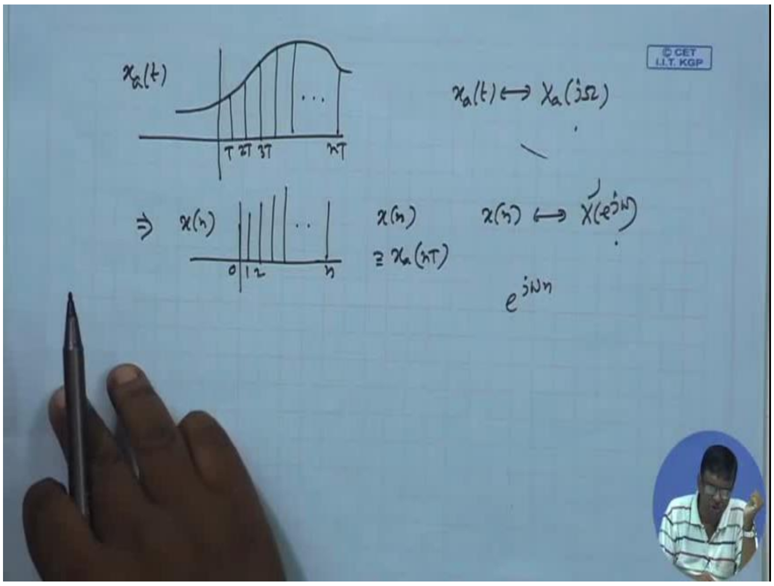
Discrete Time Signal Processing
Prof. Mrityunjoy Chakraborty
Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture - 08
Dirac Comb and Sampling Analog Signals

Suppose you have got an analog function x a t.

(Refer Slide Time: 00:28)

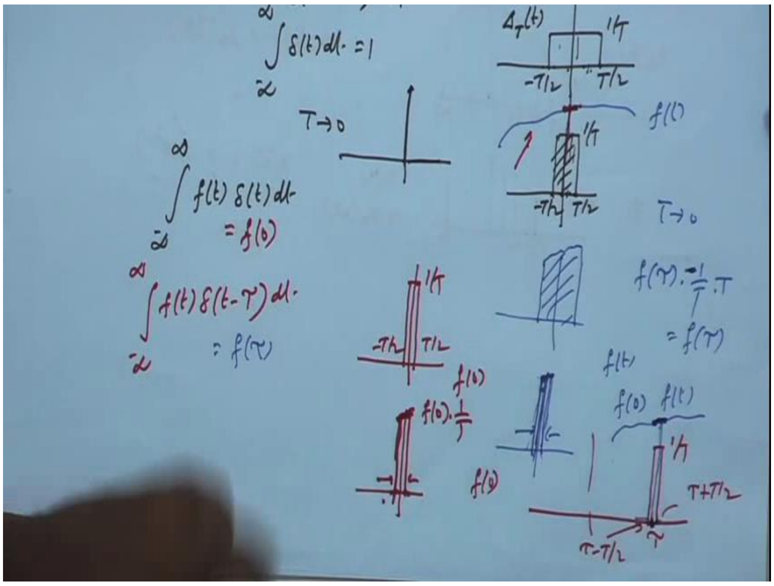


X a t has got an analog Fourier transform x capital X a j. Capital omega capital omega is the real life frequency radian per second that is 2 pi into hertz. So, this is (Refer Time: 00:40) bigger gigahertz and all those things so real life frequency. Suppose I sample it nth sample sampling period is T, so T2, T3 T like that nth sample here. So, I get a sequence x n is the n-th this is the first. For x n is the nth sample is this much that is analog function at this much point of time. And we have seen x n has got discrete time Fourier transform this, fine.

But what is the relation between this and this, because d t f t we have created we produced an Eigen sequence in time domain, calculate the system output that was Eigen function that was this d t f t times the original Eigen sequence. And there by the d t f t was discussed and inverse d t f t and all those things. But this have been created by yaws for a sequence we directly started, with an e to the power j omega n small omega was gradient we called it digital frequency. But what is the relation of this? I mean d t f t with the actual analog Fourier transform of the parent analog signal, because the analog signal which was sampled to derive at x n if that is an Fourier transform in terms of the real life frequency capital omega. The d t f t how it is related to that? This if I know the d t f t can I talk about this analog Fourier transform can I get it from there.

This is the question which is related to sampling and sampling theorem. This will be covering over these free modules down. For that you have go back to the analog signals and observe certain things.

