Spread Spectrum Communications and Jamming
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Lecture – 61
MUD - Probability of Error

Hello students. In this module our target is to derive the expression for probability of error, for the optimum multiuser detection. So, this is in continuation of the last two modules, where we have discussed and understood the concept of the optimum multiuser detection mechanism and we have also seen the receiver architecture where, the optimum data have the optimum detector is devised and how it is working. Here actually we have also understood that the efficiency of our detector optimum multiuser detector is given by the complexity involved with respect to the total number of the users present per bit. And it is a total complexity involved in the detection per bit and basically its expression is given by the total complexity and complexity involved per bit given by the total number of the users.

And here actually in some other way we will try to do the performance analysis of these optimum detectors which will be the probability of error computation.

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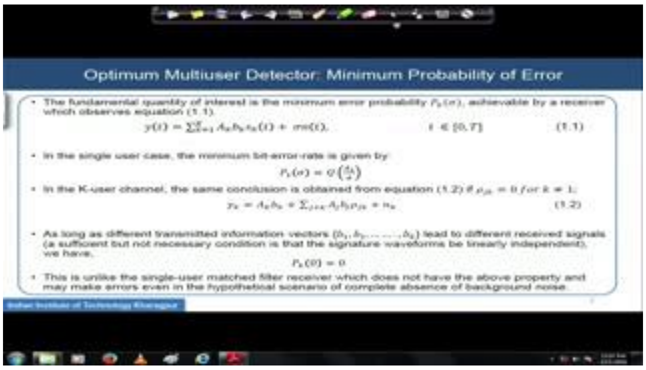

Remember one thing that the whatever the detector architecture we do, whatever the receiver architecture we propose finally, our target of the interest and our point of interest in the any receiver architecture design is the minimum, getting a minimum error probability. You minimize the error of your decision process and try to detect the transmitted signal as means perfect as possible. So, what is a error probability? Error probability calculation is the very very important parameter and very interesting parameter for all the system engineers.

The error probability here in the multiuser, optimum multiuser detector situation we will be actually defining the parameter as P k sigma and where, the k is indicating the number of the users. So, over which actually you are a carrying on and the small k can vary actually for the whenever we compute the minimum error probability it is associated with a typically user also. So, this k parameter is actually varying for the user to user and in presence of the noise actually it is important. So, it is a noise, the variance of the noise will be sigma square and. So, P k sigma basically in presence of that noise you are trying to identify. So, the fundamental quantity of this minimum error probability is achievable by a receiver with the receiver I understood that receiver structure is like this.

Where, we understand that for k number of users are present in the channel and each of them are transmitting with their transmitting their corresponding signal b b k I should say with their unique signature wave form s k associated with it and A k is the received amplitude of the signal associated with the k-th user and we. So, we have considered n t continuously over the last two modules. The n t is the Gaussian noise process associated with it and in the single suppose if I see that this is the very generate module at the frontend of the CDMA receiver in the multi user scenario. If I consider the fact that we have only a single user. So, this summation goes off and your ending up with the y t is equal to A 1 b 1 s 1 t plus n t sigma n t and in that case we understand that for a single user the minimum bit error rate or minimum beta error probability a probability of getting minimum error which is governed here by P 1 sigma can be governed by a can be given by a key function of the SNR involved.

Where, A 1 is the amplitude you have estimated, you have received from that user 1 and this is the noise power spectral density sigma is about. This expression is very well known expression to all of us because we always deal with this situation for a conventionally user where, conventionally receiver where multiple user concept is of. If I extend it now for the K user channel so then we understand that for the k-th user you will receive you have the interested part of the signal related to the k-th user and also you will receive the interference. Now from the other users and also the noise associated with this k-th user.

And so if my rho j k and rho was what rho was our cross correlation between the two signature vectors, two signature wave forms and if rho j k is equal to 0 and the for k naught equal to 1 then, actually this will also as long as this the same conclusion we can obtain. I mean this whole part will go back and you will get for the user 1 again for the error rate is equal to error probability will be given by this. Because if the rho is 0 perfectly; that means, the signature vectors are aligned to be perfectly orthogonal to each other then there is a contribution from this second part of this expression 1.2 will be completely 0 and you will be ending up with the expression equivalent expression of 1.1 and hence your error probability will be governed by this.

