Spread Spectrum Communications and Jamming
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Lecture - 13
Power Spectral Density of ML Sequences

Hello students, we will as we discuss in the last module; that in this module, we will continue with the computation of the autocorrelation periodic autocorrelation function of the spreading sequence PT. And we would not stop only with autocorrelation function calculation of periodic autocorrelation calculations, we are really wish to see how the power spectrum density of this spreading sequence look like. Our focus mainly will be on the ML sequence because we understand that that is the most it is the mother of all the sequence and it is largely used also in the practice.

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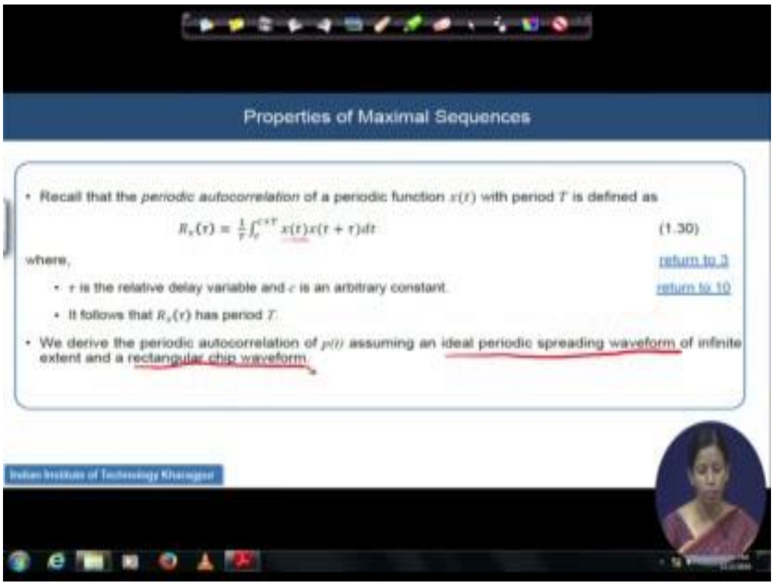

Recall that in the last slide of the last module, we have established this equation of the periodic autocorrelation. This is a classical expression where if x t is a periodic function with a period of capital T, with a period of capital T then the autocorrelation function will be given by R x tau. Remember we are also considering that x t is a stochastic process and with W S assumption which is a wide stationary assumptions that is why the autocorrelation function is only the function of the delay tau. It is not t comma tau. And c is an arbitrary function it can be 0 also, but basically the c is the integration over one period of x t and hence it ends up with the fact the period of this periodic autocorrelation function will be also capital T.

Now, we will proceed to compute the periodic autocorrelation function of p t, where we will assume that p t is having it is an ideal period spreading wave form and it has infinite extent and we will consider a rectangular chip waveform for our consideration for our analysis.

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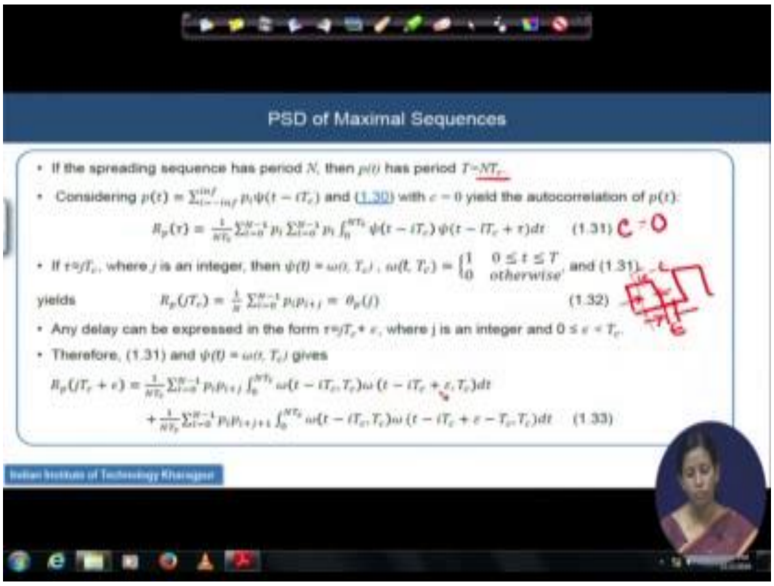

