Modern Digital Communication Techniques
Prof. Suvra Sekhar Das
G. S. Sanyal School of Telecommunication
Indian Institute of Technology, Kharagpur

Lecture – 22
Characterization of Signals and Systems (Contd.)

Welcome to the lectures on Modern Digital Communication Techniques. So, so far we have discussed source coding. And after source coding we have been discussing about characterization of signals. We have described some ways to look at different kinds of signals or how to categorize them in different ways, different forms. And then in the last few lectures we have discussed about how to represent the signals in a form, so that they can be very general. And what we found out is generally if there are if the signals are represented in form of exponential weighted sum of exponentials. So, which is usually represented through Fourier transforms then you can handle things quite manageably.

So, in the previous couple of lectures we did discuss how to represent a signal and we have taken a bandpass narrowband signal. We defined what is meant by that. And we had related the low pass equivalent signal representation to that of the bandpass signal.

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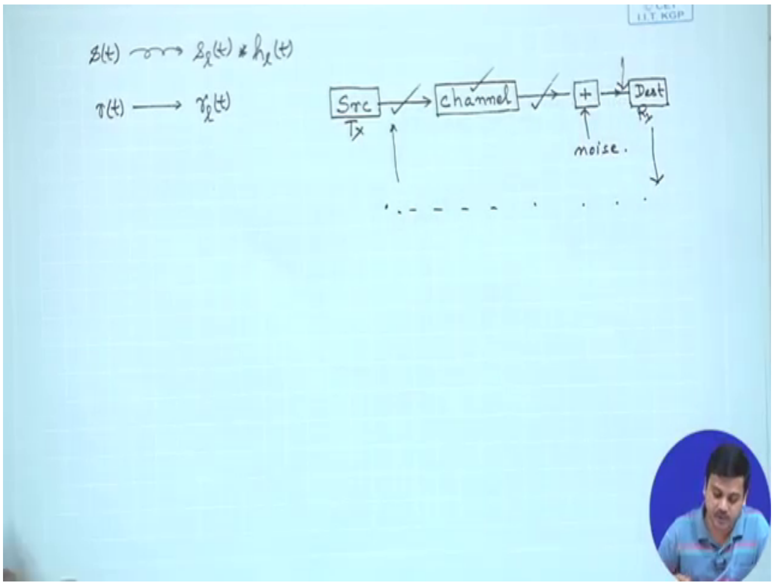

So, we had taken s of t as the real bandpass signal and then we have calculated s l of t so that means, if you would recall we had s of t and we had connected it to s l of t through the relationships of e to the power of j 2 pi of ct multiplied by with s l t as well as taking the real part of it. And we had also looked at how to represent the spectrum in the pass band in terms of the low pass equivalent representation.

So, once having done that then we said that this pass band signal would pass through a bandpass channel. And then we define that the bandpass channel in a similar form where we said that the bandpass channel would allow signals within it is pass band. So therefore, the band should be around the centre frequency of the carrier f c or f c of the carrier would be around the band pass of the channel in that form. And then we established exactly a similar relationship except the coefficient in front of it for the relating the passband equivalent pass band of the representation of the channel with that of the low pass equivalent of the channel.

Then we moved on and we said that well you have a passband signal which goes into a bandpass channel and definitely the signal that comes out of this would also be a bandpass signal and we call this r of t. So, we connected r of t to correspondingly r l of t. R l of t is the low pass equivalent and then we found that r l of t is connected, r t is connected to s t through convolution. Similarly r l t is connected to the s l t with convolution with h l of t which is the channel impulse response equivalent low pass representation.

So, having done so, we have the source we have the channel and then we started looking at instead of looking at signals we looked at stochastic process. We said what about a narrowband stochastic process we are started looking into it. Now why we are looking into such a thing, it is not arbitrary the reason being you have a source which generate signal goes into the channel, right. We said in one of the early lectures at some point, the channel is something which would change the source signal, right. In some form which is usually like a filter like when you send something into a filter it modifies the signal. And yes in the previous lecture we have seen that the outcome output of it could be r of f or low pass equivalent it is r l of f is equal to the Fourier transform of the signal multiplied by the Fourier transform of the channel impulse response.

