Discrete Time Signal Processing
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Lecture - 09
Relation between DTFT and Analog Fourier Transform

So we have established the relation between Analog Fourier between the discrete type Fourier transform and analog Fourier transform, where there is an analog function given.

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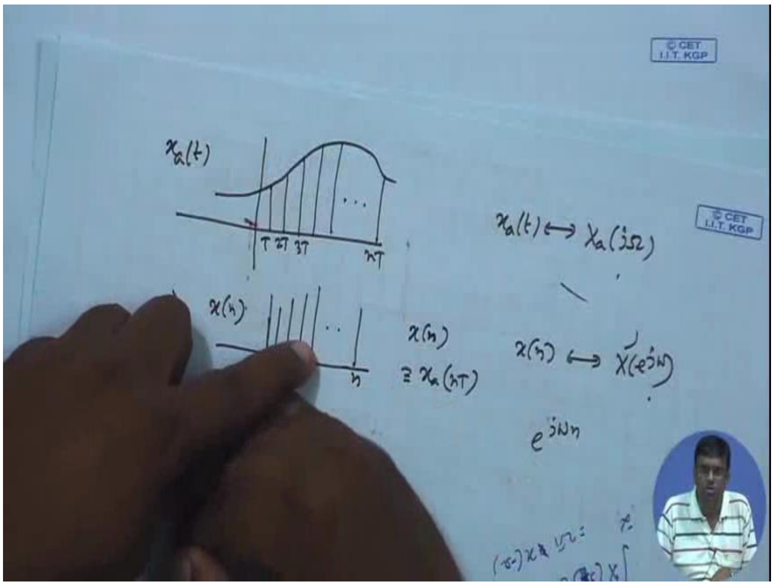

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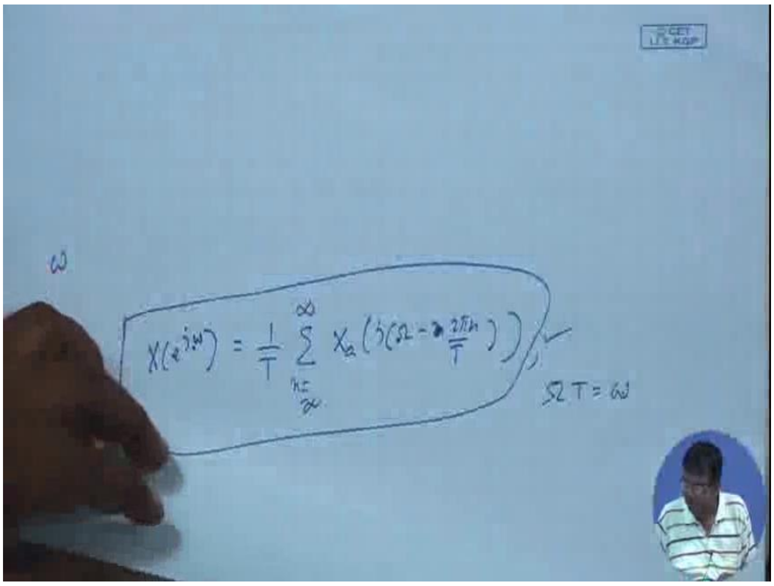

There is an analog function given you are sampling it and forming a sequence out of it. The DTFT of this sequence that a frequency small omega digital frequency is related to the analog Fourier transform, this parent sequence, parent function at an analog frequency capital omega, where capital omega and small omega related like this. So, if you want to find the DTFT of small omega, it should substitute capital omega as small omega by T. That means, radiant per second and these summation these looks bit complicated, but when we analyze it and then some interpretation still follow, but before that one small observation you know I kept it to share with you.

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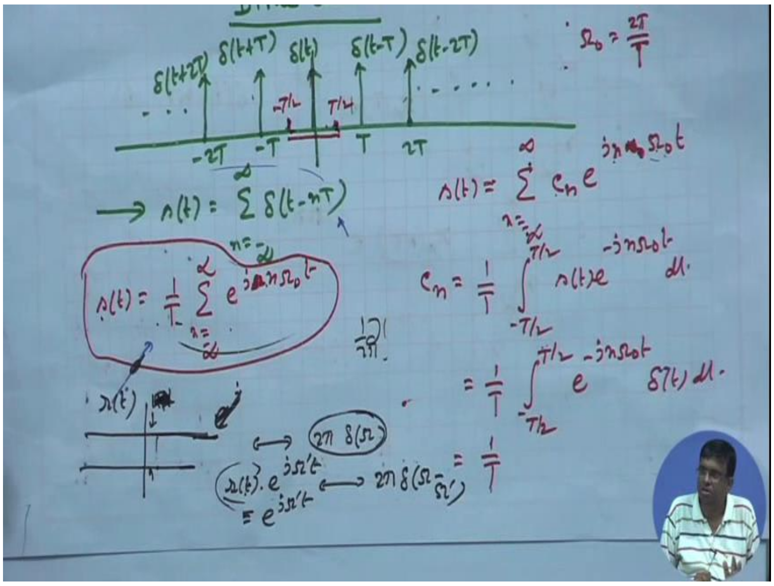