In practice, that is not the case in practice you have to deal with this whole expression where sigma k but the rho k will have some specific value between 0 to 1, and an as long as this different transmitted information vectors and in b 1 to b k that are coming from the different K and different users, they lead to different the diseased signals. We have actually that rho k is equal to 0, but remember one thing this is very unlike expression that we have received for this multiuser detection scenario. It is we never get actually in a single user matched filter because single user matched filter output will never give actually with this property is does not hold good for this single user matched filter because there it can give actually some hypothetical scenario, if something happens that the background noise is totally absent, but still it will give at the output the matched filter output will give some error. It will generate some signal at the output it may high generate.

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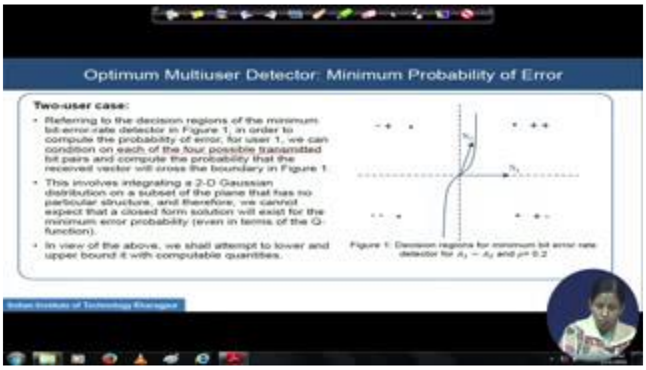

So, matched single user and matched filter output can never ensure you that you will get the P k is 0 is equal to 0 when, my noise variance is completely 0. So, computation of this error probability calculation we will start with a 2 user case because it is easy to track and easy to understand whether and in future it can be extended also for the K number of the users. We will refer again to this diagram that we have seen in the earlier discussion and we understand that referring this decision regions of this minimum error probability detected in the this figure. In order to compute the error probability it will be for user number 1 same. It will be fundamentally to device the conditions on each of this four possible transmitted bit pairs such that its specifically four possible transmit bit pairs and then you compute the probability that the received vector will cross the boundary of this decision zone, when other part is transmitted.

And once we try to do it no then it will involves actually a integration over the 2 D Gaussian distribution and on a subset of this plane and which cannot have any close form particular structure and therefore we cannot have any Qs from expression for this in integration over the 2 D Gaussian distribution and hence we cannot actually come with a nice expression, closed form expression for example in terms of the Q function in such scenario.

If we involve this 2 D Gaussian distribution integration of other 2 D Gaussian distribution. So, we will try to do what know I mean instead of showing something that does not have a closed form expression with a known form or function like the Q function and which has actually finally, solution is there because if you are not having any closed from expression you can solve that equation by numerical optimization or some numerical solutions are available, not optimization numerical solutions are available for that expression. But better approach may be taken by providing or trying to find out the lower and upper bound of those expressions such that you can understand the performance of such detectors will be bounded by this region.

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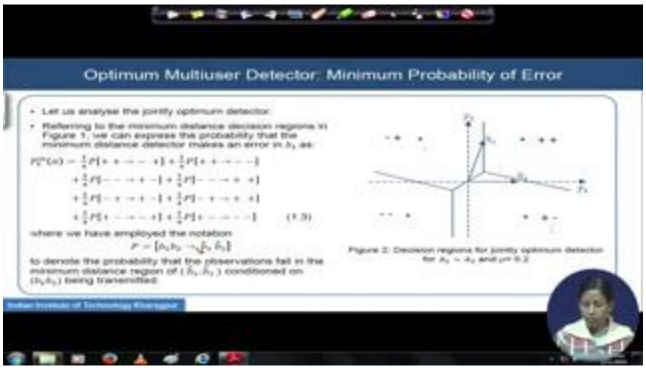

