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NPTEL ONLINE CERTIFICATION COURSE

Course
On
Analog Communication

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Lecture 41: Random Process (Contd.)

Okay, so in last few classes probably we have started discussing about random process and in the very last class I think we have proven one of the most important property of a random process which will be heavily used for our case of noise analysis, so that is called the Wiener khinchin theorem right. So that is something we have already proven we have said that if a process is at least wide-sense stationary and then what we can do is we can always evaluate the autocorrelation function that is the ensemble autocorrelation the way we have defined and then we know that that if it is wide-sense stationary then ensemble autocorrelation function should be just depending on the separation of time instants where the samples are picked.

And if you do a Fourier transform of that autocorrelation function then you get the average power spectral density, so this is this is one of the most fundamental theorem that we have proven which will be heavily used because that is how we can actually take any random signal and then try to guess what will be the spectral quality of that signal right which spectrum components are present which are not present.

So although spectral analysis or visualizing the signal from the frequency perspective that will be useful for us okay so let us try to define on that same line some more property regarding this autocorrelation function, so let us say I have a wide sense stationary process XT okay.

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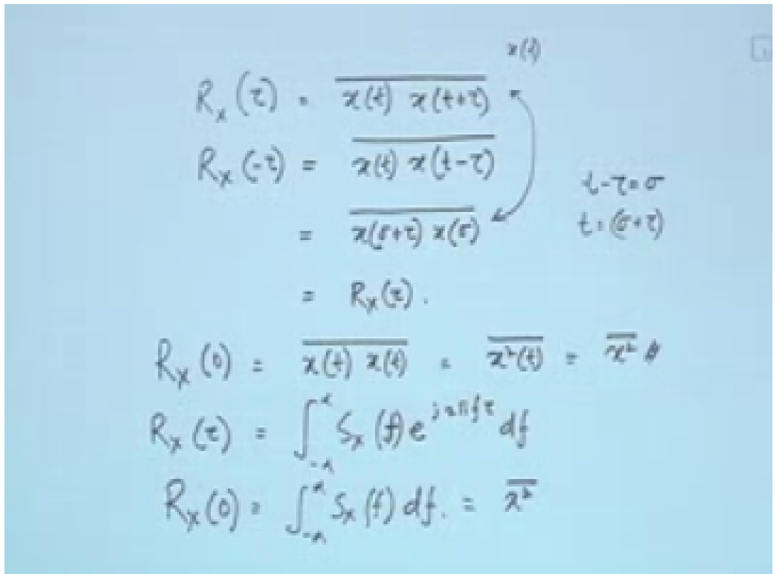

So whenever we are mentioning we are always mentioning that it is at least wide sense stationary whether it is stationary or not because we had to go for a higher order so that is something we have seen, so at least white sense stationary has to be mentioned so this is the process which has a autocorrelation function once it is wide sense stationary then we know that autocorrelation function just depends on the time difference between the samples we are picking.

So it is RX τ which we have defined as X T XT + τ and the ensemble average over that right, so that was our definition immediately if we try to guess what will be our X- τ so that should be xt x t minus τ average right. Now what we can do is we can we can assume that t - τ is σ right, so immediately we can represent t to be σ + τ right and I can write this as X σ + τ X σ average, now σ is just a time right so it can be any dummy variable so I can again put T in place of σ and it is just for evaluating that IV variable and it does not depend on basically we have told already this entire autocorrelation function does not depend on that particular value right T where it starts or σ.

So it does not matter whatever I take so I can immediately say that this and these two are equivalent so this is actually R X τ so this is a fundamental property of autocorrelation function we have also proven this same thing for a time autocorrelation part right earlier when we were dealing with deterministic signal at that time we have proven that autocorrelation function for a particular sample signal which is deterministic in nature it's always even symmetric for a real signal right same thing is happening for a random process also.

