triangular inequality.

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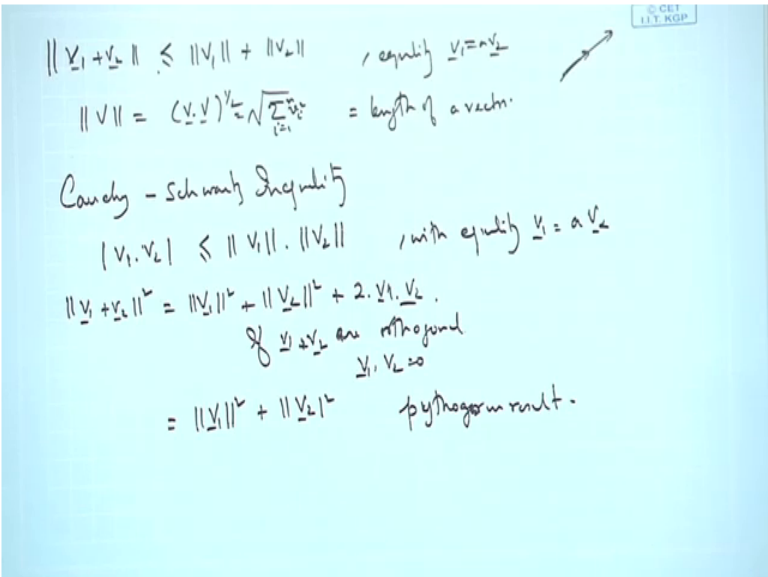

Which would say that vectors v 1 plus v 2 would be less than or equal to v 1 plus v 2 where this sign indicates it is a norm of a vector it is v dot v; that means, the dot product with the self with the square root which is equal to square root of i equals 1 to n v i squared, right.

So, this is basically the length of a vector and this less than is holds with equality, if v 1 is equal to a times v 2; that means, both the vectors are aligned in the same direction. So, that is true and then you have the very important result which will be using at a later stage Cauchy Schwartz inequality, which states that v 1 dot v 2 is less than or equal to the norm of v 1 times the norm of v 2 and with equality if v 1 is equal to some constant times this v 2, right.

So, this is typically that for the vectors that you have and the norm square sum of 2 vectors. So, again you have v 1 plus v 2 squared is equal to norm of v 1 squared plus norm of v 2 squared plus 2 v 1 dot v 2. Now if v 1 and v 2 are orthogonal then we have this term going to 0 and then you have v 1 squared plus v 2 squared, and this is basically the phythogoras result that is well known, right.

So, these are some of the important results from vector that we can recall and we will be using a similar notation for signals and that is the motivation for doing this particular lecture. And why we do this because we will be encountering signals in n dimension at some point. So, if you could represent signals at vectors then it would ease many of the computations at some point, right.

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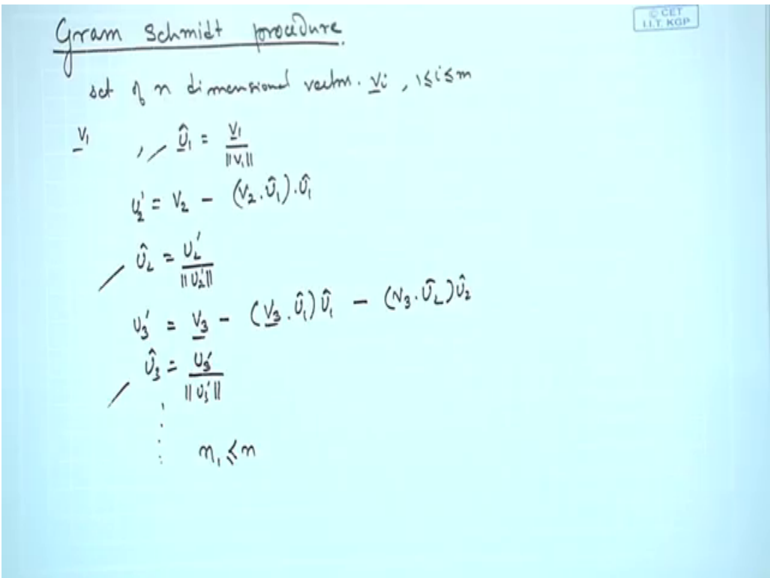