So, error probability can never go beyond these levels nor cannot come cannot be improved below this level. So, upper bound and lower bound we will try to find out now. So, let us start with and we continue with the analysis with this jointly optimum decision strategy and referring this decision, minimum distance decision regions that we have seen earlier that a minimum distance detector makes an error in identifying the b 1 as this. So, P 1 in is actually P m stands for the minimum distance detector and actually it will try to find out, it will try to detect a signal by minimize where, actually the distance is minimum that is a fundamental is about. So, the minimum distance detector giving by P m if it is a user 1 confined to user 1 that is a meaning of P 1 m is in presence of the noise sigma is a power spectral density of the noise we have already considered.

So, that error can be written like this. What is this form I have written is? 1 4th is the possibility that the probability of 1 4th and the probability is that your transmitted is both the signals plus 1 plus 1, I am b 1 and b 2 transmitted plus plus. I mean you transmitted the signal here this signal is transmitted, but because of the wrong decision or the changes in the channel you have detected it to be minus 1 plus 1. So, actually signal transmitted in these coordinate and you have detected that as if this signal is transmitted. So, another is the 1 4th is the probability that you have transmitted these and you have detected this vice versa. You have transmitted this, you have decided this one or transmitted these and you have decided to be here.

Other two is you have transmitted minus plus and you have got here or you have transmitted minus plus and you have turned to be here equivalently. You have transmitted plus minus, you have certainly been here. You are detecting you are here or you have transmitted this and you have you are detecting that as if you are here. So, actually whenever we have you are writing it basically the way this probability is written is like this b 1 b 2 was actually transmitted, but this is estimated so to denote the probability.

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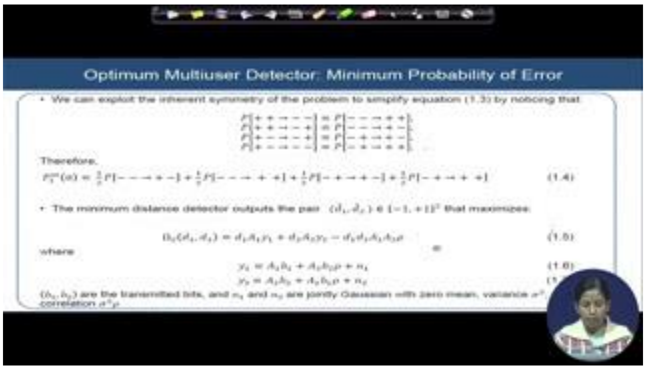

Next so we understand actually that instead of writing all this eight different probabilities. Fundamentally we will see that some are there equivalent. For example, probability of finding that plus plus two plus signals where, two plus signals where transmitted but wrongly it was detected as double minus is equivalent that double minus transmitted and detected double plus. Double plus transmitted minus plus is equivalent to double minus detected and the plus minus. Similarly these two probabilities are also equal. So, hence if I substitute this in this equation of 1.3 in the equation of 1.3 then, we will be ending up here. Here actually that whole expression of a substitution of them we will be ending there with a fact that half of the times.

So, this minus minus it is transmitted it will be detected as the plus minus and half of the time the minus or both are the minus transmitted, but detected as the double plus and plus other two terms are like this that it is minus plus transmitted, but detected as plus 1 and half of the time minus plus transmitted by detected as the plus 1, both are plus 1s. But this minimum distance detected outputs the pair, it is actually the detector is actually finally, give you the pair estimated detected pair that belongs to this minus 1 and the plus 1 and that maximizes now that being omega function once again given this 2 d 1 and d 2 to this transmitted.

Is given by this fact this expression and where actually your y 1 is fundamentally the A 1 b 1 plus A 2 b 2 multiplied with this cross correlation factor and embedded in the noise. Similarly y 2 will be given by this expression 1.7 and we are understand that the b 1 b 2 are the transmitted bits and n 1 n 2 are the jointly Gaussian noise zero mean and variance actually sigma square and this rho is sigma square into rho is will be the correlation factor. Now see one important thing in this expression of this 1.4, this first term. The probability that observed better due to this is closest to plus minus and remember when can it happen, let us first see.