So the way we define autocorrelation function we can again see that it is even symmetric that means - τ is same as our X τ right, so this is true and then we can also try to guess what is this that autocorrelation function when this τ separation is zero which is if I just put it in this definition it becomes XT multiplication XT oh that is just X^2 T average and then we have said this is wide sense stationary signal so therefore the means first-order or second-order if I just take no separation, so first order second order up to means any order mean should be our moment should be independent of time, so it should be X^2 bar and that is the means that is the standard deviation or means variance of this signal okay.

So what it says that the autocorrelation function already captures the means associated whatever underlying associated random process is there any time you actually sample it can be any time because every time is equivalent and if you take the variance of it should be that okay so as long as the mean is same this is actually becoming the power of that signal right. So that is also very true because we have also told that X^2 t if we just try to guess how much is that.

So that should be the power and in a different way also we can correlate it to the power if suppose the signal is also argotic then what will happen? The second order means this same average time average if you wish to calculate that should give me the power so that is equivalent because it is argotic signal so immediately we can say this gets correlated or related to the power of the signal okay.

So what we can understand from these things that autocorrelation is a very strong property of a signal it just gives me back many information about the signal itself okay, I do a Fourier transform I get the spectrum of the signal of course the average spectrum I put 0 in place of τ then I get the power of the signal back ok as long as the signal holds property like it is erotic it is wide sense stationary and all those things okay.

The other part is this what I know that this is our X τ is actually the Fourier inverse Fourier transform of the power spectral density, so if I write power spectral density ass X F okay so therefore it must be $e^{J 2 \pi F \tau}$ D F right, now we know that hat RX $\tau = 0$ it becomes power, so let us try to evaluate that our x 0 over here also within the integration τ must be 0 so once τ is 0 this becomes 1 so it is just - ∞ + ∞ s X F DF this is a well-known thing that that is the power so power is nothing but the average sense power spectral density that you have got for this particular random signal.

So that power spectral density like the deterministic signal you just integrate from - ∞ to + ∞ over the frequency domain you will be again getting back the power okay, so it is all sitting very nicely mathematically as you can see the way we have we have actually developed a strong tool to evaluate the random process and then once we have developed that tool we can see everything is now almost getting similar representation with respect to our deterministic signal okay, whatever we have understood in deterministic signal here also we are almost getting similar representation we know a power spectral density of course this power spectral density is a average power spectral density.

But it behaves like for a deterministic signal whatever power spectral density you are having for that also we are integrating from -∞ to + ∞ in frequency domain we are getting back power here also same thing is happening we are integrating it we are getting power back okay, so that is very nice everything is sitting nicely. Now let us try to probably in the last class we have already started characterizing noise with this random process. So we have told the most random noise that we can think of is having probably autocorrelation function.

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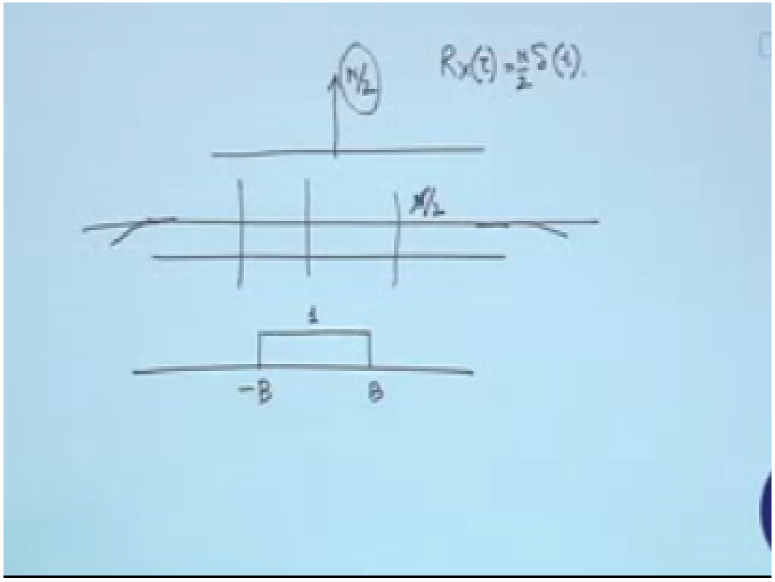

Which is RX τ which must be almost like δ okay with some strength let us say that that is related to the noise power okay let us say that strength is n / 2 okay so therefore that strength must be n / 2 okay, so we are just saying that this is probably the representation of noise there were some reasoning behind it that we were saying this is the noise because it is a kind of signal where in the current time if I have some if I observed some voltage level or some current level or power level whatever it is next time instance however in finitely similarly small or closer it is to this current time instant I will have no predictability that means I will not be able to say that it should be within this range I will have no predictability it can take any value any value possible from -∞ to+ ∞.