Remember here one thing, if p t if the spreading sequence has a period of capital N then definitely p t will have a period of N into T c, because T c is the chip period, capital N is the length of the spreading sequence. So, the total time that you are having for this period the time that you are having for this spreading sequence, it is equal to capital N multiplied by T c. We know the situation is well known to us, because it is infinite duration sequence p t is. So, he will be given by the series of the bits here or the chips it is not bit it is a chip the pi multiplied with the waveform associated with the psi t minus i T c. We have seen this equation earlier when we were considering the direct-sequence spectrums system.

And if we consider c is equal to 0 and we try to calculate the autocorrelation function using the classical expression that we have shown in the last slide, then we will be ending up with the equation 1.31. What we have done here instead of your x t and x t plus tau i have just written this pi and p a l, and corresponding psi’s are written for pi the psi was t i minus T c. And for l it is replaced t minus l T c, and it should be added with the have a delay of tau. And the summations are kept outside of the integration because they are not related with any dt. And integration is running over 0 to N T c, because the c value that we have considered here for the analysis is equal to 0. So, definitely if it should be average over capital T earlier we saw here my capital T is N into T c.

If I consider now that this tau is equal to j in to T c that means, there is a j is a integer value and the delay is an some integer multiple of the chip duration T c. And let us also consider that this psi t is a rectangular chip waveform. And for rectangular chip waveform, we understand that value will be 1 if for the duration of 0 to capital T; and it will be 0 otherwise. So, once we put all this consideration here then R p instead of tau it will be j T c, it will be ending up with p i into p i plus j because this whole expression will give you the value is equal to 1. We define this R p j T c as equal to theta p j.

Now, situation is such that this delay that delay for which you are computing this autocorrelation function did not to be always the integer multiple of the chip duration. Situation may happen such that you are getting a delay which is equal to integer time of T c plus small increment epsa, epsa is always less than T c. So, with in durations, you are getting, so if this is the chip duration, and you have got some j number of the T c shift then after that it may happen that you have got another shift of amount epsa which is basically a part of the next chip duration. So, for this, what will be the concentration what will be the computation of your R p or theta p equivalently is our point of interest.

And again if what this analysis also we will consider that psi e is having a rectangular waveform. Here see we are following the way we have constructed equation 1.31 instead of this tau j T c, I have placed j T c plus epsa. So, what will happen for one component of w you will get for i T c or it is j T c is equal to i T c for t minus i T c that shift will be will be there. And another one we will get for the extra shift for this epsa. And it will be added with a situation that see you will get one portion one will be the from the existing chip and another will be from the adjacent neighboring chip, where actually due to the shift of the extra shift of the epsa you are entering into it. So, the autocorrelation function now is only not only confined within this T c, it is having some extra contribution from the neighboring chip; and that extra contribution is also getting reflected in both the cases.

So, situation is you are fast integrating over a duration of say epsa and then the remaining part is T c minus epsa. So, one part is T c minus epsa, another part is epsa. So, your autocorrelation function will run for both of these, so that is why the additive term is coming; one is from the epsa, and another is from T c minus epsa. And these are the two staff that you are basically taking care of for using it actually inside the autocorrelation. One is the original one and another is the epsa delayed portion, another is what is the tau minus T c minus epsa delayed portion you are trying to find out with autocorrelation. So, incoming signal is as if coming and because of this j T c plus epsa delay part of it is getting collapse with the T c minus epsa and part of it is getting collapsed with the epsa. So, the same waveform you are actually now taking with the shift to different shifted version of this pulse.