That is we consider this function, this is called a Dirac comb because as such it is look likes a comb, but this is another beautiful property. That this is the s t, we how do we obtain it is a periodic function, in time over a period T. So, we expanded in Fourier series, not transform Fourier series of these form nth harmonic n times fundamental frequency of omega naught came, omega naught was 2 pi by the period T; corresponding coefficient on C n; n from minus infinity. C n was given by the formula standard formula in Fourier series. We got C n to be common 1 by T for all n. So, substitute that it comes out of the summation this is what you have; this is a s t. Now if I take the Fourier transform of this s t, then that means the right hand side also we are putting the Fourier transform, but you know e to the power j any frequency, actually if you take at D C function.

It is Fourier transform is 2 pi delta omega. It is covers for the reciprocity, let delta in time domain give raise to flat in frequency domain, flat in time domain gives as to delta in frequency domain, but there are 2 pi cos because if the inverse Fourier transform 1 by 2 pi cos before the integral that create this problem 2 pi comes. So, if I am got a function. So, I call I give it a name say u t not u, u stands for unit impulsion unit state function. So, let us give some name say r t. Now suppose I have got if I multiply r t with some frequency e to the power j some frequencies some omega prime t. What will be the

Fourier transform of these? It will be Fourier transform of what is shifted to the right by omega pi right, but what is this function? What is 1 set is 1 all through out? DC function. So, 1 into these, this is equivalent to this function itself. So, these function it is Fourier transform is the original Fourier transform of r t 2 pi deltas, but instead of omega it will be omega minus omega p. It is shifted right. So, if I apply this here, Fourier transform the left hand side is same as Fourier transform of the right hand side analog Fourier transform and therefore, what is the analog Fourier transform?

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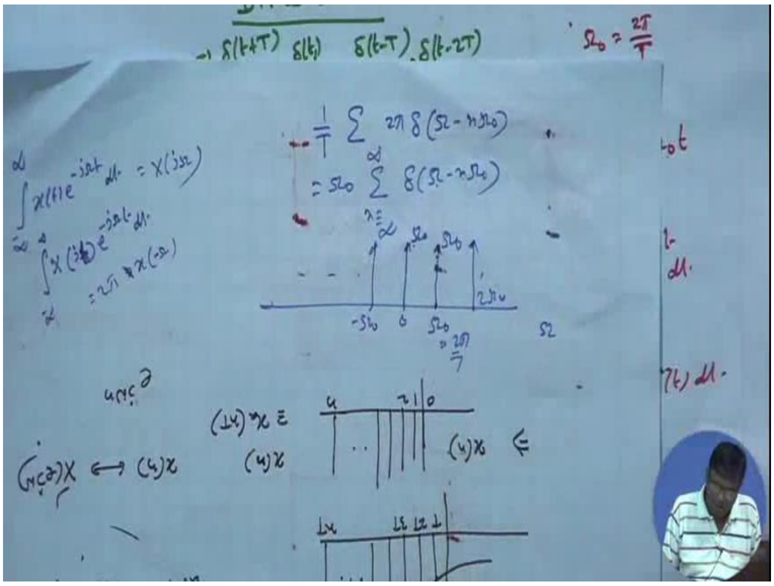

It will be 1 by t summation every such component will have 2 pi delta, capital omega them shifted by this amount omega prime t; omega prime is n into omega naught. So, we shifted by n into omega naught. So, 2 pi goes out 2 pi by t which is actually omega naught. So, this is again another Dirac comb, in frequency domain also. At n equal to 0 it is delta omega. So, delta omega means it will have any impulse at 0 and not presented anywhere. That n equal to 1 it is delta of course, is height will be omega naught, I mean it is strength is omega naught, if you integrate you will not get 1, you will get omega naught.