So that is probably the most random noise that can come across whenever we are transmitting or that we can come across whenever we are transmitting some signal right, so that is why we say that is basic noise which is something which is completely random with respect to our signal transmission and the autocorrelation function immediately is very high correlation whenever we are not shifting it but a slight shift in fact even if it is in fact simile small that should take the articulation function to zero okay.

So it should not be correlated because I cannot have any predictability with respect to the current observation, so that should be represented as δ function immediately what we can see for noise the power spectral density becomes because we know that if it is a δ function of ε / 2 if you just do a Fourier transform that's what we have understood that Fourier transform autocorrelation function should be giving me back the power spectral density, so that should be this so it should be with ε / 2 or n / 2whichever way you represent so that should be the noise.

So this is the most random noise that you can counter and fortunately most of the noise you'll be seeing looks almost like that within the band of our interest of course it will not be flat over the entire band because then noise power if you start integrating it from – ∞ + ∞ that what we have understood that noise for will be from - ∞ + ∞ you will have to integrate but if you start doing that it will be infinity right.

So noise power will be infinity in that case, but within the band of our interest probably this will be flat okay so that is a very important understanding that it will be probably flat of course at a very higher band it will start showing some low-pass effect okay, but within the band of your interest this will remain flat okay so that is the characteristics of a general noise. Now let us try to see most of the time what we will be encountering in the communication means whenever there is noise in communication what we can see if the noise is like this suppose from the channel it is coming like this so if I just allow the entire noise then huge amount of power because you integrate over the entire frequency domain huge amount of noise power will be coming into my signal and that will of course contaminate the overall signal.

So I want to reduce the noise power because I want to increase enhance the signal-to-noise ratio or signal-to-noise power signal power to noise power signal power means whatever message I am transmitting related to that the power related to that and noise power is whatever is coming out okay out of this process. Now I wish to suppress this noise power because otherwise my signal-to-noise ratio will be very high which is not good because if noise is bigger than signal I will not be able to means decode my signal okay.

So if this is the case then what I will be employing most of the time I know my signal will be within a band of interest if it is modulated it will be in a band if or maybe a in a pass band if it is not modulated directly transmitted in the baseband it will be a low-pass signal okay, so whatever it is I will have to employ either a low-pass filter or a band pass filter so that is why you will see that most of our energy now will be concentrated on characterizing this low-pass noise and band pass noise okay.

So that is something we will be concentrating on so first let us try to see what happens to the low-pass noise so I know that noise spectral density is something like this which is flat okay, now I let us say I pass it through a ideal low-pass filter which has a characteristics of this from minus B to B it has transfer function one and rest of the places it is just zero okay. So if I just pass this through this process what do we get okay so this is something probably we will have to now see that a particular random process if it is passed means passing through a particular transfer function or a particular system how the output will look like okay.

So that something will try to characterize now and after that probably will characterize this low-pass, so let us say I have a system.