(Refer Slide Time: 02:45)



One is the Dirac delta function which is also called unit impulse function delta t. It is different from the unit sample function will considering in the discrete case. Delta t is defined as is it is 0, if t not equal to 0 and if you integrate area under it is 1. So, delta d actually is a continuous function, a function of continuous variable. And this means we have wall strategies that suppose they start with a function. This may be t by 2 this maybe t by 2 and height is 1 by t. So, 1 by t this like t area is 1, I call it delta capital delta as a parameter capital T function of t. This does not qualify for delta t because it is not 0 outside the origin, this says it should be 0 outside the origin it is not 0 for here, here, here, here, it has got this much value. But second equation is satisfied the area that is one.

So, what we do we try to compressed, height goes up that is capital T we make smaller and smaller, so it is width goes down, height is 1 by t that goes up area remains 1. Still this is not 0 outside the origin like here, here, here, at all this points it is so much. So, you make it further narrower and it is goes up further so finally as t goes to 0 you have an impulse like function and that is what this delta t is. It is another property if I have another functions say f t that f t delta t if you integrate, what will you get?

That means, suppose I have got an f t and you have got this function. If you multiply you will get something like this and then integral you will get something. But this is not delta t it order to make delta it slowly I will make this t go to 0 it will become narrower and narrower. If it becomes very narrow then this will say, pressure this is suppose very narrow then this function f t does not change much we did this heavy narrow period because even though my diagram shows it is so much actually it will be approaching 0. So, f of t will be I mean f of 0 all throw out. That means this is so narrow that whether this much or this much or this much all values are approximately f 0 there is the value at the origin.

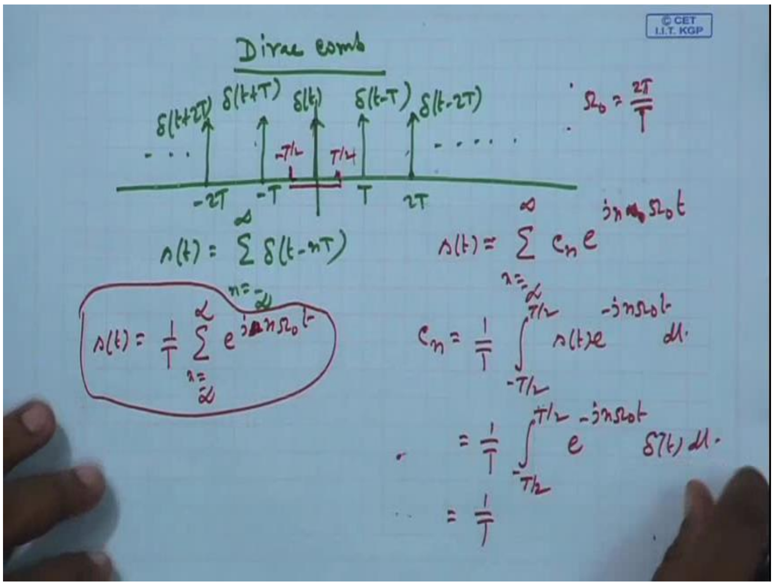
So if that be, what is the area? I am repeating again that is suppose I am made this pulse very narrow and I am multiplying this by this by f t this f t I am multiplying this by this, it will be very narrow. If this width is very small the function f t over this entire period minus t by 2 root t by 2 its value will be approximately f of 0, it will not vary much because the width is so small. So, if you approximate the height to f 0 area under this will be f 0 into what, if height is f 0 into 1 by t because pulse height was 1 by t width was minus t by 2 plus t by 2, there was the pulse I multiplying by f t.

So, new height will be approximately f of 0 into 1 by t there I am assuming that for all the points from minus t by 2 t by 2 function value does not change, it is a very slowly varying function it is value does not change whatever value at the origin it reverse small is that this. So, height will be roughly approximately f 0 into 1 by t width is 1 by t so area is product of the two f 0. Even if capital T goes down further and further and 1 by t height goes up area will still remain f 0. If the limit this will became f 0. So, f t delta t d t integrated is f 0. You can then say f t delta t minus some points say tau d t. This is nothing but same thing, but the impulse is shifted instead of here impulse is shifted to some point tau so the same phenomena will come here.

I have got two ends tau plus t by 2 and this side this tau minus t by 2; that is tau is the replacing origin, height is 1 by t and the same thing here this will be multiplied by the function. So, function over here if you over here if this is the function, if the function f t goes like this what this window it is like this this much, but you can take approximate this you can assume that since this width is very small the function is almost constant and it is the value and it has at f tau.