So that means the multiplication in the frequency domain and it is a convolution in the time domain. So, that is what it does it changes the signal as if there is a filtering process. Now then there is the receiver or the destination, it is good to call this the transmitter it is good to call this receiver. Probably we have said before or you can say now also. If you would look at the several receiver components this will be filters, this will be oscillators, this will be noise amplifier low noise amplifiers and so on and so forth which would have several passive as well as active components.

Now, these components that are present would generally add noise into the system. And generally in one of the models well accepted models there is additive noise present in the system, right. And we briefly stated that noise is the realization of noise could be through a sample function. And which can be thought of as a bandpass stochastic process, right. And of course, we will we will talk more about it

So, before actually studying noise would like to study the characteristics of a bandpass stochastic process. So, that we have it is equivalent representations and corresponding relationships between the time domain and frequency domain properties of it, so that when we analyze the system we have modelled the signal here, we have modelled the channel, we have also modelled the output here. So, once we are able to model this then we will receive the signal at this point then our job would be to process the signal finally, with the aim to reconstruct what was available at this point as closely as possible. So, we have covered these parts we will now, we are now in this particular section.

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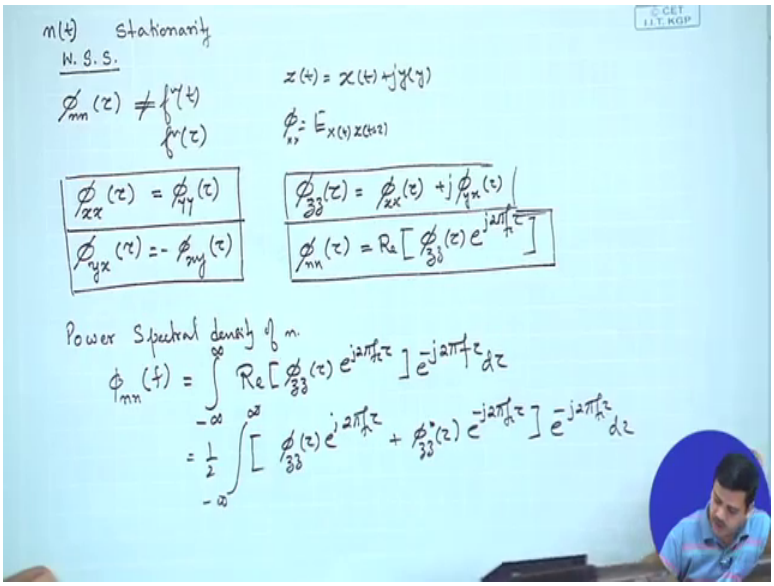

So, what we have done in the previous lecture was we had taken n of t as a sample function of narrowband stochastic process. And we discussed about it is correlation and we invoke the stationary condition. We said that let n t be a wide sense stationary process.

Now, for a wide sense stationary process we said that the typical thing that we know is, the correlation function is dependent only on time lag and not dependent on the time. So that means, if we say phi nn of tau, where tau is the time lag. It is not a function of time it is independent of time, but it is some function of tau. So, going by that we did arrive at a few relationships like phi xx of tau is equal to phi yy of tau where x and y are the i and q components of the corresponding low pass equivalent quadratures of phi and t, that is what we said.

So, we also said that if we said t which is the complex envelope of n t is represented as x t plus jy t at phi is the expectation phi xx is the expectation of x of t with x of t plus tau, that is what we said and similarly phi yy t. So, this is one of the relationships that, we had and we also had phi of yx of tau is equal to minus phi xy of tau.

So, these are the 2 conditions that we had established in the earlier discussion. And then since, we also had the relationship that phi zz of tau is equal to phi xx of tau plus j phi yx of tau. And we could connect that phi nn of tau is equal to real part of phi zz of tau, e to the power of j 2 pi f c tau.

So, these are some of the relationships which we had established in the previous lecture and this would help us of course. So now, what will be interested in is, so here if we look at things we said that these, this is deterministic this is deterministic, but since typical of these things are outcome of stochastic process there is not much sense in actually taking the Fourier transform of these, these z ts or n ts. Rather what would be interested in our correlation properties and the power spectral densities which are duels of each other in terms of Fourier transform.

