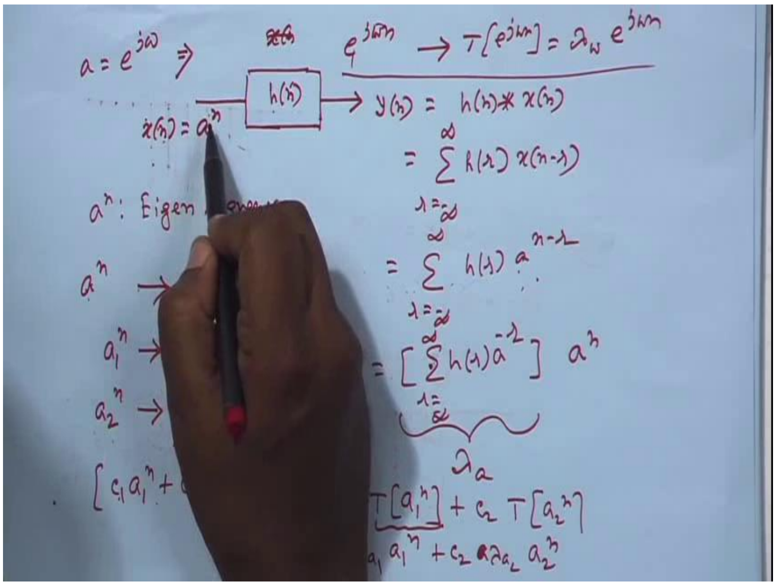
Discrete Time Signal Processing

Prof. Mrityunjoy Chakraborty
Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture – 06
Discrete Time Fourier Transform (DTFT)

So in the previous class, I started with a concept of Eigen sequence, of a linear time invariant or linear shift invariance system. This was the page, h n represent the unit sample response of a liner shift in various system. My claim is if I give a sequence of the form a to the power n.

(Refer Slide Time: 00:43)



That the corresponding y n will be of the same format to the power n, just amplitude will change, which means give this you will get the same thing at the output, with the change amplitude. These like a matrix Eigen value, Eigen vector problem, where if a matrix it either Eigenvector say x, with the matrix time x will come back as x only multiplied by this scalar constant lambda, the all the elements of x get multiplied by lambda. Similarly a to the power n if I input, output will be a to the power n, but every sample of a to the power n, for each n only multiplied by the same constant, some constant if I called lambda. Why because we showed there if you carry out the convolution between y n between h and x n, because output is convolution between x and h and equilibrium h and x, but I told you with the beginning, I always preferred convolution between h and x this format, because then derivative is becomes easier, it is what I have found I mean arrears. So, I will always prefer these.

So, if we know h r which is been h r x n minus r summation r minus infinity to infinity. Now x of n is a to the power n. So, x of n minus r is a to the power n minus r. And now this is coming in the power n and r they are coming in the power and this is separable easily, a to the power n minus r is a to the power n in to a to the power minus r a to the power n can go outside; because it does not this summation is our r. So, this a to the power r does not dependent on r. So, it can be taken as a common it can be pushed outside.