So, we transmitted double minus, but we actually detecting that it is plus minus means we transmitted this, but we are detecting that as if we are here. So, when it can happen? It can happen only if the observations are very close to plus minus then actually the minus minus. I mean the due to some reason actually I am detecting that I am here because it is a detected minimum distance decision region best detector right. So, he tries to actually find out where the distance is minimum, if your detector symbol is such that, its distance from here to here and from here to here.

If I try to compare it minimal distance from the detected symbol to the plus minus zone. If this distance is closed then you will detect that as if the transmitted signal is plus minus, instead of minus minus. So, the decision is becoming wrong because actually you detect a signal is or the received signal is fundamentally closed to this zone. It is not close to the double minus zone.

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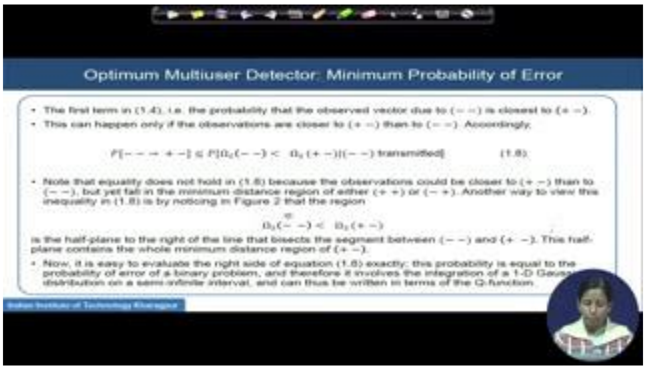

Or the (Refer Time: 16:14) number 4 and that is a only reason why actually this first half will come into picture, which is a wrong decision first wrong decision will come into the picture and this situation is accordingly leading us to the fact that the probability of double minus transmitted, but you see plus minus. It is actually the less than equal to the probability that double minus we transmitted this omega bigger omega for the two number of the users with double minus is less than the plus minus. Given actually you have transmitted the double minus. So, note that here this equality cannot does not hold good because this observation could be closer to plus minus than to minus minus, but yet fall in the minimum distance region of either plus plus or this minus plus.

So, another way to view it, realize it is something like this. In the figure we have shown that this is the zone right actually, this is the boundary zone of the decision and. So, you try to see here and map it like this. In the figure 1, the region of this double minus, it is less than the plus minus. It is so just the half plane of this right of the line that bisects the segment between this and this. So, this half plane contains the whole minimum distance region of this plus minus in the figure.

So, it is now easy to evaluate means once I actually write down this expression or the concept like this. So, this right side is easy to evaluate and because is a 1 dimensional Gaussian distribution on a semi the infinite intervals and we can easily trace it and we will be seeing that, we can actually easily express this expression or this probability functions by the non Q functions.

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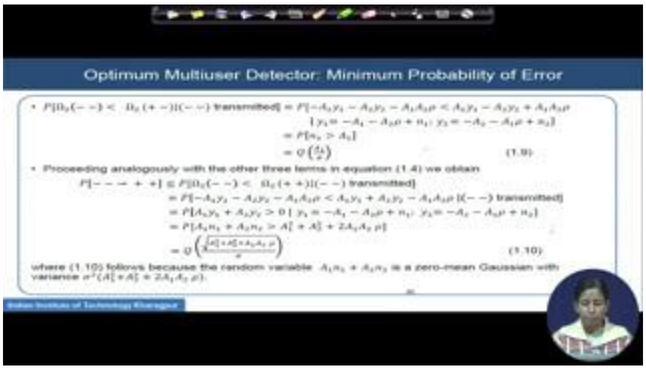

So, now we are here, that probability of that capital omega 2 of transmitted double minus is less than equal to this given that double minus was transmitted. That can be written like this expression. Given that y 1 was equal to this and y 2 was transmitted like this. Fundamentally once we write this expression no basically you are trying to find out whether the probability of n 1 is greater than A 1 and if this is the probability you are can actually justify it by a Q function given by the SNR, given by 1 and the noise. Equivalently when that by going by the same logic till transmitted double minus, but your receiving double plus it is the cross change that can only happen if this is the situation that is happening and.