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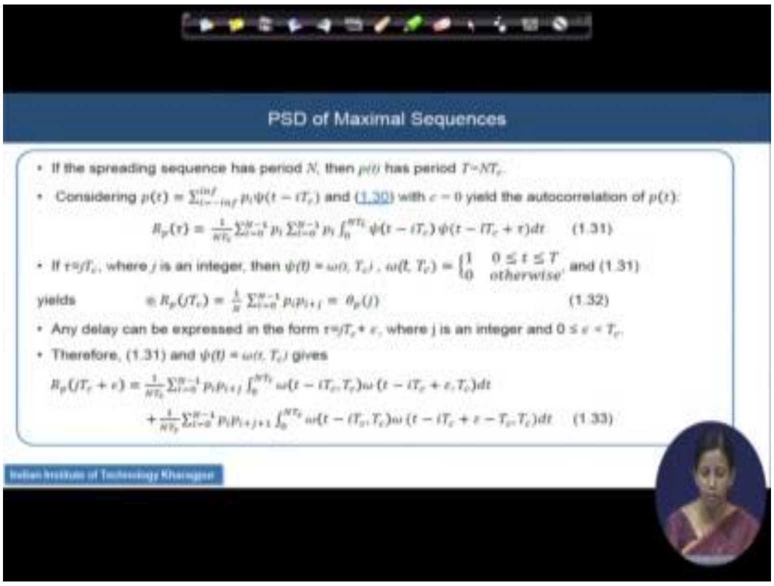

So, if we are again taking the value of the w T c, it is a rectangular pulse as we have discussed earlier, we will see that the earlier R p j T c plus epsa will boil down to this point. If we substitute the value of this theta p and theta p plus theta p j plus 1 from where from the consideration of of these two equation the way we the logic which followed here, and if I follow the same logic, we will be ending up with two values theta p and theta p j plus 1. Remember actually for T c minus epsa duration normalize to T c, you are actually getting the autocorrelation value with theta p. And remaining epsa normalized to T c, if that that period you are basically giving autocorrelation value for j plus 1th chip, so that is why this two contribution are logically also it is followed and you are getting that. Now we will recall that for maximal sequence, there is a substitution we generated for the random binary sequence.

What is the autocorrelation value? We saw that it gives as a triangular function. So, by calling that one we will see that the maximum sequence by substituting from this 1.29 which was actually already discussed in the last slide on the last module. We will come back here and we will be able to see that this triangular function is coming up and this guy will end up with a minus 1 by N part, if my del value or the tau value - the delay is N T c shifted by N T c by 2. The value of this triangular function also already we have seen and if I rearranged this by keeping minus 1 by N earlier and this N plus N by N. And if we see that there is a it is not a one period, it is a multiple period going on a for which we are computing the autocorrelation function and this period is running from minus infinity to and i can run from minus infinity to plus infinity whereas your pt is also running from minus infinity to plus infinity. So, this triangular function now will be summed over this i value and it will there will be a delay of i into N minus i into N T c by T c.

Over one period, this autocorrelation function periodic autocorrelation function of ML sequence will boil down to the lambda to the triangular function of tau by T c which is basically the autocorrelation function of a random binary sequence that we have already seen earlier. In the next slide, we will plot both of them I mean the autocorrelation function periodic autocorrelation function of the ML sequence and the one period autocorrelation function of random binary sequence both are having a duration of T c. We wish to plot both of them. So, we wish to superimpose both the plots in the next slide only because we wish to compare the performance of this autocorrelation function of both the sequence.