So, height now will be f tau into 1 by t after multiplication width is again t, so area under that will be f tau into t into 1 by t is the height and t is the width so it will be f tau. So, area under this will be f tau. And even if as t goes to 0 build because very small and small height goes up, but area remains the f tau. That means, f t delta t minus tau it will f of tau.

(Refer Slide Time: 08:48)



Now, with these two properties I bring something very interesting which is called Dirac comb. Suppose, I got a periodic strain of impulses this is delta t then at a gap of capital T I have got delta t minus t, there are two t I have got an another impulse dot, dot, dot, on this side also delta t plus capital T and minus t minus 2 t dot, dot, dot, dot so it is the periodic function and period is t. Mathematically you can write as though it is a supreme imposition of impulses delta t minus say n into t n can be 0 1 2 minus 1 minus 2 say this minus infinity to infinity. Let we call it s t.

Now you see s t what is it is, so this is this a periodic function. Any periodic function and it is period is from say minus t by 2 to t by 2 this much period is capital T any periodic function can be represented in terms of Fourier series of this form. S t this is from you know ordinary Fourier series knowledge so this was we able to know otherwise just go back Fourier region read it a little s t will be summation of this complex sinusoids where the basic frequent sequel fundamental frequency; sorry this I capital omega naught, capital omega naught is 2 pi t and n times so n is harmonic n times these into time.

So, it is a complex sinusoidal function of time, it is a function of time it is a complex sinusoidal it has got cosine component sin component. If n is 1 it has a frequency the omega 0 radian per second. We call it fundamental frequency otherwise if n is 2 second harmonic n is 3 third harmonic like that n is minus 1 minus 2 that is also possible, I mean write in terms of cosine sign then n will take only 0, 1, 2. But these are more general form for any periodic function not only these for any periodic function of period capital T you can write in a Fourier series form like this.

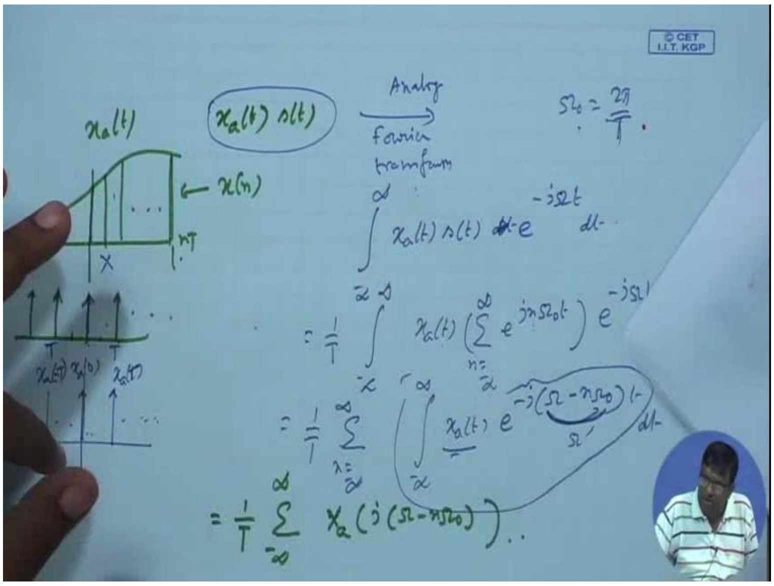
So, c n is the nth Fourier coefficient, nth Fourier series coefficient that is the amplitude part associated with the nth harmonic. N-th harmonic has this form n into capital omega naught in the frequency so e to the power j n into capital omega naught into t transferring coefficient this c n is a super imposition of such coefficients. But you have discrete coefficient because either c 0 or c 1 or c 2 there is either e to the power j omega 0 t or e to the power j twice that twice omega 0 t or e to the power j size omega 0 t so on and so forth, or minus 1 into omega 0 t so on and so forth.

So, some discrete frequencies omega naught twice omega naught minus twice omega naught minus 3 omega naught plus 3 omega naught like they are present. And in that case c n n-th coefficient is given by this integral, this is derives the basics of Fourier series that everybody knows and in case you have forgotten you can just see the thing. You multiply s t by the nth harmonic with the minus sign and integrated over a period you get c n this is a (Refer Time: 12:41).