Which will lead you directly towards the another Q function expressions governed by the expression 1.10 and where as you see actually when we have derived this no, we have utilized the fact because at the random variable A 1 n 1 plus A 2 n 2 where actually this is the equation where probability that this is coming greater than this whole quantity. It is the zero mean Gaussian and that the variance will be governed by this. So, the variance of this total signal is coming as a power of the transmitted power of the signal section and this is the variance of the (Refer Time: 19:48) density of the noise. So, given the (Refer Time: 19:50) density of the noise you will be ending up with the variance of this A 1 n 1 plus A 2 n 2 and. So, finally, the Q function will be governed by this expression.

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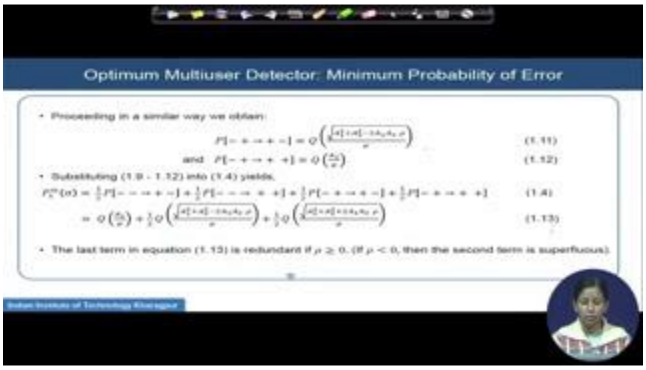

Next is the third term where, similarly from minus plus transmitted plus minus we are ending up with will be again with and other Q function where, this is the variance of the another terms of A 1 n 1 plus A 2 n 2 even n 1 minus A 2 n 2. And this is actually minus plus transmitted, where we are ending up with the double plus it will be also given by the another Q function. So, if I substitute all this values in the fundamental expression of this probability of getting error, we will be ending up with the equation 1.13 where, two times we got this Q function of A 1 by sigma. So, that is why that term is coming is Q of A 1 by sigma other terms are remaining with the corresponding multiplication of half.

So, this last term in this equation it is redundant it will be if my rho is greater than equal to 0. Equivalently this term will be redundant if my rho is less than equal to 0. So, accordingly actually you are of which zone of the SNR you are dealing with based on that you can either discard the second term or the third term and the calculation of the error probability will be ending up with any two terms over the zone of the operation.

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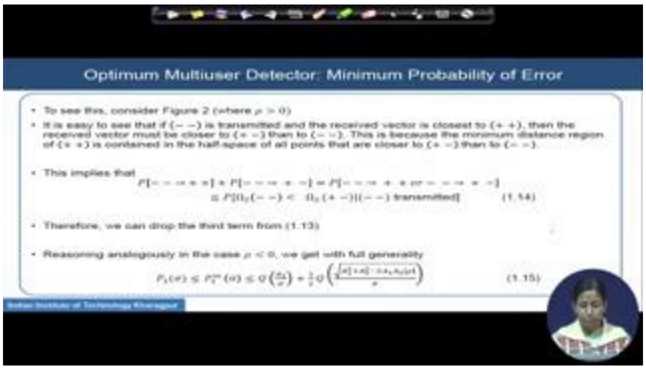

So, that can be also expressed by the figure. And finally, actually we are ending up where; we are ending up with the fact with that if I am over the rho less than 0. So, that rho less than 0 is a situation that where the cross correlation values which be actually are with basically the cross relation values are the very very low and in that situation we can finally, ending up with the fact that the probability of error for the user number 1 will be basically given by these two terms which are coming from the fact that you are here. Now this is the upper term and so, it is how we are going I had so you are upper bounded by this.