Then at n equal to one delta omega minus omega naught; that means, it will be shifted to omega naught; which is 2 pi by T and it is strength n will be omega naught then again next one at twice this, this I also and dot, dot, dot, dot.

So, you see in time domain it was a train of impulses, separated by T. In frequency domain, it is a train of impulse is separated by omega naught, where omega naught is 2 pi by T. And here height was 1 there if you integrate any impulse you get a 1. Here if you integrated impulse, you get omega naught. That is why it is called Dirac comb beautiful function. This train of impulse is in time domain on Fourier transformation give raise to again another train of impulses in frequency domain, this just a site observation I thought of passing on to you. Now again I come back to that relation we have derived. And you know see interpretations and all that.

So, if now considering example I re-write the result here. The result which you derived was this that.

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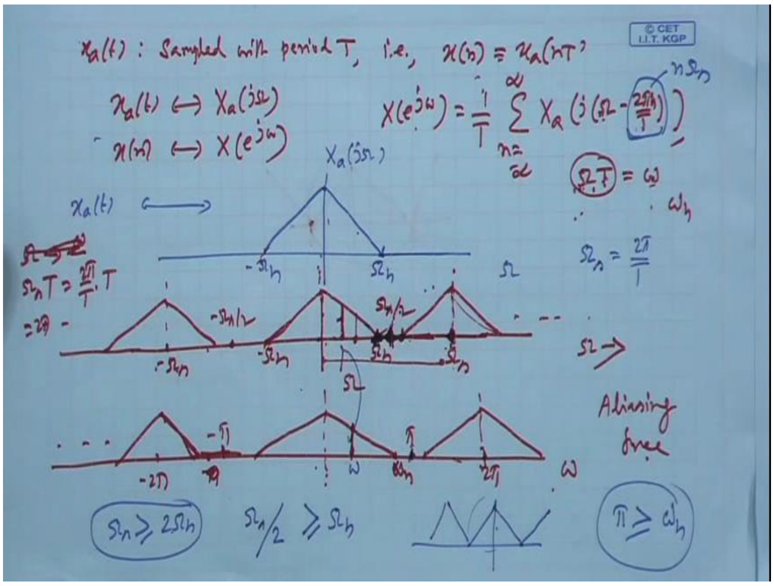

If suppose x a t, is sampled with period T that is nth sample, is X a n T an (Refer Time: 06:40) n times and that I call x n. X a t it has got analog Fourier transform capital X a omega, capital omega x n it has got a discrete time Fourier transform, capital X e to the power j omega. Then we have established that they are related like this. For any small omega of your choice this is nothing, but 1 by t summation capital X a these analog j. You can put capital omega minus 2 pi by t I am taking just j common, but this capital omega T is small omega, that is capital omega is small omega by T, you can put this is a relation. Now what is the implication for that; suppose as an example; we have a band limited function say X a j omega again (Refer Time: 07:45) question.

This analog Fourier transform is a complex function of omega. So, I should to plot it correctly I should have 1 plot even the magnitude versus omega another phase versus omega, but just to explain my point of view, I am not going into that description I am just writing it, X a j omega. Whatever logic I use here that will be applicable if you draw the magnitude per separately and phase per separately.

So, but this (Refer Time: 08:14) question. Suppose, it is goes up to omega h; h for highest and these outside this is a 0. There I saw it is a band limited function, band limited up to some capital omega h and now suppose I am sampling it these a X a t, these a theorem. We are sampling it at a period capital T. Sampling frequencies omega not so long I said. So long I used the term omega naught. Now I will call it omega s; s for sampling 2 pi by t same thing which I so long called 2 omega naught. I am just changing the name to omega s, where s is for sampling. So, this is sampling frequency. This is you know, 2 pi by T all right 2 pi by T radiant per second omega s. So, this 2 pi by T, I call omega s nothing else. In that case, what is the DTFT of that sequence? That is after sampling I am getting sequence, I have to find out the DTFT, these DTFT.

What is that? So, I have to use this formula of which 1 by T is a scale factors I am not bothered about these, I will early consider this. So, first I plot right hand side as a function of capital omega. Then I have to finally, plot it as a function of small omega because I want the DTFT to a function of small omega, but to start with I plot the right hand side is a function of capital omega. That is why I always preferred to keep capital omega on the right hand side, small omega on the left hand side. First plot it in terms of capital omega, and then change over from capital omega to smaller omega. This way it becomes easier. Some books you know brings small omega here. So, that left hand is the function of small omega, right hand side also visibly clear, but that way you know I mean handling it because difficult. So, suppose I am plotting it. So, what will I get for n equal to for there will be a n here this n (Refer Time: 10:17) out.