Now this is the periodic function, for this periodic function s t will be this. Now from minus t by 2 to t by t pi integrate I have only one impulse others are not coming, so which means it will turn out to be just this function e to the power minus j n omega naught t into 1 delta t only others fellows are not present because the range is from minus t by 2 to t by 2. And we have already seen a function f t times delta t integral you can now integrate from minus infinity to infinity you will get the same result, because delta t is outside I mean except for the origin you have the same you know it has the value 0. So, if I know integrate it will be value of the function at t equal to 0, but value of the function at t equal to 0 means e to the power 0 which is 1. This integral will give me nothing, but 1.

So, c n is 1 by t which may and this is independent of n. So, for every harmonic and fundamental c n is common it is 1 by t which means s t is 1 by t this n into omega naught nth harmonic e. This is one important result; this kind of s t periodic function is given by this. Now I will come to this these function actually is called Dirac comb, it is like a comb. And if you take this Fourier transform not Fourier series it will be another comb I am I am coming to that, but before that this result now why I am so bothered about doing all this is the following.

(Refer Slide Time: 14:54)



Suppose, I have this one analog function say x a t and I multiply this by this s t. That means, basically means I had one x a t and I have got this impulses dot, dot, dot, if I multiply I will get one impulse but its height will be x a 0, then another impulse x a 1 like that. That is if we integrate this you will get x a 0, maybe under that will not be 1 will be x a 0 say x a 1.Who is a strength that is if we integrate this delta x a you will not get 1 you will get x a 0, x a minus t x a t like that. If you multiply these two this into this is you will get this.

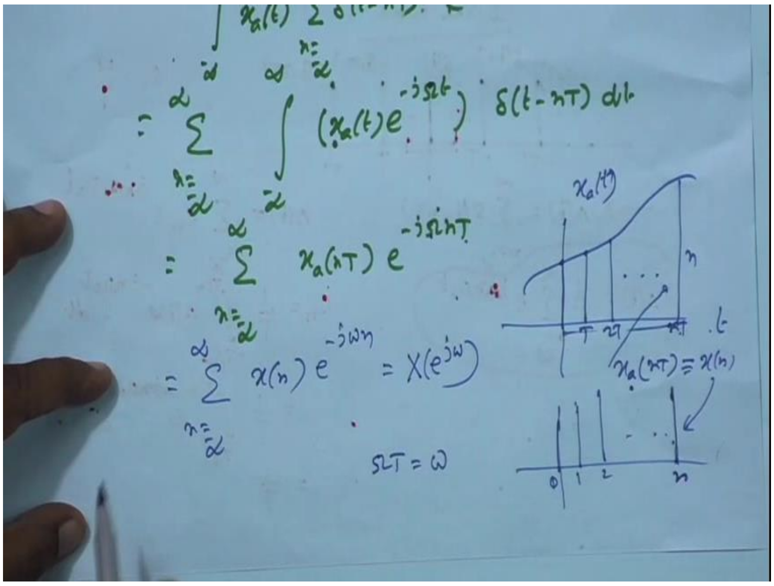
So, this is a function suppose I got I am just multiplying performed again on this product function suppose I have into take it is analog Fourier transform, what will we get that is the question. That means, we have to carry out this integral x a t s t d t minus sorry, e to the power minus j we have to find out this at a capital frequency capital omega. We have to find the Fourier transform of this at analog frequency capital omega, so you carry out this integral.

Now, x s t instead of this form I can write it by this form also, so this I bring inside this form. So, 1 by t this is the summation inside x at s t I am writing so x a t as it is s t will be this 1 by t has gone out e to the power j n some capital omega naught t where omega naught as I told you is the fundamental frequency for this that is 2 pi by t, t is the period sampling period. T is where this is this deduct this impulse strength is periodic over a period t, that capital T I am taking. And from that I have derived the fundamental frequency capital omega naught that capital omega naught we have here n times that was the nth harmonic the summation over all the harmonics, 1 by t outside this is one part and then e to the minus j omega t d t.

Now, I told you whenever you come across double summation next step is to interchange the two. This discrete summation can go out and here we have got x a t for every n n is getting within fixed and then you are entering this integral for that n you write a x a t e to the power j you combine the both, capital omega minus n omega naught t d t. But what is this? This is nothing but analog Fourier transform of this guy at this much analog frequency. If you call it capital omega prime so it is x a t e to the power minus j capital omega prime t, so it is analog Fourier transforms of this at frequency omega prime. Which means, this is capital X a for the analog Fourier transform so I denote the (Refer Time: 19:25) capital as X A J capital omega minus n omega naught.

This is one way, but that is if I take this product function carry out it is analog Fourier transform we get like this.