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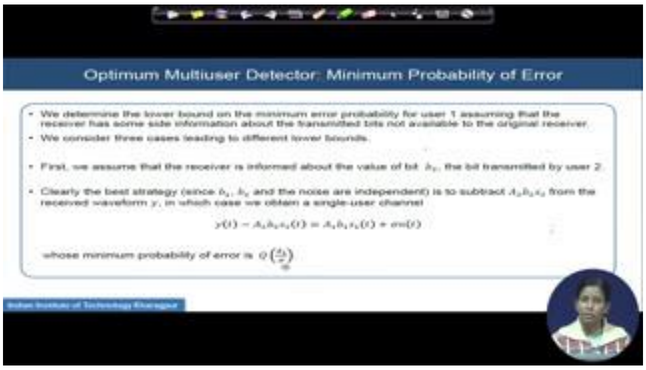

So, the expression he saying that you there is a maximum possibility of the errors possible given by the whole expressions. So, this is the upper bound as if on the minimum distance based error calculated in the optimal multiuser detector. And now the task is to give the lower bound also on the minimum error probability for this user 1 and we assume the receiver has some side information. Now otherwise actually computation of the lower bound is not feasible and this inside information is related to the transmitted bits, but not available to the original it is some side information about the transmitted bits are there.

So, we will consider the three different situations. Where the, to calculate this different lower bounds; first condition is that we assume that the receiver is informed about the value of this bit 2. So, we understand actually what the bit 2 is transmitted at this movement. It is a bit transmitted by the user number 2 whereas; I am interested to compute the error probability of user 1. So, the now condition is something like that you compute the lower bound on the error probability of user 1, given the information about the bit 2 which is the user to transmitted bit. So, the best strategy is what? Best strategy is that as b 1 and b 2 and the noise all are independent of each other.

So, I can subtract the component related to the b 2. I mean A 2 b 2 s 2 t, you subtract with from the received signal and you will be ending with the simple the contribution of the noise associated with the contribution of the user 1 or intendant users and this one as is very easy conventional user structure. And we can approximate the error probability by the Q function given by A 1 by sigma.

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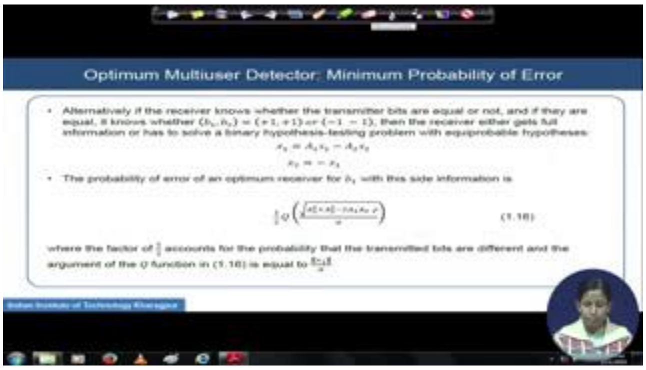

Alternatively, if the receiver knows whether the transmitted bits are equal or not so he does not know what exactly b 2 transmitted, but he understands such one information is that which says that the transmitted bits are equal or if they are not equal. If they are equal then the receiver nodes whether the b 1 b 2 is going to be plus 1 plus 1 or it is minus 1 minus 1. Then the receiver either gets the full information or has a solve binary hypothesis testing given by this. The probability of error in such situation also can be finally, found in the form of a Q function given in the equation 1.16 with the side information that the bits are equal. And when the factors of this half it accounts of the probability there they transmitted bits are different they are not equal and the Q function will be in that situation equivalent to the mod x 1 by sigma.