So, what we will be studying is the power spectral density of this noise n t. So, so we will get in phi nn of f that we are going to get through the Fourier transform of the phi nn of tau. So that means, we will be directly writing it real part of phi zz of tau e to the power of j twice pi f c tau times e to the power of minus j 2 pi f tau d tau. So, this is what we have with us now typical this expression we have seen before. So, we can straight ahead said that this is equal to half integral minus infinity to infinity, then bracket phi zz tau e to the power of j plus phi zz tau conjugate, e to the power of minus j phi 2 pi f c tau.

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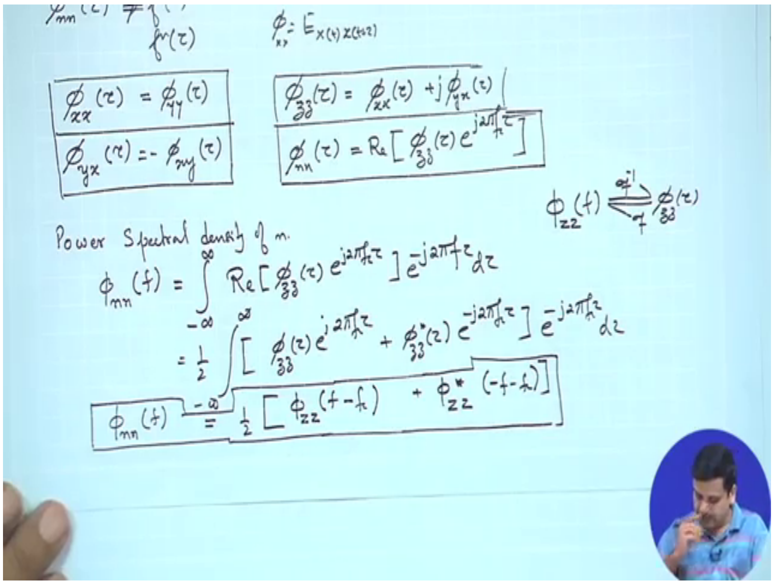

This is what we have been doing earlier. So, it is not new for you and we want to get phi nn of f is equal to, if phi zz of tau has a Fourier transform of phi zz of f, if this and these are Fourier transform pairs.

So, then what we have is this whole thing becomes this, right. Where we can say phi zz of f is the Fourier pair of phi zz of tau. So, this is the inverse Fourier this is the Fourier relationship. So, naturally this we have been using is of course, phi zz conjugate minus my f minus f c right. So, this is the expression again relating the autocorrelation of the band pass stochastic process to that of it is equivalent low pass forms where z is the complex envelope of n. So, this is the basic relationship which we have established. Now using this we look at a few more properties of this noise process.

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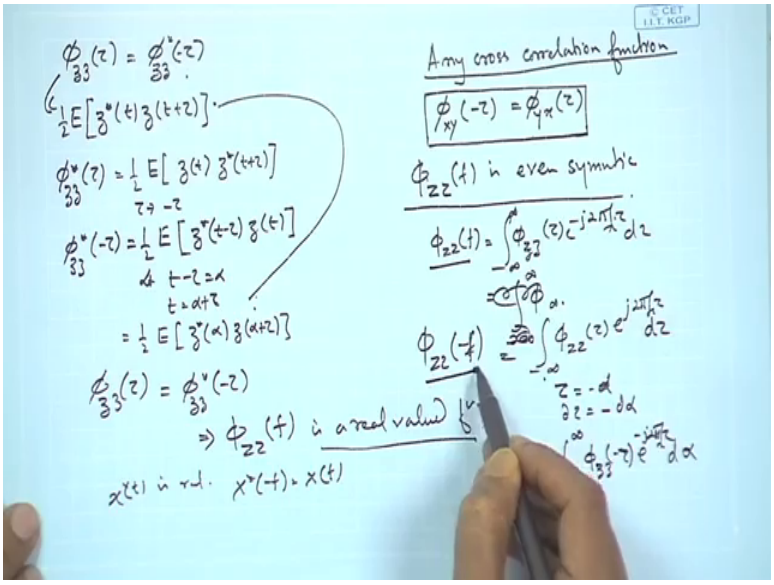

So, one of the important properties that we can try and see is that phi zz of tau is equal to phi zz minus tau conjugate, right. This will help us in arriving at some of the important things. So, phi zz of tau this left hand side is equal to half, e is the expectation operator z conjugate of t z t plus tau, right. And phi zz conjugate of tau would be simply conjugation of that half e of z of t plus z conjugate of t plus tau.