So, n equal to 0 this is not present. So, it is only this original one. Now suppose, my analog frequency omega s that is this is nothing, but this thing this is nothing, but n times omega s, 2 pi by T into n, 2 pi by T is omega s sampling frequency into n. So, n equal to 1 now I am considering, earlier I took n equal to 0, out of these summation. So, capital X j omega it was just originally one. Now I am taking n equal to 1. So, there will be 2 pi into 1 by T. So, this plot will be shifted to the right pi, 2 pi by T; 2 pi by T is omega s. So, instead of origin, origin will shifted here now. So, there may it is like this. I choose this capital omega s 2 is sufficiently far. So, that these do not overlap with this, but if omega is not sufficiently far to the right, they will overlap. For the time being I am not considering that situation.

That n equal to minus one also you can take. So, it will be capital omega plus 2 pi by T. This will be shifted to the right. Originally move to minus omega s. So, on and so forth; and then we multiplied this scalar 1 by T does not change anything. So, this will be the summation function plot it here dot, dot, dot, dot, dot, and dot. Then you have to write these as a function of naught capital omega, but as a function of small omega. So, wherever capital omega; capital omega here is to be replaced by small omega by t. Capital omega is to replaced by small omega by or rather. Wherever there is capital omega you multiplied capital omega by T, it will among to that is my small omega.

So, this is capital omega h and likewise. So, it means at any capital omega if it has so much of value, the same value if I am plotting versus small omega, it will go to which small omega? That capital Omega into capital T, radiant per second into second, so that much radiant second, second cancels that must radiant, you find out same value will move here. That is how one plot will be converting into another plot all right. So, if this corresponds to small omega, if this corresponds to small omega here the same guy will move here. So, likewise, it might be likewise omega s.

So, all points if you are get omega h at omega h this is 0, but omega h if it is 0 capital omega h, if it is 0. What is the corresponding small omega, that capital omega e into T, radiant per second into second? Second, second cancels you get radiant. So, you can call it is small omega h, if it is capital omega h. So, this will be small omega h which is capital omega into t; at that point this 0 will come. So, at every capital omega whatever value you have that will come to this plot at that capital omega times T so much of digital frequencies. This is how? One plot will be converted to another one. Now one very important thing here you see is this, capital omega s this value. This capital omega s will get moved to what? What digital frequency?

So, capital omega s into T, but capital omega s is 2 pi by T. So, 2 pi by T into T. T, T cancels, 2 pi radiant. So, these will always move to 2 pi radiant and it is independent of the sampling period T you have replied whatever will be the period you applied. This sampling frequency analog sampling frequency will always map to 2 pi, half of it. So, this if it is omega is s by 2, this will map to pi. Say, this is to the right of omega capital omega h pi is to the right of small omega h. Say this is to the left of this point this is left of this point also here, otherwise shape will be same. By the same way, if it is capital omega is s by 2. So, it will minus capital omega s will map to minus 2 pi and minus omega s by 2 which I call half sampling frequency that will map to minus pi. This is in between the 2. So, this was in between the 2 this will you map like this. My diagram is not very good. So, this point is minus pi these minus 2 pi dot, dot, dot, dot. So, 2 pi then there will be 4 pi 6 pi minus 4 pi minus like that.

Here, the function was band limited. So, it is becoming 0 and after a while again it is starting you can like there is a periodic reputation. But there is no overlap between them. So, if I see the DTFT, it is a DTFT and DTFT is periodic. So, this is turning on to periodic from minus pi 2 pi whatever you have that is repeated.

So, if I just take out this function from minus pi to pi. Suppose I throw away the right hand side and left hand side just take out this, this becomes a replica of these function only from small omega you have to go capital omega, there is a small omega whatever the value you reflect it to small omega by t that much capital omega same value. So, you get that this. Now of course, this 1 by t was used. So you have to multiply by capital T now, to cancel that, but you can get back. So, all information about original Fourier transform is retained here it is shape has not changed. We say aliasing has not taken place, there is interfere form here to here inside from here to here. If there are interfere there overlap those shape would have been lost, that interferes of the overlap is call aliasing. So, this is aliasing free.