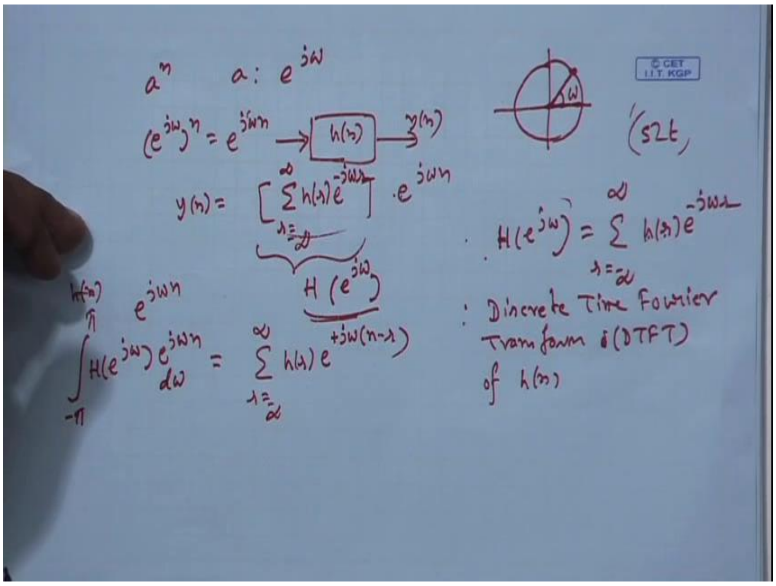
Inside we have got these summation h r a to the minus r. This is a summation or is a local index 0 1 to minus 1 minus 2 all that, and you get a constant independent of n, independent of the index n. There is no index n here it is a constant, its true for every n you can give it a name lambda as a function of a, because a is your choice, so this value will change from I mean in occurrence with a, if a changes this overall value will change. So, is a function of a. So, I called lambda a, is like an Eigen value. So, if you give a to the power n as input, you get lambda a into a to the power you know the output. what is the advantage of this, Let us suppose I have got one sequence some constant a 1 to the power n or if I input, it gives me an output it a 1 to the power n only, but multiplied by some constant, some Eigen value lambda a 1 of this form that is in this summation instead a, you put a 1 whatever you get lambda a 1.

Similarly, a 2 to the power n if I input I know from this I will get output lambda a 2 a 2 to the power n. What is lambda a 2, here instead of a put a 2. Then if I give neither these nor these, but a liner combination of the (Refer Time: 03:38) sequence that is this a 1 to the power n is a sequence, every sample multiplied by c 1 a 2 to the power n is a sequence, every sample multiplied by c 2, and there is 2 had added you get a reality sequence. If I give this computation at the input to calculate output, originally I would be requiring convolution, there is I have to carry out a convolution between h n and these total thing, but since system is linear and (Refer Time: 04:03) sees system is linear, this appointed t that is the system, working on a summation, will be same as summation of res individual responses; that is summation of response due to this part, and summation of response due to this part.

Response due to the t working on c 1 a to a 1 to the power n, because of linearity c 1 is a constant to c 1 will go out, t will work directly on the sequence, this response multiplied by c 1. And again c 2 into a 2 to the power n, t working on that c 2 is a constant, because of linearity c 2 will be post out. it is same as t what working a 2 the power n first then multiplied by c 2, and these two are added, but I know what is t working on a 1 to the power n. If a 1 to the power n it is input, what is a output t a 1 to the power n. I do not have to carry out any convolution I just straight away right in terms of Eigen function, because this is Eigen sequence a 1 to the power n. So, I know it will come back as it is a 1 to the power n, just multiplied by Eigen value the risk Taylor constant lambda a 1.

Similarly, t of a 2 to the power n, I do not have to calculate. It will be same as a 2 to the power n, because is an Eigen sequence just amplitude will change, it will be lambda a 2. So, calculating the output becomes dam easy, I do not have to carry out any convolution sum. My question is, if I give any arbitrary sequence, not basically a sequence which are linear combination of Eigen sequences just (Refer Time: 05:28) arbitrary sequence. Can I, what can I do then, because directly Eigen sequences are not coming. So, what can I do then?

(Refer Slide Time: 05:50)



Here I will say something very interesting that. Suppose I take a, like I was taking a to the power n, I take a to b a constant, a complex value complex constant e to the power j omega; that is a complex number, which magnitude one phase omega, magnitude one phase omega; say the j-th domain is a complex plain if you have an unit circle; that is circle with unit radius, this will be this angle omega, so this is a to the power j omega. This much is cos omega, this much is sin omega cos omega plus j sin omega. If I give that as input there is e to the power j omega whole to the power n, which is nothing, but e to the power j omega n, if I give that as the input to a linear and shift omega and system, my y n. What will be y n. y n will be, I know from these one Eigen value time e to the power j omega n only, but what is that Eigen value, I have to carry out these sum, where a is e to the power j omega.