So, we could change this things and then if we would replace tau by minus tau, what we are going to get is phi zz conjugate minus tau is equal to half e. So, I will I will write it here. So, tau replaced by minus tau conjugate t minus tau 0 of t. And now we can say that let t minus tau be equal to alpha. So, what we have is t is equal to alpha plus tau. So, what we get is half e z conjugate of alpha times z alpha plus tau.

So, if you look at this expression and this expression they are similar. And this is equal to this, right. The left hand side is this now we are seeing that the, right. Hand side which is this expression is same as this expression in the left hand side therefore, we can say clearly phi zz tau is equal to phi zz conjugate minus tau right.

So, from this one can establish that phi zz f is a real valued function. I would like you to recall when we did the Fourier relationships we had if x of t is real, we said that x conjugate minus f is equal to x of f, right. That is what we had established. So, now, similar thing applies over here if this is true then this is the dual in terms of the frequency domain or the Fourier transform we get this result that phi zz of f is real valued.

So, this is one of the important properties that we will be using as we require. So, now, a small another small activity that we need here is any cross correlation function would satisfy the relationship that phi xy of minus tau is equal to phi yx of tau. So, this particular relationship one can try proving it or otherwise, we may fill this gap at some later time.

So, we have these 2 things handy with us and with this we can move forward. And we could also say that phi zz f is even function or even symmetric right. So, we can look at that as well. So, you have seen this is real valued we will just need to see that this is an even function. So, what we could check is phi zz of f is equal to minus infinity to infinity phi zz tau e to the power of minus j 2 pi f c tau t tau, this is by standard definition, right.

So, since this is real valued, phi that that is real valued we would go here sorry, we have this thing. So, we could also go by that just a second phi zz f is equal to phi zz tau, we could also say this is because of conjugate symmetry, we could say this is equal to minus infinity to infinity phi sorry, what we need to do is at this point we will calculate phi zz minus f. This is what we are going to calculate, phi zz minus f. Because for even symmetry we need to prove phi zz f is equal to phi zz minus f. So, minus infinity to infinity this will be phi zz tau, e to the power of j 2 pi f c tau.

So, we have simply changed f 2 minus f and this becomes plus d tau. At this point we would like to change over tau is equal to minus alpha. So, if we let tau is equal to minus alpha d sorry minus alpha d tau would be minus t alpha. And this range of integral would change from infinity to minus infinity; however, because of this minus again the sign is going to reverse. So, you are going to get minus infinity to infinity d tau can be replaced by a d alpha and there you are going to get phi zz of minus tau e to the power of minus j 2 pi f c tau is equal to this. So, through this we could establish that phi zz of f is equal to phi zz of minus f; that means, it is an even symmetric function.

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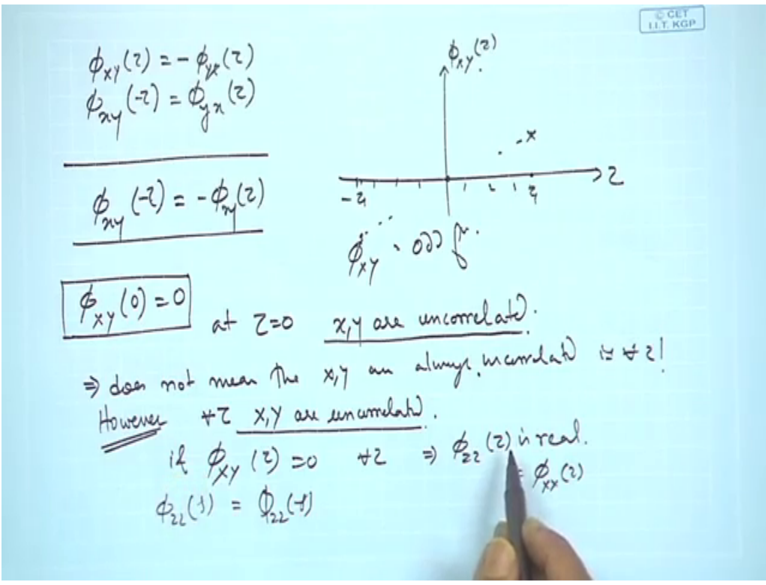