(Refer Slide Time: 20:41)



But on the other hand, if I use this form here instead of the other form what do I get, this form all right e to the power minus j omega t d t. Again I interchange the two summations this goes outside, this is inside what you have is x a t e to the power minus j omega t this part this into this into delta t minus something. Now a prime just (Refer Time: 21:42) consider this some function f t into delta t minus tau if you integrate it becomes f of tau, so that means this is nothing but this function at n t.

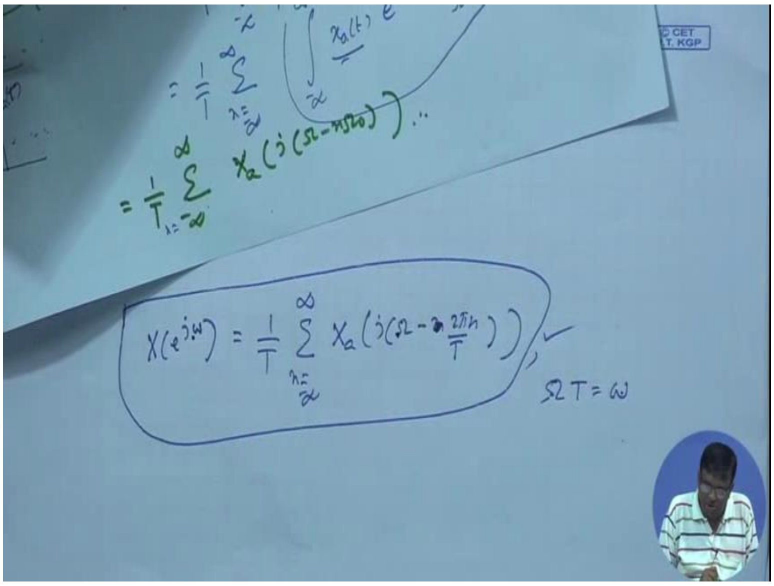
Now what is this x a n t? That is this function at n into t. This point this is what I say nth sample x n, is not it. I mean how I constructing them. If this is an analog function x a t you take one sample then at t another sample. Nth sample that occurs actually at time point n into capital T, that is x a small t if it is n into n into t then this value I may call it x n nth sample. When I draw sequence of them is a same fellow x 1 1 sorry, x 0, 1, 2, dot, dot, dot n.

So, this x n is here and x a n t is here but they are same. Here at this axis was time, so n into capital d point time whatever the value that is this and that I rewrite here, but as a function of index n only nth sample but they are same. That means, this is nothing but x n and capital omega t radian per second into second is a digital frequency only radian equal to omega, so I left with e to the power j omega n which is d t f t. So that means, we get a fantastic thing that if we multiply certain x a t via strain of impulses is a periodic strain impulses with period capital T, then take Fourier transform of it analog Fourier transform at analog frequency capital omega. What you get is nothing but d t f t of the sample strain sequence at a digital frequency small omega, where capital omega and small omega are related by this. That comes from this approach.

We have carried out the integral twice; once we replaced s t by this formula and here I replaced s t by what is given that is impulses. By this approach what I found out is that if you take x a t multiplied by this impulse strain then relating function the product function if you take Fourier transform analog Fourier transform at frequency capital omega analog frequency what you get is nothing but discrete time Fourier transform of this samples. If you put this samples one after another you get a sequence discrete time Fourier transform of that at small omega, where small omega is related to your capital omega by this that is the d t f t. But this is from the other approach equal to this one which means I can equate the two.

And say x e to the power j small omega from one approach it is same as that is the what is this, this is d t f; t d t f t of what if you take this samples form a sequence like this, this 0th first no time here all the index values only n so the this sample sits here this sample sits here this sub (Refer Time: 26:08) form a sequence call it nth sample x n, x 0, x 1, x 0 x 2, x n they are nothing but those fellows only, but not as the function of time just as the function of index n there you get a sequence and that is what I obtained here. I replaced nth sample of nth sample value of the original analog function by just nth sample of the sequence.

(Refer Slide Time: 26:47)



If you take the samples and they take their discrete time Fourier transform at a frequency small omega that is same as this n minus infinity to infinity that should have been n here. Capital X is analog Fourier transform, j capital omega minus omega naught which is 2 pi by t, so you can write 2 pi n by t. Where, this capital omega and this small omega they are related like that. If you are finding out d t f t or small omegas then you take the analog Fourier transform where capital omega is small omega by t and they superimposed. So, this is the relation between analog Fourier transform and discrete time Fourier transform. This is what we will expand on in the next class.

Thank you.