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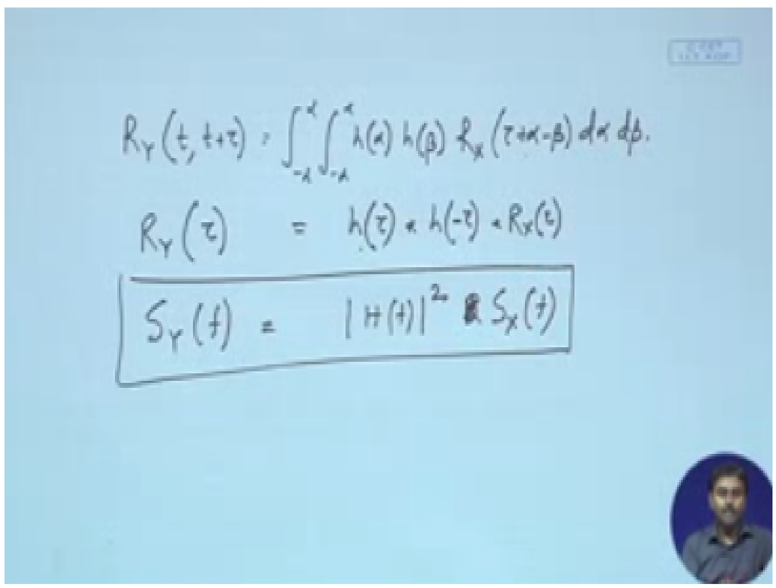

Which has a transfer function of H(F) okay, and the associated impulse response which is the inverse transform of this which is H(T) okay, so I know this and I do give our input X(T) which is let us say a stationary or at least wide-sense stationary random process and I want to get some output and I want to now characterize this output now it would also will be random okay it should be a random process but I want to characterize this random process okay.

But what we know that if for a given XT and if I know this HT this should be for a linear time-invariant system it should be convolution of these two, so I can immediately evaluate YT should be -∞ to + ∞ H α X t - α D α right this is something I know okay because I have told that this particular system through which I am passing my signal that is linear time-invariant and we have already characterized that for linear time invent this is what happens output and input get this relationship as long as I know the transfer function or the Associated impulse response okay.

So this is the case now I want to characterize this output process so what I have to do how do I characterize so far we have discussed that characterizing any random process requires calculation of autocorrelation function, so therefore I need to evaluate also YT + τ right it is all about that so YT + τ will be just its when I am passing my XT + τ that should be because it is linear time-invariant so time wise it should be just YT + τ because it is time invariant so YT + τ must be according to our understanding of linear time-in variant system that that should be X T +τ – α yeah right this is something we have already understood that linear time invariant system should work like this.

So therefore now all I have to do is find out the autocorrelation function, so our Y τ I will try to find out which is nothing but Y T YT + τ and symbol average right which is nothing but these two and symbol average so - ∞ + ∞ H of course there are two integration I will be merging these two integrations so this integration variable inside variable I will take any I can take any dummy variable so I will take α for 1 and β for another one right so the first one I will pick α x t – α D α second one I will pick as β, so that should be H β X T + τ - β D β right and I need to do this.

Now these two integration are not dependent on each other so I can actually Club them together now this ensemble averaging that is really on X that has nothing to do with this internal variable of α and β so therefore ensemble average can be taken inside so I can write this as H α does not depend on that so which is not dependent on that X random process so H α remains as it is H β remains as it is and XT – α X T + τ - β I take an average of that D α D β I can write this right.

Now what is this? This is if t -α I take as something okay let us say σ and this should be σ plus something okay I can immediately write that as σ plus something, so this is just nothing but autocorrelation function of X which I know already and I know X is stationary see for this probably I have written a wrong thing I still didn't know while doing this whether the output process will remain stationary or not I had no idea I have written this but I should not have written this I should have written this as t T + τ because I do not know whether this will be stationary or not okay or wide-sense stationary or not.

But X I know for sure because that is the into input process I have given and I have chosen it to be wide sense stationary right, so if that is the case this must be function of just out which is the separation between these two right. So I can write this as ry so let us write it correctly t t + τ must be -∞ + ∞ -∞ + ∞ this is H α this is H β and this becomes this fine as this if I just do so that should be T will be cancelled and I have τ + α – β so that should be τ + α - β D α D β okay if you just carefully see this is just to basically convolution nothing other than that.

So it can be written as H τ I I would not do that this is take that as homework this is just a -convolution we already know single convolution if you just do it two times this will be a convolution of H τ H - τ and rx τ okay this is the convolution of these three things. So therefore it is just a variable of τ only nothing else so therefore I can now write my output process that are why that is just a variable of τ that has nothing to do with T because T is already getting cancelled due to the property of X where X was wide sense stationary okay.