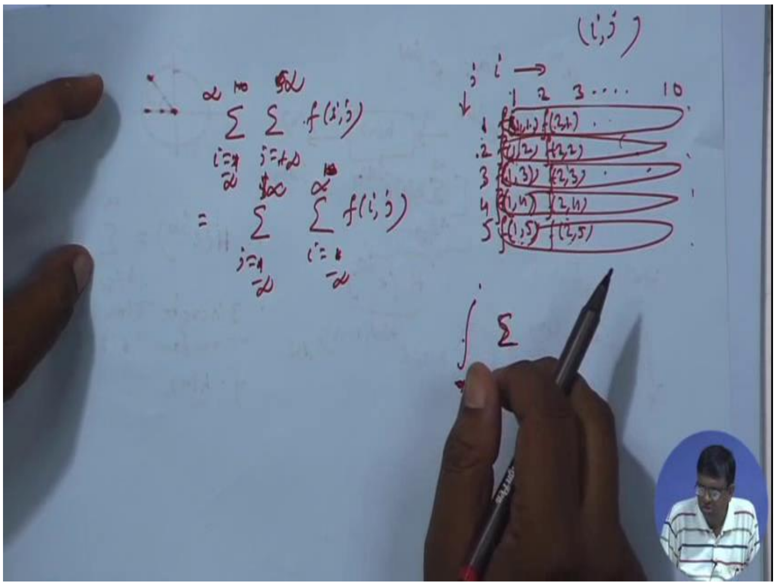
So, this some will be h r e to the power minus j omega r. You see h r a to the power minus r a is e to the power j omega, e to the power minus j omega r. This quantity will be a function of omega. So, I could write this as some lambda omega or something, which is Eigen value function of omega, because cos is a local value variable, you are just putting variable value of r. So, it not a function of r, it is a function of omega, but we give a different name to it. We write it has a function of e to the power j omega, and write it in capital, if it is small h it is capital H. Now, you see actual it is a function of small omega, because e is a constant e is a universal constant exponentially, j is square route minus 1 which is a universal constant. So, if you have to plot it, you can plot only versus omega, but instead of writing you can see function of omega, we write it as a function of e to the power j omega, just to emphasis the fact, that in the definition of this that is these expression, we get a series, you get a power series e to the power j omega r minus j omega r.

So, e to the power j omega e to the power j twice omega e to the power thrice omega e to the power minus j omega e to the power minus j 2 omega e to the power minus j 3 omega, various powers of e to the power omega come, it is a power series, these summation is nothing, but a power series of e to the power omega. Just to remind has of that fact we write it as a function of e to the power j omega, but actually it is a function of this omega only, and what is small omega, its unit must be just an radiant, because n has no dimension, and this omega into n must be an angle radiant. Since n has no dimensional omega must be radiant only. So, omega is called a digital frequency as I told you the other day. Analog frequency radiant per second, we denote it by capital omega radiant per second, because they are omega into t comes t has a unit second. So, omega per second into second, second, second cancel whole thing becomes an angel radiant.

Here n has no dimension no unit with dimension less; that is why omega has smaller become must be radiant only, so product is radiant. This omega is called digital frequency. Fine, so there is nothing new that has come up, so far. I am giving you e to the power j omega n output I am getting again that Eigen value writing in a different form using defined symbols H of e to the j omega into same sequence e to the power j omega n. This sequence by the way, this h e to the power j omega, which I rewrite, this is called discrete time transform; that is DTFT of this sequence h n, because you just give an name discrete time Fourier transform. In fact, this is related to the analog Fourier transform; that is if this sequence h n or h r has been obtained by sampling an analog function, then the Fourier transform of that analog function, analog Fourier transform, and this discrete Fourier transform, they will be related, that will see later.