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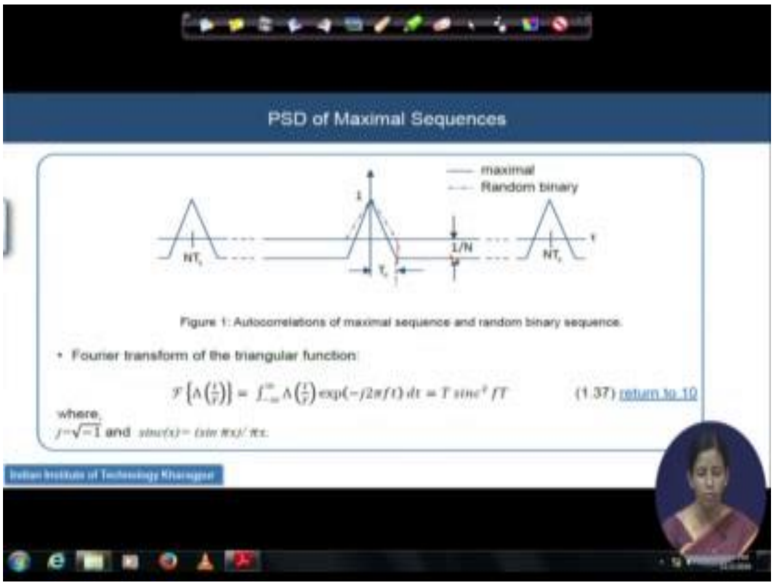

Remember random binary sequence will be ending up over the period of the T c; this is a over the period of T c the random binary sequence will be ending up with this value. And maximum value is definitely always equal to 1. And maximal length sequence, he will actually when T c is beyond you are going beyond the T c value, so for that duration, we saw that it will give a value of minus 1 by N. Where from it came from the fact that here we have realized that even if actually i is equal to completely 0. And there is not existence beyond T c if you are trying to find out, so you do not have any autocorrelation portion like this, but you will be ending up with the value of minus 1 by N always.

So, hence this minus 1 by N is contributing in the figure like this. So, beyond T c you are ending up with this minus 1 by N, and other values are going like this. And now as I told that we are interested to check the power spectral density of this triangular function and we will take the Fourier transform of this triangular function to check the power spectral density. Any triangular function if you take the Fourier transform of it, it could be written like this. And we will see that it ends up with the t sinc square function, a sinc square is given by sin pi x by pi x and j is equal to minus 1.

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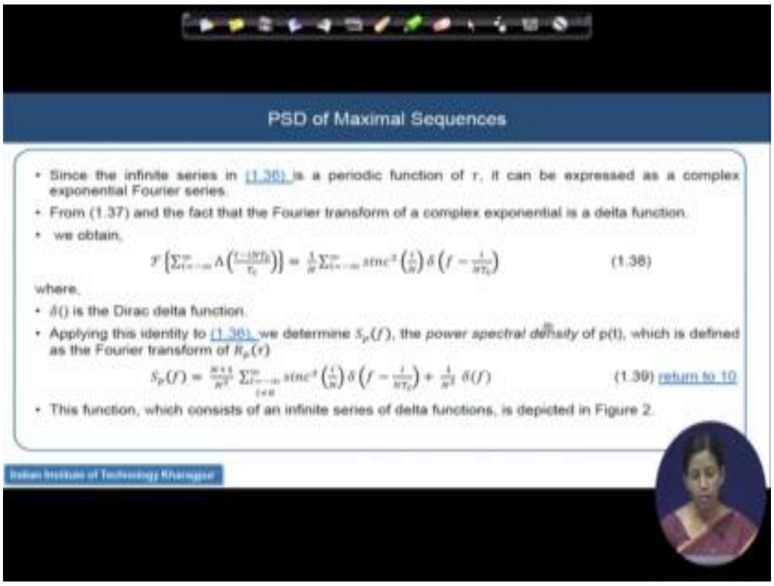

So, Fourier transform of a triangular function ends up with the sinc function. Now, we have seen that the series of a periodic autocorrelation which I am saying here this is an infinite series. And hence this infinite series that was coming in the periodic autocorrelation function, it is a period of tau also and so it can be expressed as some exponential complex exponential Fourier series. And as we understand that the complex Fourier transform of the complex exponential will be always the delta function. So, for this infinite complex Fourier transform of the complex exponential will be turned down to a delta function. So, infinite series autocorrelation the part of this triangular function who is having an infinite duration; the transform of that will lead us multiplied of sinc function multiplied with the direct delta function.