So, we move forward and we would like to talk about the gram Schmidt procedure. So, what the gram Schmidt procedure does is that suppose you have a set of vectors and you know that these vectors span the vector space, but you have a very large number of vectors let us say. What you are interested in is a set of orthonormal vectors which could be sufficient to span the space. The need for this is you would be creating basis vectors, and once you have your basis vectors then you can project your signals on those and you can do many, many stuff.

So, basically a gram Schmidt procedure is a well known we will just summarize that. So, suppose you have a set of n dimensional vectors suppose there is a set of vectors v i, i less than equal to m and 1. You start by selecting an arbitrary vector v 1 and you would make u 1 as a vector in this form. That is vector v divided by the length of the vector. So, it is a unit vector and then you take v 2 and take away from v 2 the projection of v 2 on u one, right. So, if you do this then you create a temporary vector u 2 prime and then you would create u 2 cap as u 2 prime upon the length of u 2 prime.

So, now you have created another unit vector, right. And you proceed further u 3 prime is equal to v 3 minus that is, you taken a third vector projection of v 3 on u 1 is in this direction minus projection of v 3 on u 2. So, once you have taken this away you would create the third unit vector, right and so on and so forth. And you will find you have created n one set of orthonormal vectors in the sense that, these vectors they are independent of each other.

So, basically if I have a very large set of vectors then we do not know whether we have they are whether all the vectors are independent of each other. So, if we have the basis set they are basically the minimal set of vectors that are required to represent any vector in that n dimensional space, right. So, that is going to help us in signal representation anyway.

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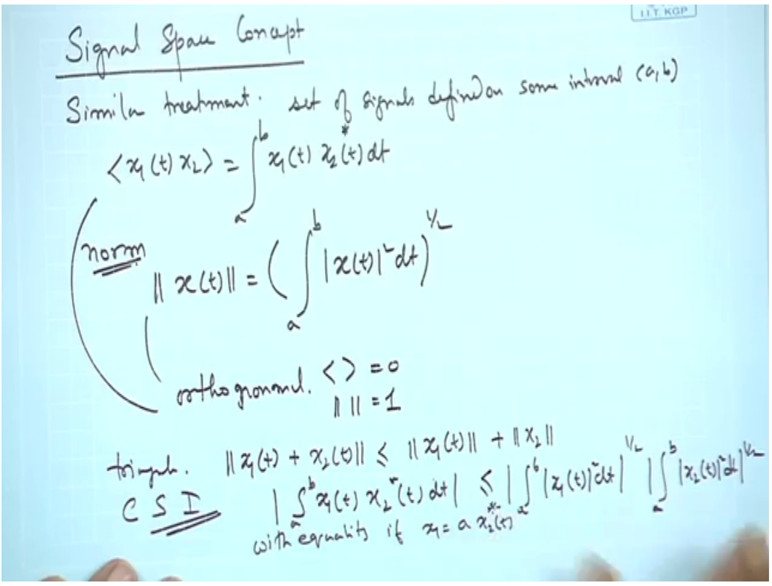

So, now we move forward from this to the signal space concept. So, if we are in the. So, having summarised the vectors we would like to see how signals can be represented in a similar form. So, we go to the signal space concept and it is a similar treatment, as that of vectors. So, we say that let there be a set of signals defined on some interval a b. And then the inner product of the signals would be defined as integral a to b x 1 of t x 2 of t dt. And you would often place a conjugate if it is a complex signal, right. And then you would define the norm of the signal as notation wise it is similar, this defined in this interval the absolute square dt and finally, a square root of that and in case of orthonormal, normal orthonormal would make it one and if you are talking about orthonormal; that means, this would result in 0 x 1 and x 2 inner product and this length would together you can define orthonormal, where you would say that the inner product would be 0 and the norm would be equal to 1.