Fine before I come back to this linear shift linear system, I want to, I will see 1 1 nice thing that suppose this is giving to you H e to the power j omega DTFT is given. Can I from these recover these samples h 0 h 1 h 2 h r h k h l h n the samples, and that is e s I can recover. Fine, how to recover? Suppose I have to recover h n for some n of my choice. There is you ask me get me a small h n from this discrete time Fourier transform. what I do, left hand side right hand side both I multiplied by e to the power j omega n, left hand side right hand side, which means l h has becomes, r h has becomes, e to the power minus j omega are already there e to the power plus j omega n and plus j omega n, no problem. Then suppose I integrate these over a period minus pi to pi which has to 2 omegas. So, here also I have to carry out integral, this is a function of omega over omega. Now as I told you integral is one summation and inside another summation, double summations whenever you come across in d s p, next times is to interchange them.

(Refer Slide Time: 12:38)



Interchange them two summation means what I told you, the other they using this discrete case; that suppose you have got one function f i, j. Here suppose i equal to a finite range 1 to 10, j equal to say 1 to 5 and you are adding you can take minus infinity to infinity minus infinity to infinity, but just for I mean exponential purpose I am writing this. What does this summation means? Every time you fix y. So, i equal to 1 put that here if transport in one. then take j equal to 1 f 1, 1 j equal to 2 f 1, 1 2 j equal to 3 j f 1, 3, like that f 1, 5 add them get a result. Next again change i 2 from 1 2 2. So, f of 2 then 1 f of 2 2 f of 2 3 f of 2 4 f of 2 5 add them get another result, and push it like that an add all of them. These are bringing up these summation, but you see it is like these i and j i 1 2 3 dot, dot, dot say 10, and j 1 2 3 4 5.

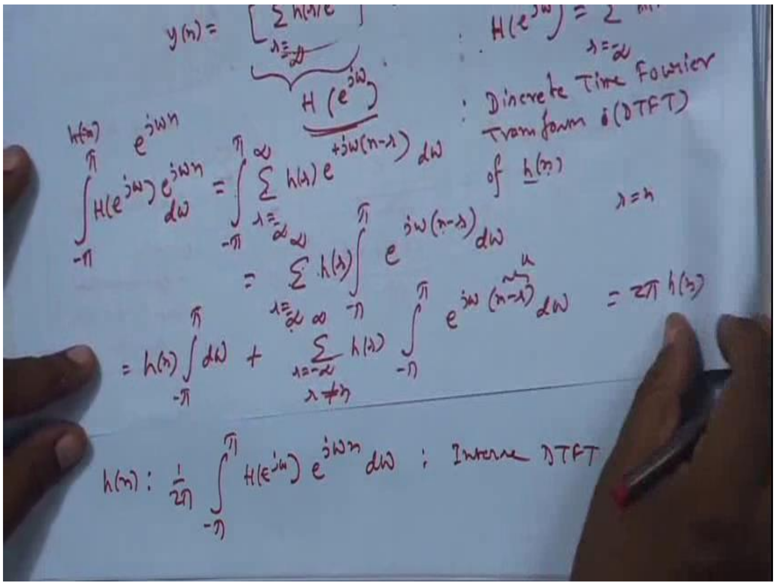
So, here I means i 1 if first pick up j equal to 1 this; 1 then 1 then 2 3 4. So, 1, 1 1, 2 1, 3 1, 4 1, 5, then again 2, 1 2, like it means 1, 1 1, 2 1, 3 1, 4 1, 5 then 2, 1 2, 2 2, 3 2, 4 2, 5 dot, dot, dot, dot, in this summation it means, first you add this column that is i is fixed at 1 from outside j is moving. The second is j I am writing in the form i, j. So, first point is first value is i second value is j. So, first i is kept fixed at 1 j is varying. Then again I move to 2 j is varying 1 to 3 4 5. So, I add this column, then this column then this column and then i at those individual results over i.

So, I am adding over j first or 1 i. I am adding over j again one another I, and I am adding over j again one another i, and then this individual values summation results I had over various is that is i at the results; that is the meaning of this, but I will get the same thing if I add like this is row, then this row, this row, this row, this row, and add that result what this j x is; that means I fix j equal to 1 i 1 1 then i 2 then 1 then 3 1 4 1 up to 10 ones. So, i is varying j is fixed i sum, then again i move j to 2 there again i is varying 1 2 3 up to 10. Then i move j 2 3 again i varying 1 2 3 up to 10. So, I add this row this row this row. So, every row means j fixed second coordinate is fixed second value, first value is moving that is i. Second value is fixed i is moving that this is what is mean, what is mean by interchanging the summation that is first you fix j.