And applying this identity and what on that autocorrelation function, and taking that we understand that power spectral density and autocorrelation function they are the Fourier transform of each other. They will take the Fourier transform of equation 1.36 now. We will take the Fourier transform of these now, and we will use the fact that Fourier transform of these infinite triangular function will be governed by this expression. And hence we will be ending up with the expression 1.39, where the sinc squared and del is the contribution from this continuous triangular function with infinite series of a triangular function which is periodic over tau. And this guy N plus 1 by N square will be the contribution from the other side and 1 by N square into del f that is the part coming when my series is not coming for i equal to 0 this is a contribution. And for i not equal to 0 running from minus infinity to infinity, we will be ending up in this section. We can prove this one by actually putting i equal to 0 also how this part is coming up.

Now, if I try to plot this power spectrum density PSD, and in the next figure, we will show us how is as it look like.

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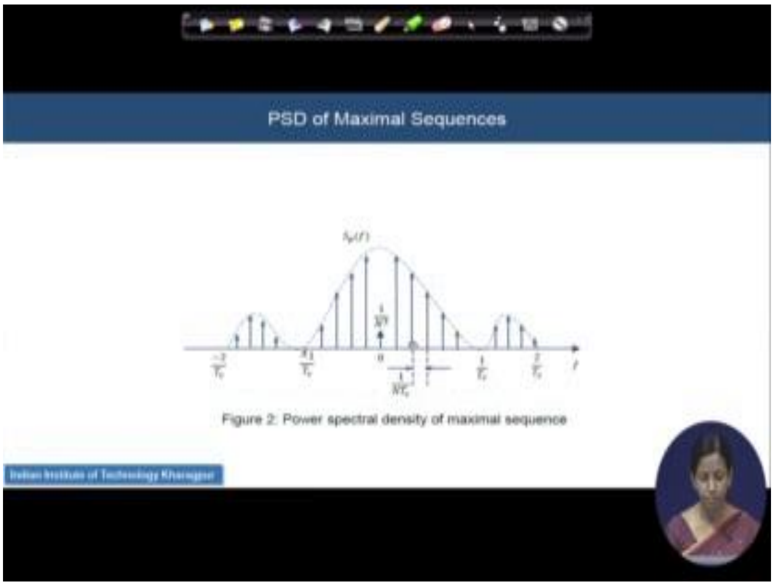

See, we are having a series of the dirac delta function, but it follows the envelope of a sinc as usual we have seen. The maximum value of that envelope will be governed by this N plus 1 by N square. Whereas for i equal to 0th location this is a point of i; for i equal to 0th location, we will be ending up with 1 by N square a delta function whose amplitude should be 1 by N square. So, we are ending up with this expression, which are ending up with this figure.

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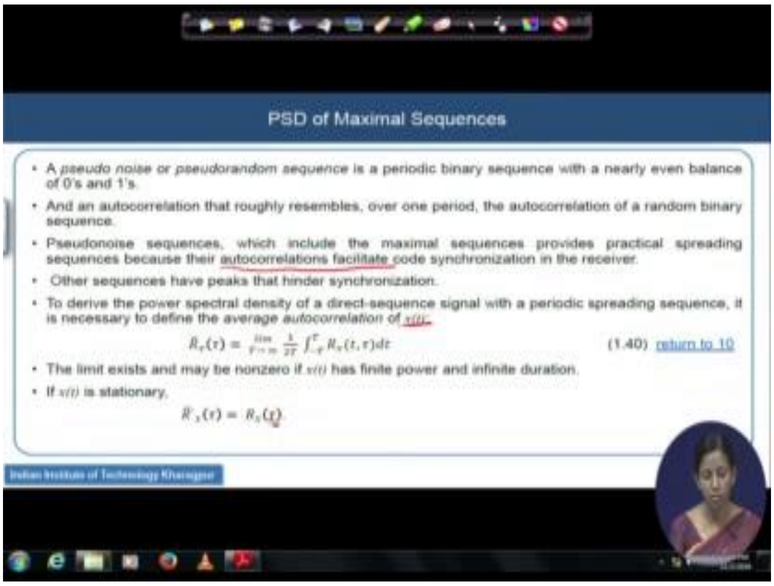