So, now since we have established these few relationships what we could further state is that since we have phi xy of minus tau is equal to phi yx of tau, this is one of the terms that we have. We also have phi xy of tau is equal to minus phi yx of tau. So, with these 2 relationships we could basically establish, that phi xy of minus tau is equal to minus phi xy of tau. So, what this basically means is if we would have this tau axis phi xy of minus tau. So, if let us say phi xy at some value of tau, let us say tau 1 has a value here and then I want to find that this be phi xy of tau of minus tau. So, phi xy at let us say 1, 2, 3 sometime interval, right.

So, here I want to find it minus tau 1 phi xy at minus tau 1 is equal to minus phi xy of tau. So, basically you are going to phi xy of tau with a negative sign, going to go there. And if you let tau equals to 0. So, you want to get phi xy of 0 is equal to 0 which is basically should be equal to 0. So that means, if you look at this it will be an odd function phi xy is an odd function, right. It is it is an odd function that is what we get and for odd function what it would mean is that, phi xy of 0 is equal to 0, right.

So that means that at 0 delay you are going to get no correlation between the x and y component it means at tau equals to 0 x and y are uncorrelated. All right, but now please remember, this does not mean that x and y are always uncorrelated that is for all tau it is not necessarily true. So, but however, however at this point let us make an assumption that for all tau x and y are uncorrelated right. So, if we make this particular assumption that x and y are uncorrelated what we get over here in this particular, in this particular case is that x and y are quadrature components as well as they are uncorrelated. So, we will we will take this particular assumption. And we will see what happens So that that helps us in characterizing noise finally, right

So, with this assumption that if on the resumption or what this particular case if for all tau they are uncorrelated; that means, phi xy would be 0 for all tau, that means, if they are uncorrelated; that means, that is if phi xy for all tau is equal to 0 right. So, this in turn would mean phi zz of tau is real right. So, if this is real then this will be equal to phi xx of tau, right. It is not equal to phi xx of tau plus j phi yx of tau. And this would also help you in further proving that phi zz of f is equal to phi zz of minus f as well, right.

So, we have this things, this thing that phi zz of t is real only if this condition holds true. So, now, in this case we take we are moving slowly into this noise. So, in the special case where.

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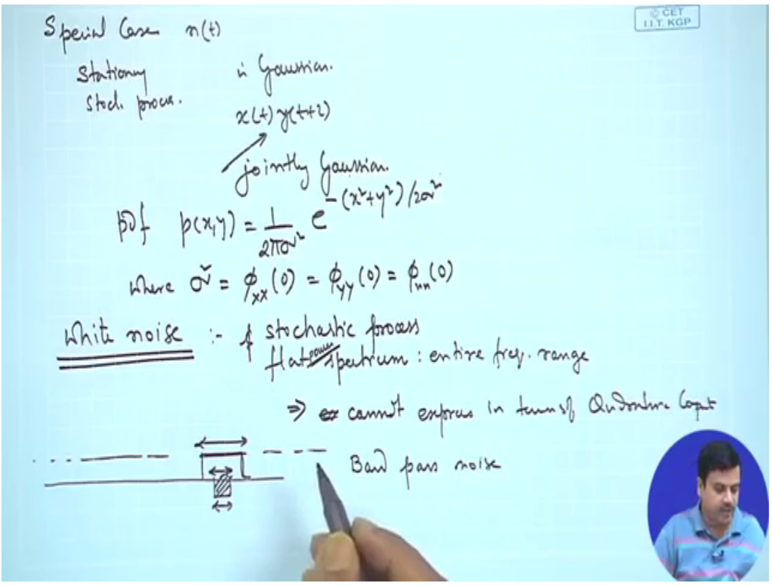