So, that is the definition you would put triangular inequality would satisfy in a similar way as in vectors. And you have this Cauchy Schwarz inequality I am writing in short it is integral a to b x 1 of t So again, x 2 of t that would be a conjugate if it is of complex is less than or equal to integral a to b x 1 of t squared dt to the power of half. So, that is how you get the norm. So, remember this inner product is less than or equal to the norm of the first and norm of the second.

So, this is slightly different in notation, but it is the same thing that we have, raised to the power of half, right. So, and that with equality if x 1 is equal to a times x 2 and if it is a complex then of course, you have a complex. So, that is how these relationships hold.

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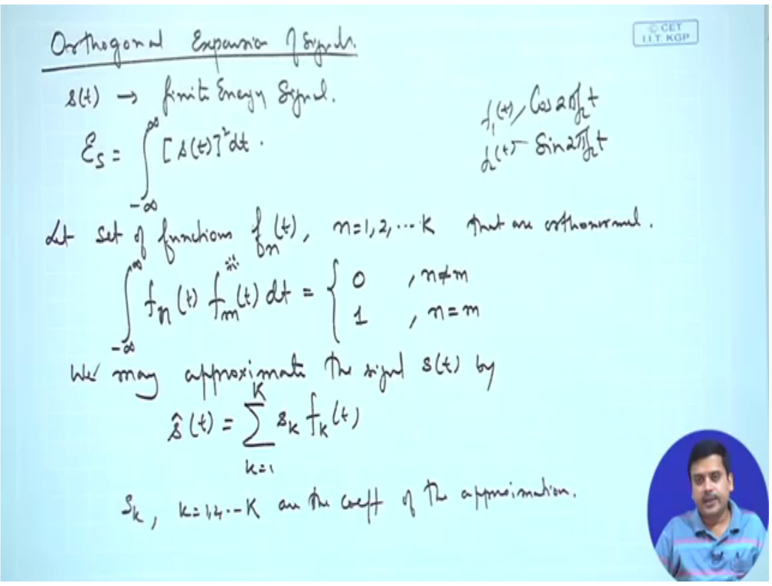

And then we have the orthogonal expansion of signals. So, what we finally, mean is we have to represent signal in terms of the basis signal set or the basis vector set.

So, we proceed in a similar way and suppose and we begin with s t which we define as a finite energy signal. We say that it is a finite energy signal and e of s is defined as we did this before, right. This is what we are with what we have defined. And now we say that suppose there is a set of function that let there be a set of functions which we mark as f n of t n is equal to 1 to up to k that are orthonormal, right. So, when you say orthonormal what we mean is that minus infinity to infinity f n of t times f m of t dt is equal to 0 for n not equal to m n is equal to this for n equal to m.

So, then what we say is that we may approximate the signal s of t by s cap of t which is equal to sum of k equals 1 to capital K is k times f k of t where s k, k equals to 1 to up to capital K are the coefficients of the approximation. So, what we are trying to say here till this point is suppose, we have s of t as a signal and there are some other signals which we define it as f n of t so, where we have the relationship that these are orthogonal signals. Orthogonal means if you are integrating them and taking the inner product so that means, if they are complex if you take the complex of it. Then you will get a 0 if these 2 functions are different and it will be one if they are the same. So, then you can say that I can reconstruct I can represent s cap of t as in this form and what we will get is that these coefficients are basically the projection of this on f k of t.

So, only if these span the signal space then only your representation would be complete otherwise there might be errors.