So immediately I can see the output process which is being generated it is also a wide sense stationary process, so if the input remains wide sense stationary I can pass it through any LTI linear time-invariant system I will always get output process which is also a wide sense stationary process okay. And the process is related to this now I can do a Fourier transform of this, so immediately I will get s Y F because this is the autocorrelation function so Fourier transform of that Wiener khinchin process tells me as long as this is wide sense stationary which is the case I will be getting my average power spectral density so I get average power spectral density.

Now if I do Fourier transform of this should be HF and this should be H - F or H * F so that must be mod HF^2 and Fourier transform of this must be sxf because this is the autocorrelation function of the input signal that is a very nice fundamental result we have got okay, so two things we have derived through this process one is that I give up input to a linear time-invariant system remember all these things has to happen it has to be linear time-invariant we have put all the formulas of YT YT + τ knowing that it is linear time anyway otherwise I could not have written that okay.

So the process has to be linear time-invariant and the input should be wide sense stationary at least if this two condition happens properly I know that the output process also will be white sense stationary at least okay and on top of that I can get the power spectral density which is nothing but the transfer function square of that linear time-invariant system or mod^2 of that linear time-invariant system multiplied by the input average power spectral density okay, so this is a very fundamental result which we will use for characterizing that low-pass system okay. So now try to just see what we were doing for that low-pass system I was having a noise.

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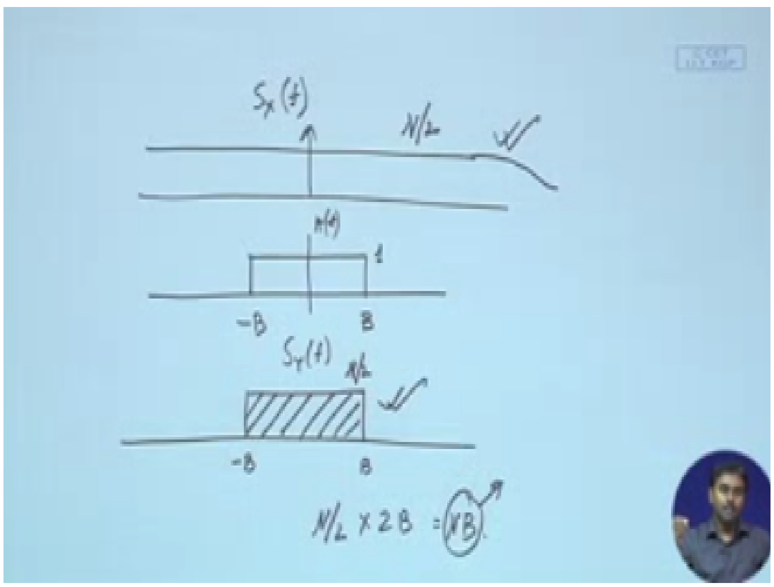

Which was characterized by this ideal noise of having power spectral density so this is my input let us say SX F, so this is my input which has a power spectral density of this which is constant over the entire frequency range having value ε / 2 okay, now I pass it through a ideal low-pass filter having bandwidth B. So what do I get this is one so therefore I must just get this into mod if this is HF mod of this square because this is all the one between minus B to B it will remain 1.

So I will just get ε / 2over that band and rest of the cases it should be all 0 so my output that s why F must be like this it is ε / 2 from - B 2 + B and everywhere else it is 0 right and then actually I can talk about this process I cannot talk about the power of this because as I have told that probably we are just representing it but it is actually showing in finite power because from -∞ + ∞ you integrate it will be infinity but actually it is not like that it should have a low pass characteristics at a very high frequency probably it will show some low pass characteristics but I am not bothered about that it is flat in the region of interest of my frequency okay.

And then I wish to see what will be the output noise power now I can integrate this to get my output noise power so that should be nothing but ε / 2 x 2 B because it is all constant, so integration will be just multiplication of this the band and that overall power spectral density that should be ε x B so this happens to be a low-pass nice power okay. So whenever you have a noise which is most random possibly that can be and most of the cases it is okay so and that noise if you pass through a low-pass filter with all our derivation winner from starting from wiener Khinechin to what happens if I pass it through a LTI system.