So, we consider this special case where the process n t the stationary stochastic process, Is Gaussian, right we take this to be Gaussian and that means, x of t and y of t plus tau we take this to be jointly Gaussian, right. We take to be jointly Gaussian and that means, they will be statistically independent so that means, the pdf of such a function is given by p of xy equals to 1 by 2 pi sigma squared e to the power of minus x squared plus y squared by 2 sigma squared, where sigma squared is equal to phi xx of 0 which is equal to phi yy of 0 which is equal to phi nn of 0, right.

So, once you have this then we can use this further. So, what we now define is for what we now go into is white noise. So, when we talk about white noise it is a stochastic process; that means, it is a process which produces values which are random and they keep on changing with time. So, that is what definitely it is. And it is defined to have a flat spectrum, flat power spectrum and call it a flat power spectral over the entire frequency range, right. And so basically this condition would mean that you cannot express this in terms of quadrature components.

So, what it means we should not we will not be able to express these in these terms neither use these expressions; however, when we when we have done that what we can say in typical communication system, your signal generally passes through a narrow band of frequencies and if our signal is covering a band which is much smaller than that and then we have whereas, originally noise was across this whole thing.

But now if we say that it passes through a band pass filter where the width of the filter is much wider than the width of the signal in that case we have a white noise over the band of the signal right. So, in that case we can treat the noise as band pass noise right; that means, the noise which is passed through a particular bandwidth.

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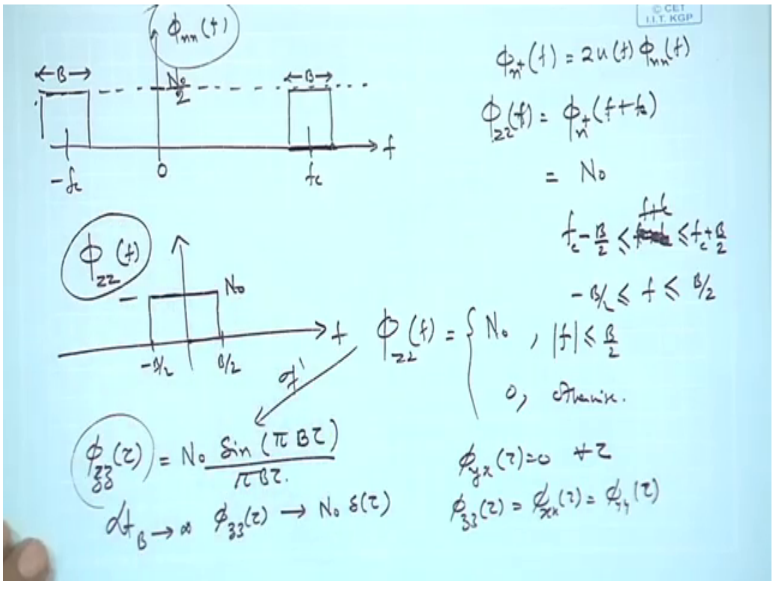

So that means, for such a case you would define phi in an f as otherwise it was covering this entire band with the constant power, but now we say that it is covering only a particular band which is cantered around f c and let is band B.

So, similarly on this side if we merge this is minus f c is covering this band with this thing and the value is n naught upon . So, this is the standard representation of band pass white noise. And what we are interested now is calculating phi zz of f, which is the equivalent low pass form of this. And clearly from this you can see that it is symmetric and hence we can reconstruct the whole thing from, this is 0 of course, from one side and we will use typically as we had done before that phi n plus; that means, only the positive frequencies will take to u of f phi nn n of f. 2 because there are 2 on both the sides and you means we are going to take only the positive.