So, j equal to 1 then move i, j equal to 1 move i what is entire range get this summation, then take j equal two put the 2 here 2 2 2, and get i equal to 1 i equal to 2 up to 10 get this result, and then add all of them over j, you will get the same result, because all these values have to be add, function f of 1, 1 f of 1, 2 f of 1, 3 f of 1, 4 f of 1, 5 you can put if you want to you can put an f here. So, either you add these columns function f of 1, 1 f of 1, 2 1, 3 1, 4 1, 5; there is I fixed j varying finish this column hold a result. then change i 2 2 f of 2, 1 f of 2, 2 up to f of 2, 5, so is varying i fixed 2 at, again hold a result and this results is it add over i, because this is i x is i 1 i 2 like that that was here you will get the same thing if you add this y first f of 1, 1, because purpose is to add all these values.

So, either you scan like this and compute all the points, or scan this away compute all the points. If you scan this away; that means, first j is fixed i is varying, then another j i is varying another j i is varying; that is what is happening here, you will get the same result. So, here it was discrete summation. Now it could have been from minus infinity to infinity same thing any limit you understand this logic is same. Then it can be between two summations; one is continuous integral from some limit to some limit, and may be minus infinity to infinity, and a discrete sum some limit to some limit. This two also sums. So, they can be interchanged by the same page of it, and in d s p variable you come across double summation.

(Refer Slide Time: 18:01)



Take my, what? Very next studies you interchange that two summations, then again next steps will follow. So, here I put double summation after I multiplied both side by e to the j omega n, then I mean integrating from minus pi to pi which is shift omega. I have got double summation; I take this inner summation out and outer summation in. Outer summation is with respect to; this is integral which shift omega is. Only e to the power j omega n minus r these contents omega, h r does not contents omega. So, h r does not have to be within integral it can be taken out as common only here, and this is summation over r. now look at these summation, from outside r is changing r is fixed. No, r is not fixed your moving r for every r you are carrying out this integral obtaining the value. Then taking another r putting that r here n is already fixed n is your choice. You wanted to find out h of n for a particular n. So, n is your choice, or every time fix from outside, that h r times this integral put that r here n is fixed carry out the integral, and then another r put that r here do the same thing and so on and so forth. Now as r varies from minus infinity to infinity, it will become equal to n; that is r is equal to n ones, and all other locations it will be near equal to n.

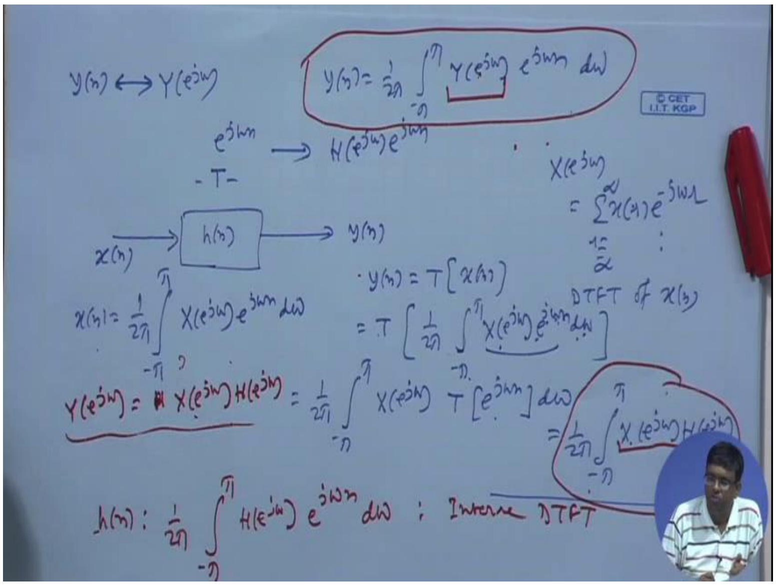
So, r equal to n k size separate out, when r is n n minus and 0 a to the power 0 1. So, then I have put is integral equal to h r equal to h this term I am writing separately minus pi 2 pi. This is already e to the power 0 1, so just d omega. and remaining sum, or has before minus infinity to infinity, but right or not equal to n, is moving on to n, but all it does not these value and these integral. Now since r is not equal to n here, n minus r will be an integer, positive or negative does not matter. e to the power j you can call it k. So, it is e to the j omega k, e to the power j omega k if we integrate over minus pi to pi it will be 0, why, e to the power j omega k be cosine omega k plus j sin omega k. Cosine omega k will have k number of full cycles from minus pi to pi there is a (Refer Time: 20:45) e to 2 pi