So after doing all these things this simple understanding we have got that the output noise power I will be able to evaluate this is going to be a very important part of our noise analysis further you have to be very careful about doing these things correctly okay, so once this is being done let us try to understand some more property of this whole processes okay. So one thing is that is called in when we were talking about random variable we are actually trying to show the joint PDF of two random variables right.

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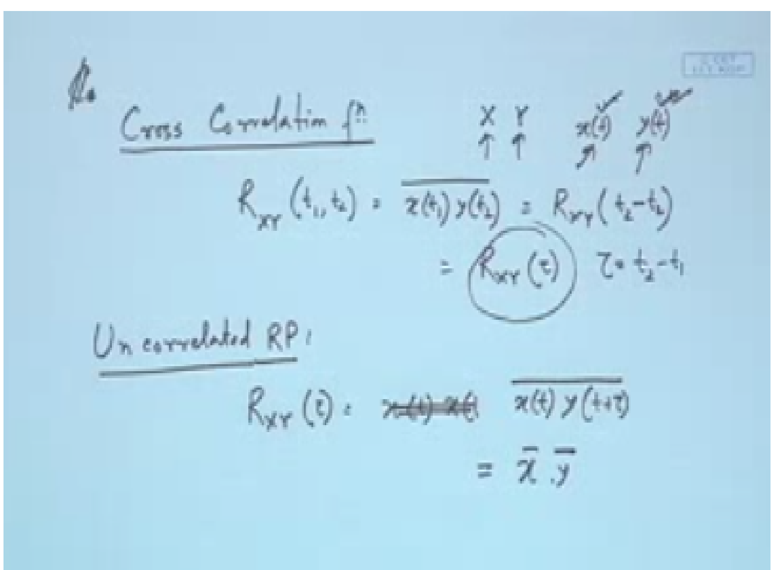

Same thing will now try to do over here also, and when we will be doing noise analysis you will see why that is important it is really required that we take two processes and then try to see their means either correlation or cross correlation whatever you term them okay, so without proof I will give some of the theorems which are required for us because if we start proving all of them probably it will be means we would not be able to cover the major part of analog communication in the duration of this course.

So I will give some property of it so the first part is cross correlation okay, so cross correlation function between two random processes x and y both of them we assume to be wide sense stationary at least so both x and y or I should say XT and YT they are independently means I should not say independent over here because that is another property I should just say that they are probably means they are actually who at least wide sense stationary okay up to second-order I know that they are stationary then the cross correlation is defined as this okay.

Still I have not exerted the property of there being means wide sense stationary if they are then I should be getting this just a difference of these two so this must be our XY just a function of t2 - t1 it should not have the sense of origin that it should not be dependent on sense of origin it is just the difference between these two time instant it should be dependent on that okay or I can write this as our XY τ where τ is defined as t2 - t1that time difference all right, so this is what it should be.

Now and this will happen see both of them are probably white sense stationary but if this has to happen then there is probably I have not mentioned another extra property which is required that's called they should be jointly stationary as well so independently they might be and separately they might be stationary wide sense stationary but they should be also jointly either fully stationary or white sense stationary that has to happen then only their cross correlation will be dependent on just the separation of the time instants we are taking.

So there is a now you can see that there is a concept of joint stationary okay, now next thing will be defining which is called uncorrelated we have already seen it for random variables now we will see for random process what do we mean by uncorrelated it is just by definition okay the uncorrelated random process at that time this are X Y τ that we have defined if they are jointly stationary okay.

So this is as we know it is XT and sorry okay, so what should happen this must be okay, so when they are uncorrelated I should be able to just get the autocorrelation joint or cross correlation function to be just the multiplication of their respective mean where okay and because it is stationary these two processes are stationary so there is no notion of time because mean at any time will be same okay.

So therefore if two processes are uncorrelated this must be happening okay, so what we will do? We will try to define some more property of these things in the next class and we will see how they are means they will be utilized in our further processing okay, thank you.