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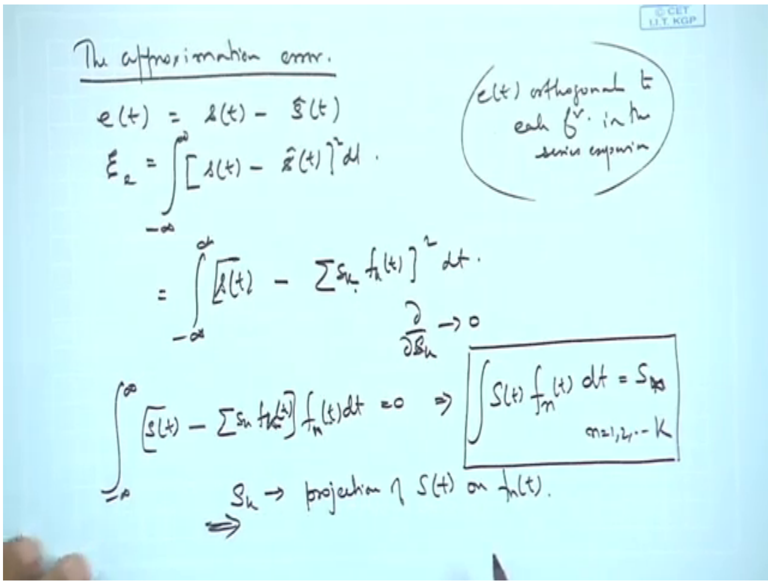

So, the approximation error that you can say, now you know just a note at this point. Simply you can think of that cos 2 pi fct and sin 2 pi fct if, if it is sounding out to be too abstract these 2 functions I can call this f 1 of t I could call this f 2 of t. So, then I could say that what is the projection of my signal or can I reconstruct my signal using these 2.

So, the answer turns out to be yes in quite a many cases, right. And there could be many other functions as well. So, this is one particular example of the situation where these are normal, orthonormal functions there could be n dimensional orthonormal function. So, then we go to the approximation error. So, the approximation error e of t is equal to s of t minus s cap of t. And the energy, the approximation error energy is of course, you integrate the whole range dt, right. And this you would get the result as minus infinity to infinity s squared of t minus summation s k f k of t squared sorry dt yeah. So, we have replaced this with this, right. And then the optimum coefficients if you have to find you have to take derivative and set it to 0 and; that means, derivative with respect to s k set it equal to 0 and you can get the solution.

The other option is you do not follow this procedure, but what you can do from geometry is that e of t is orthogonal to each of the functions in the series expansion. This is from the MMSE criteria so, the geometric criteria. So, if you do that you will land up with s of t minus s k, f k is the error on f n of t dt this will be equal to 0, right.

So, if you work this out you are going to get s t times f and t on this product and you are going to get this term along with this term. So, if you look at this integral with s k and f k so, s k is a constant term which is not a function of time. You are left with f k t and f and t. So, f k t and f and t when integrated we have said that we will select them in such a way that they are orthonormal; that means, only when they are equal to the same value it is equal to 1 otherwise it is 0.

So, using that over here if k is not equal to n then this turns out to be 0. So, only when this is equal to f k then we have s of k with integral f k squared. So, what this means that these 2 terms yield integral s t f n t and this term is integral s k f k and s k because, f k and f n would turn out to be 0 f k, f k remains. So, if it is f k or f n integral is 1. So, basically this is equal to f n. So, for k equals to n this is for only for k equals to n. So, you have this relationship. So, what it means is that you s k or s n that we have over here they are the same. So, only it is a matter of index what we mean is that n is equal to 1 to up to capital K so that means, f n or the s k s that we get are basically projection of s of t on f n of t.

So, it is similar to the vector projection and what we will find is that, this helps us in the few in the in the few lectures later.