Now, coming from there entering into a PN sequence; a pseudo noise sequence or a pseudo random sequence it is also a periodic binary sequence where actually there is as a nearly as even balance between the number of 0’s and number of 1’s that are there inside that sequence. And this autocorrelation of this PN sequence over one period usually resembles autocorrelation of a random binary sequence and this PN sequence which includes the maximal length sequence also. These are heavily used in practice for spreading the sequences, because these autocorrelation properties that really very useful for the code synchronization in the receiver, we prefer the PN sequence a lot.

And why actually a lot such preference is there, because for other sequences, we will see the lot of the if there is no not only a sharp peak, there are several peaks associated with the main peak of the autocorrelation function and which hinders the synchronization giving false log to those peaks. And that is why we will be a preferring the PN sequence from now onwards in our discussion. And to derive the power spectral density now with the direct-sequence spread signal which is the spread by a periodic spreading sequence. Now we will enter there, we would like to see what is the spectrum of direct-sequence spread spectrum signal, which is spread by a periodic spreading sequence.

We would like to start from the fact that now this is not only the autocorrelation we have to look into, we have to look into the average autocorrelation of the x t of the random binary process random binary sequence x t. And by definition of the average autocorrelation function of x t, we will be given like this which is basically the autocorrelation function with that of a delay tau which is integrated over minus T to plus T period and over total 2T period. And this is the averaging that T tends to infinity. And the limit exist and maybe nonzero if and only if the x t - the random binary sequence with which you are starting, it has some finite power and some infinite duration. If you think that the x t is a stationary process then definitely your average value will boil down to the autocorrelation value of a wide stationary process that is R x tau.

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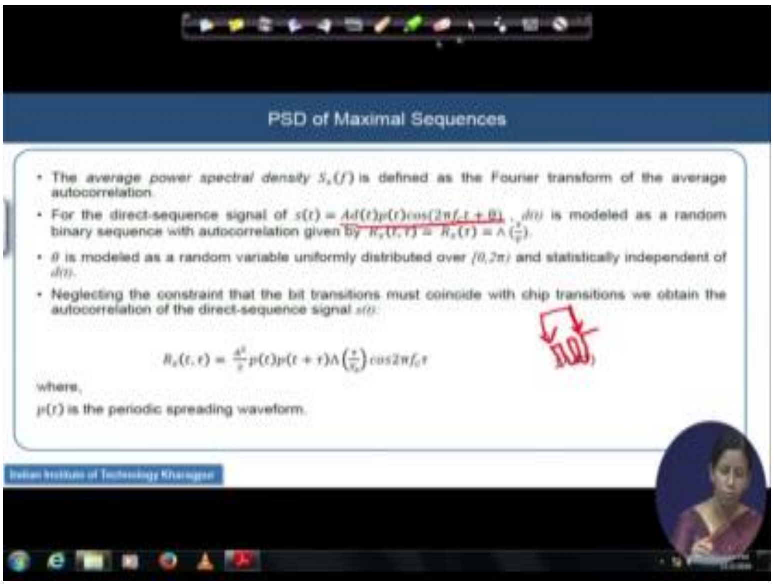

The average power spectral density, which is defined by the Fourier transform of this average autocorrelation, we will be defining it. We know that this is our standard equation for the direct-sequence spread spectrum signal x t was given by A d t p cos 2 pi f c t. We knew that d t is the modulated data modulation. And it is model as a random binary sequence, because all the data that we deal with an practical communication system, there are random in nature. And there are binary sequences considered for this analysis. P t, which is a periodic spreading sequence, f c is the centre frequency.