So, this is one way of doing it you could do it in another way and then we would take phi zz of f as we translated one of this. So, phi n f plus f c, if you do this just see at phi zz of 0 is phi n plus; that means, this thing at f c so that means, it is now got translated this is taking this frequency. So, if I take phi zz f plus B by 2; that means, here it will be B by 2 plus f c so that means, now it is gone here so that means, your phi zz is between here to here which is B by 2. And also on the other side minus B by 2, right. And clearly this particular one would result in n naught right. So, 2 times n naught by 2, 2 times n naught by 2 is going to n naught. For the range of frequency f minus B by 2 less than equal to f plus f c, because this is only positive less than f plus B by 2, right. F c plus B by 2 that means for f.

So, f plus f c this particular set of frequencies in this range f c plus B by 2. F c plus B by 2 and f c minus B by 2, only in this range is this value equal to n naught and otherwise it is equal to 0. So, if we now work on this equation what we will get is for f lying between B by 2 and minus B by 2 this is equal to n naught.

So, in other words you can say that phi zz of f is equal to n naught for f less than or equal to B by 2 and 0. Otherwise now, from this you could calculate the inverse Fourier transform and you could get phi zz of tau which is equal to, now this appears as this phi zz and this is f right. So, this will be n naught sine pi B tau upon pi B tau.

Now, if in this one you let B tends to infinity; that means, you take this whole bandwidth of your consideration then you would find phi zz of tau almost tends to n naught delta tau. And the power spectral density is symmetric as we are seeing we have also seen that phi y of x of tau is equal to 0 for all tau, So that would mean, phi zz of tau is equal to phi xx of tau equals to phi yy of tau.

So, with this what we complete is the relationship between the passband noise process the autocorrelation function and that of the phi nn n of f so that means, yeah. So, we could relate phi n of 2 phi zz of f; that means, given the power spectral density of the noise in the passband, we can also calculate it is equivalent power spectral density at the baseband.

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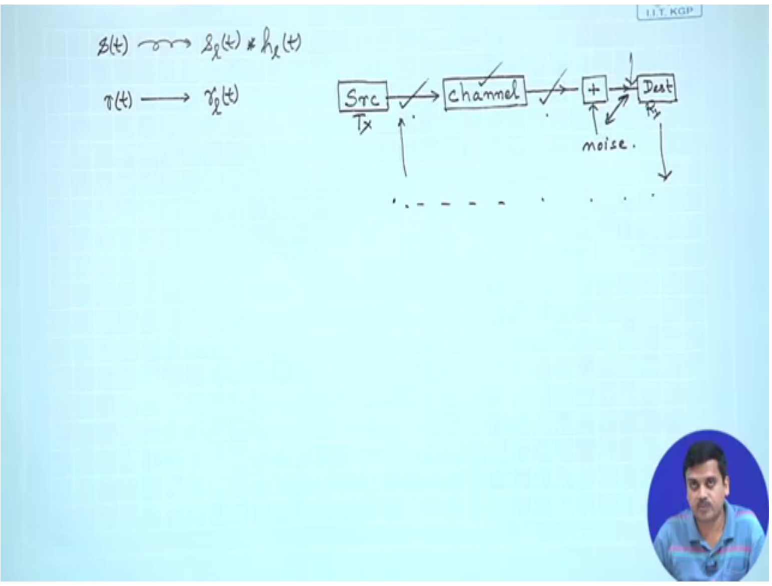

So, using so now, using all of this what we have done in summary, in summary we have taken the frequency relationship of the passband and low pass equivalent of the signal of the channel and the output of this channel when excited by this. As well as the properties of stochastic band pass process which we some early say as noise. So, now, we have, but of course, for noise we have not talked about the Fourier transform the sample function, but rather we have established the relationship of the autocorrelation function as well as the power spectral density.

So, these particular set of expressions that we have developed are very critical and we will be using them for the establishing various receiver components and writing down expressions of transmitted signal in the rest of the course. We need to do one more important thing that is the signal space and the gram Schmidt orthogonalization which will give us a picture of the vector mode of viewing a signal. And then these once we are through with these things I think our representation or the way of unambiguous statement about a communication system will be easier. So, this particular part might appear to be a bit confusing, but clear statement is we need to do these things by yourself a couple of times. So, that you are used to this way of notation and you can be confident of using this notation in all future setups that we handle.

Thank you.