When can that happen? If these becomes 0, then we have got something this, then the next one starts. That is, if the half sampling frequency is to the right of these or at least equal to these, even if these starts from here even if in the extreme case.

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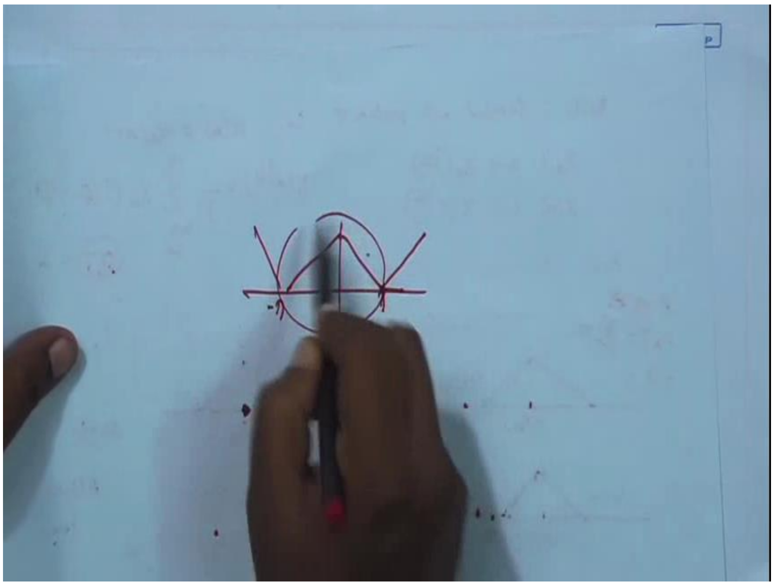

If you have things like these there is no gap it starts directly here. So, this is your pi half sampling frequency these minus pi even then is fine, I will see from here to here, this part. I will not see right hand side left hand side this will be replica for original Fourier transform which means, if this omega s by 2 is either to the right of. that is a either greater than or at most equal to omega h, omega s by 2 if it is greater than n is equal to omega h and that omega is this should become 0. So, if it becomes 0 it is start again. So, there is no overlap. If this happens, then we get a situation like this or in the extreme case. If it is equal to you get a situation like this where it is start from here, there is no gap left this point then matches quantize with this and with these therefore, you get things like this, but you can then this shape is the same as original shape, so no loss of information.

So if this happens, then we can recover original Fourier transform from here and get back my original function. So, this condition should be satisfied, this is called Nyquist condition. Nyquist sampling rate condition or Nyquist theorem, it says that for perfect reconstruction for I mean it not to be able to get back by analog function from the sequence because if I know the sequence, I know the DTFT. But from DTFT can I get back to the original analog Fourier transform? And therefore, by inverse channel of Fourier transform on it get the original time function X a t. For that, this kind of situation must prevail. That is there must be aliasing or overlap. This happen only this which means omega s must be greater than equal to twice omega h. That is analog sampling frequency must be greater than equal to twice the band limiting frequency. First it should be band limited. It must be 0, somewhere that I call band limiting frequency and then I should sample it and analog sampling frequency omega s greater than equal to twice omega h. This is called Nyquist sampling rate or this is Nyquist sampling theorem.

All right in DTFT domain it means, pi should be greater than equal to this omega h because pi transform omega is by 2, analog omega s by 2 maps 2 digital pi because you multiplied by capital t omega s is 2 pi by T replace all that you get pi. You get pi here this is the sampling theory. On the other hand, if this is not satisfied. If suppose this is not satisfied, this is not satisfied, what will happen is an alias thing.

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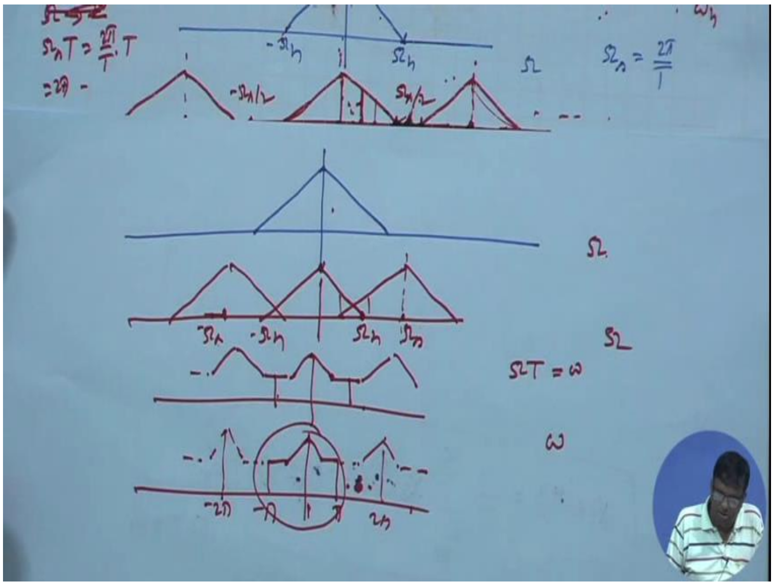