And remember if we are having a random binary sequence, so the autocorrelation of this section the autocorrelation value of this data will be given by a triangular function that we have already understood. Theta is a random variable, which is uniformly distributed over a duration of 0 to 2 pi open integral of 0 to 2 pi. And it is statistically independent of dt. So, neglecting the constant that we understood that there should be a direct transition the transition of the chips and the transition of the data should go inside and that for both the direction, if we are neglecting that constraint on it. Then the autocorrelation of this direct-sequence spread spectrum signal will be given by this.

How we are writing it, so it is the autocorrelation. So, the same s t is getting correlated with s t and so you will get A square by 2. This p t will be given p t with multiplied with the delayed version of it, because of dt you will get a triangular function associated with it; and then cos 2 pi f c t will be there. And phi are not related there independent and identically distributed, and they are not having any dependence on the value of the d t, so we can exclude that also. And where this p t is a periodic spreading waveform, why we are interested in let us see in the next slide how that autocorrelation from this autocorrelation function of 1.41 we will be ending with the spread spectrum sorry the power spectral density.

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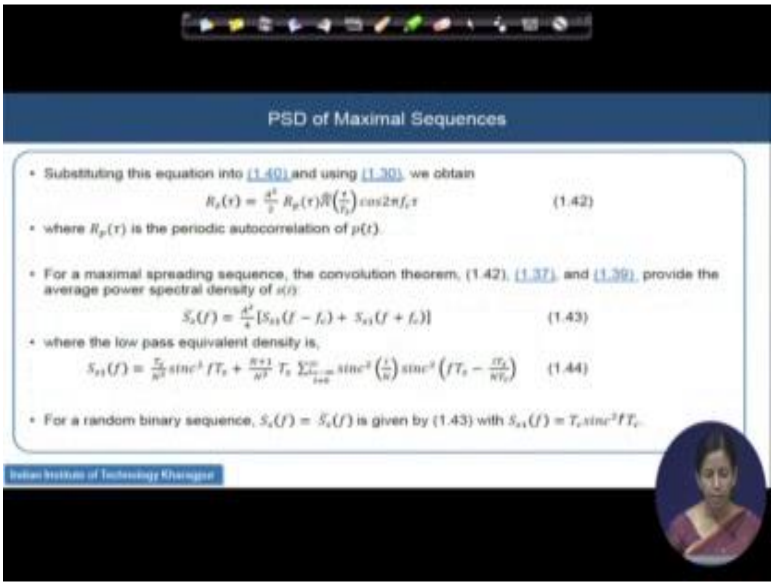

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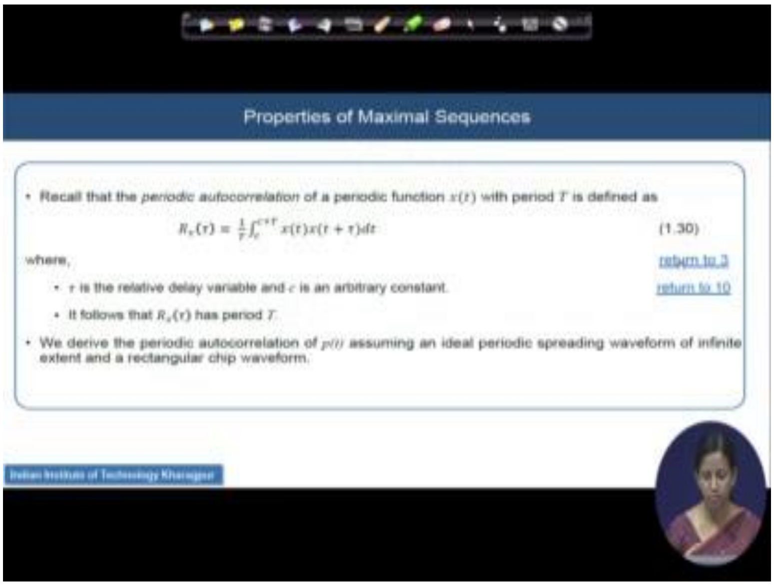