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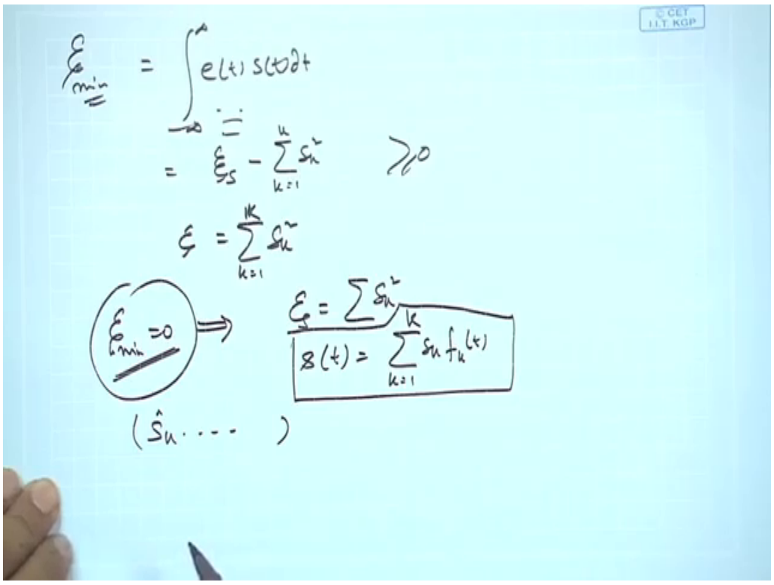

And if you would calculate the e minimum; that means, the minimum energy that would turn out to be minus infinity to infinity e of t, s of t dt and this will work out to be within a few steps e of s minus summation s k squared k equals to 0.

Now, since this is an energy this should be greater than or equal to 0. So that means, this whole term should be greater than or equal to 0. And in the best case this will be equal to 0. So, in the best case you will find that e of s is equal to e of s k squared equals to 1 to capital k. So, if we have to set the criteria that e min to be 0. So, we can only say that this is 0 only by the condition that the energy of the signal can be represented by s k squared, where you can construct s of t as s k f k t k equals to 1 to capital K.

So, you should be able to construct the signal using this form where these are orthogonal basis functions in the minimum mean squared error sense ah, that is the signal energy matches the energy of the original source signal, where these s k s can be found by this relationship over here.

So, once we have this then what we can clearly state is that, we have found these coefficients s k s which are the coefficients of the signal and they could give us some hints towards signal representation.

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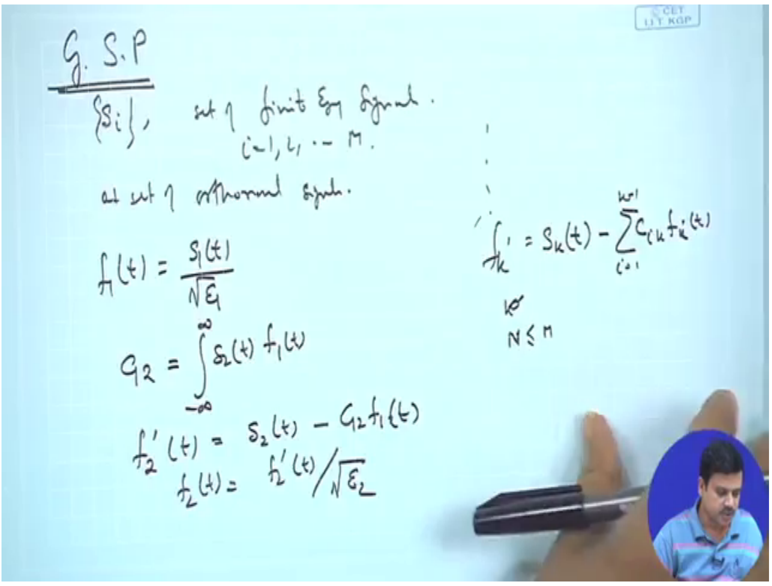

So, what you have at this particular point what one should typically do is, one should look at the gram Schmidt procedure for the signals. So, in this case we will summarize by suppose s i is a set of finite energy signals, is the set of finite energy signals for i equals to 1 2 up to capital m. And we wish to construct a set of orthonormal signals, right from this set.