This is given a function of capital omega first I am plotting. This is your this diagram I am re drawing, but omega is not that far, omega is a suppose here. So, it is around omega s same thing will come, around minus omega s same thing will come. After over lapping, it will become like this you know like that. So, if you see, over this range. If you now plot it as a function of small omega, you replace every omega by omega T equal to small omega and you get a plot, you will have these up to here pi and minus pi and this continuous. This is 2 pi like this, but from here this is not this. So, your information is lost. So, this is the essence of this thing his Nyquist theorem.

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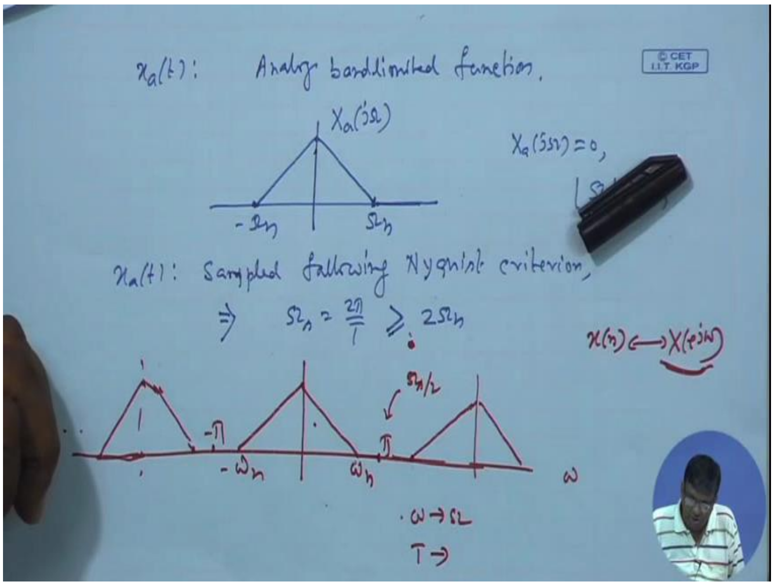

Now, this list to a fundamental question and there is a fundamental theory is D S P. So suppose, there is an analog X a t is an analog band limited function. That is X a j omega like this, that is X a j omega actually again I tell you I am plotting a complex function versus omega; capital omega. So, for correct plotting I should take the magnitude versus omega and price versus omega. But again I repeat that whatever logic I am trying to build up just for explaining that plotting like that is, technically is not correct mathematically is not correct, but as per as conveying the information or explaining is constant you can get all that information from here only that is if you plot magnitude and face separately the same logic will hold good, all right. So, this is given to be 0 for omega for mod omega greater than omega h.

That is omega greater than omega h or less than minus omega h. But I am sampling it X a t sampled following Nyquist criterion, that is analog sampling rate omega s, which is 2 pi by T, T is a sampling period. That is say, greater than equal to twice omega h. So, we have seen it does not lead to any aliasing. So, I have seen DTFT would be like this replica of that will come here. It will be omega h and it will go up to pi and minus pi, this corresponds to omega s by 2 analog simplification frequency by 2. In digital domain it maps to pi and minus of that is maps here and again dot, dot, dot.

Here also, dot, dot, dot no aliasing. So, these shapes retain the original shape. That is X a t band limited analog sampled following Nyquist. So, DTFT is like this question is: This DTFT is given in terms of samples? That is x n if I know, I know this. But from this DTFT if I take out only functionally speaking if I take out this much nothing to not to the right nothing to the left. So this functions then next small omega I replace by capital omega and you remember DTFT up to DTFT for analog we had multiplied by a factor 1 by capital T. So, 1 by capital T times this will be this. So, when I go the reverse direction capital T times, then that will be giving me this function and there I apply inverse analog Fourier transform over it, I get back X a t. So, this is a way X a t I can get back from x n, and that is the famous formula Nyquist interpolation formula which I consider in the next class.