And now if I substitute the equation of 1.40 - this one, and we use this relation then we will be ending up with the R s tau function, where we understand that this is the contribution came from the data part. And p t p t plus tau can be now written as a form of R p tau cos 2 pi f c t was there. This, R p tau was the periodic autocorrelation function of p t that we have already derived earlier. For a maximal length spreading sequences and using the convolution theorem and using all the other preliminary equations of 1.37 and 1.39, so if I take the Fourier transform of this guy we will be ending up with the average power spectral density of this ML sequence. Whereas, this is S s 1 or S s 1, S s 1 f minus f c or f plus f c in one of them or S s 1 any f value can be given low pass equivalent density can be computed like this.

For a random binary sequence, if we are taking care of then if the sequence is not ML, if it is a random binary sequence on this average value will be ending up with the S s f value. And then this S s f value is given by this expression T c sinc square f T c. So, this is the understanding of this power spectral density of a transmitted direct-sequence spread spectrum signal, so that will be the spectrum will be a combination of two different autocorrelation, two different power spectral densities PSD of the data multiplied with the PSD of the periodic autocorrelation function of the spreading sequence. And hence we will get a combined effect of both of them on the spectrum of the data sequence spread spectrum signal.

So, it would not be only either a it would not be actually the autocorrelation function only be only triangular or it would not be only governed by this R p tau. It will be always the combination of both of them. See, actually why we are ending up here why we are so interested about the spectrum density computation of a direct-sequence spread spectrum signal is because once you are unknown about the fact of the spread signal spectrum, then designing the frontend section designing actually the frontend different receiver inside the receiver frontend different components and identifying the optimal bandwidth especially for the wideband filters that you need to capture would not be feasible.

Automatically the way the spreading operation will go on. And the after the spreading operation, how will you design the optimal width of the filter to capture the bandwidth of the signal of your own interest. And the portion that you need to cut off which is not of your interest. And to design that filter that we call the interface rejection filter also. And to design that interference rejection filter, you should have the knowledge of the spread power spectral density.

So, spread power spectral density will help you to design the wideband frontend filters both in the transmitter, before transmission and also in the receiver to capture that intended bandwidth. And then while coming to the spreading operation after this spreading also once the spectrum is shrinked or is a that intended signal spectrum is shrinked then to design the interference rejection filter properly we need to design, we need to know the optimal width of the signal that is getting shrink. And what exactly was the total PSD of the spread signal, so that we can nicely reject the interference.

Because, once actually we are multiplying the incoming signal with our intended PN sequence or the spreading sequence, we understand that though our intended signal bandwidth will be shrinked, but the interference the bandwidth of the jamming signal or interference signal will be spread over a bandwidth of the spreading sequence. And the power spectral density, there will be also the same power spectral density and the same computation of your autocorrelation or the spectrum whole spectrum consideration of that will be continuing with that section.

And hence the filtered design will be will be totally governed by this PSD of the intended maximal sequence. And also this two different kinds of the calculations with ML sequence and with random binary sequence we will help you to understand that the spectrum is really heavily dependent on the spread spectrum or PSD is heavily dependent on the choice of the sequence. Once we are changing the sequence from the maximal to random or random to maximal, see the spectrum consideration is totally getting varied. So, the design of the wideband filters also will be keeping on varying based on the choice of the sequence.