Now, why we want to do this; that means, you are given a whole bunch of signals and you would like to construct the basis set from this. So, that any signal in that signal space could be represented in terms of the basis function in this form, where you have this coefficients. So, please try to understand this that we have a whole set of signals given to us and we if we can construct the basis set from this using gram Schmidt procedure then we can find the projection of the signal on this basis which will term as our coefficient which will serve as the coefficients for the expansion of the signal.

So, to do this we will we will similarly follow that let f 1 of t is equal to s 1 of t upon the square root of energy of 1. And then you would calculate c 1 2 which is the projection of the second signal on the first signal. And then you would calculate f 2 prime of t which is equal to s 2 that is, the second signal that we have selected minus c 1 2 f 1 of t. And finally, f 2 t is equal to f 2 prime of t upon the square root of energy of 2.

So, if you look at the process it is very similar to the one that we did for vectors. And similarly you will go on to find f k prime which is equal to s k of t we will take the k th signal minus c i k f i of t, i equals to 1 to k minus 1. So, in this way you will be constructing k prime. So, which or you can go up to let us say some value n which is less than or equal to m. Because you have m signals similar to the other case for which you can find this orthogonal basis set. And you will be given through a tutorial how to use this.

So, finally, once we have these then any of the signals that are available with us we can find a projection on these f 1 and f 2. You can find the similarity of the expression here. So, these f 1 and f 2 that you see can be used to construct s of t using coefficients which can be calculated using this expression; that means, if you are given a whole set of signals s is of t it means, if you are given a whole set of signals s i of t over here from which you construct this orthonormal basis set. Once you have constructed this orthonormal basis set then you can expand the signal in this form wherein, you will use s k s from this expression and all of this follows with the minimum mean squared error criteria. And then you can say that you could represent some k th signal.

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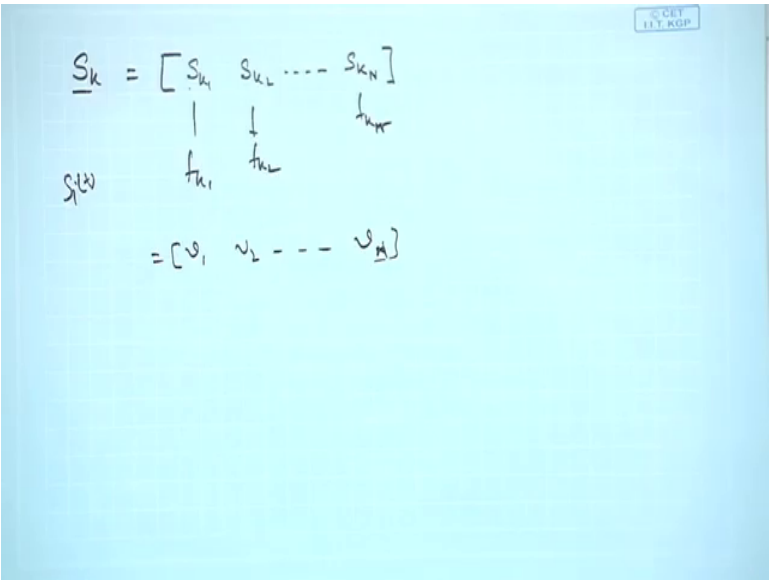

From this original set; that means, any signal which is in this which is from this. Or in this signal space as s k 1 s k 2 up to, let us say s k n. So, if you are representing a signal in this form basically this is the coefficient in the k 1 th function.

This is in the k 2 th function, this in the f k n th function. So that means, you could represent a signal which was s i of t in a form of coefficients like the way we have represented vectors in the initial set where we said vector could be v 1 v 2 upto v n, right. So, now, if we are able to represent the signal in a vector form then we could use the linear algebra or we could see it in n dimensional vector space and then we could proceed with this in designing transmitters as well as designing signal processing mechanisms at the receiver. We can use some of the results of vector analysis to find solutions of receivers which we will see in a few lectures from now.

So, we summarily have completed characterization of signals and systems as well as noise and we will move into designing of transmitters or analysis of transmitters from the next lecture onwards.

Thank you.