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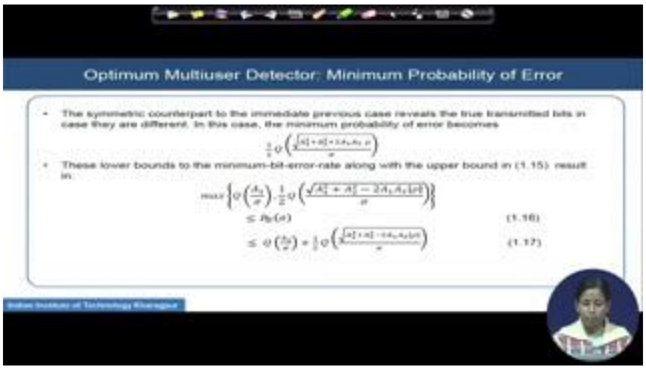

The symmetric counter part of it is to the immediate previous case. It reveals the fact that the true transmitted bits in this case they are the different. If they are different no in that situation the minimum error probability will be given by the Q function like this. So, by getting all this inference we can find out we are now ready with to find out the lower bound on the minimum error probability, along with the upper bound already computed by 1.15. So, which results in that the maximum of this of finding out these two Q functions is equivalently actually will be the maximum fall this will be less then equal to the error probability of the user 1.

And error problem of the user 1 is given by the expression 1.17 and if I try to plot now the single user matched filter detector output along with this newly computed to lower bound and the upper bound error probabilities. We will be ending up with the figure that is shown in the figure number 3.

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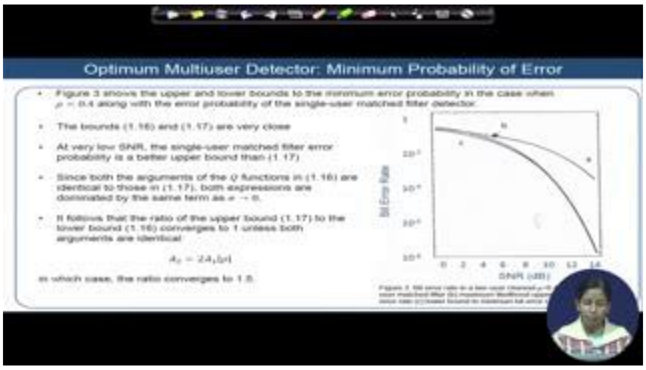

Where this graph a is talking about the single user matched filter detector output and this b is actually showing you the upper bound you have computed, c is the lower bound that we have computed. And we can see that there is not much difference between the upper bound as well as the lower bound, that we have computed over the for the optimal multiuser detectors for two user case and. So, at a very low SNR what we see? The very low SNR we can see that the single user matched filter error is the minimum. It is better to use the single user matched filter at a low SNR zone. Its performance is far better compared to this other two and both the arguments of this Q function that we have found in the equation 1.15 as well as 1.7, 1 7, they are dominated by the term when the by the same term as the sigma tends to 0.

Sigma tends to 0 means you are actually in a very high SNR zone and that is dominated by the multi user expression, multi user interference and both the terms are almost the performing the same. The upper bound and the lower bound say emerge. And we can actually have a would relative ratio of this upper bound to the lower bound. It follows from the fact this ratio of this upper bound to the lower bound it will converge to 1 unless both the arguments are identical and if they are not and if the A 2 and A 1 are related by this expression then the (Refer Time: 28:20) ratio will converge to approximately 1.5.

So, this is the learning about the two user case using the two users we have tried to finally, calculate the error probability and tried to analyze and we have understood that error probability calculation involves the integration over a 2 D Gaussian probability and which does not lead us to a closed from expression like the nice Q function expression we see for the single user cases. That is why we are not ending up there we refer to go ahead with giving a bound over the probability of upper bound as well as the lower bound on the error probability calculation. That is a way we have derived and we have seen that in the lower SNR zone this bounds are having some effect they are different from each other, but as the sigma tends to infinity.

I mean noise effect reduces for the higher SNR zone. Both the bounds mark both the bounds merge. Moreover we also see that for a very low SNR zone; that means, when the sigma tends to infinity and sigma tends to infinity in such situation the multi user at the single user matched filter output performs the best compare to the other two bounds.

So, the final conclusion should be that if you are working in a noise dominating zone, you should relay on or that architecture should go back to the matched filter, single user matched filter based architecture rather than the joint optimal decision strategy that we have discussed for this optimum multiuser direction.