You can easily see it, directly you can work out some k positive or negative minus pi to pi. If you work out the integral where is the integral j k e to the power j omega k omega pi or minus pi. Now e to the power j pi into k; that is you have k pi what will be this e to the power j theta minus e to the power minus j theta. So, 2 j sin k pi by j k j j cancels, but what is sin k pi, k is integer either positive or either. I mean positive or negative does not matter if it is negative minus will go out, but if it could be even or odd, but sin up even number into pi or odd number into pi is always 0, sin pi is 0 sin 2 pi is 0 sin 3 pi is 0 sin 4 5 0. So, this is always 0 for any k. actually what is happening, e to the power j omega k means you have got cosine omega k plus j sin omega k, each of them cosine omega k further matter it is periodic over k, or a period of how much 2 pi, that is if you have k plus 2 pi you have got cosine omega k plus omega into 2 pi, cosine omega into 2 pi plus omega k. Sorry, I am writing wrongly. It will be suppose cosine omega k plus.

So, it will be periodic over what, if omega is changed is omega is move to omega plus 2 pi. That is omega changes; it is e to the power omega. If omega changes by 2 pi, so it will be omega k plus 2 pi k, but there is same as cosine omega k. So, this is periodic over omega over a period of 2 pi same is this; that is a over a period of 2 pi it will have one full cycle, and if you integrate it over one full period it will be 0, which positive cycle negative of cycle, positive of cycle negative of cycle. So, now, here this integral this cos omega k plus j sin omega k, but your integral what a total period of length from minus pi to pi is 2 pi, there is a one period, so this part cos omega k which have 0 value, and sin omega k integral will be 0 value.

So; that means, this is 0. So, we are left with only first term. first term we give us to if you carry out the integral it will be give omega or minus pi to pi 2 minus pi so 2 pi, 2 pi h n which means h n you can get back as 1 by 2 pi. then this side, this is call inverse DTFT; that is given any sequence h n, if you carry out these summation, you get discrete time Fourier transform, and giving discrete time Fourier transform DTFT if you carry out these integral for 1 a n of your choice, you get a sequence sample at n th h n is called inverse DTFT all right. It is not only for h-th that is the 16 impose this for it can be (Refer Time: 25:16) for any sequence including x n, which has lot of property, but I am coming to that later, but what is the implegation of these, that now (Refer Time: 25:26) that linear shifted linear system.

(Refer Slide Time: 25:37)



If I given you arbitrary x n, earlier I was given either directly Eigen sequences or they are linear combination. So, competitions of computation of output was fine, you do not have to carry convolution you can directly write output inters of the individual Eigen sequence their responses; lambda a 1 a 1 to the power n lambda a 2 a 2 to the power n like that, but if I cannot have, if I do not have a exchange there is a linear combination of Eigen sequence or if it is arbitrary x n. then can I still use this fact this will give me the answer.

Now suppose I have got x n arbitrary x n and I have to find out y n. one way is to carry out convolution; suppose I do not have carry out convolution, x n also can be writ10 in the same manner, using its discrete time Fourier transform X it is j omega into j 1 omega n, where this is with the same formula x of r e to the power minus j omega r same thing for this sequence, which is nothing, but discrete time Fourier transform DTFT of x n. So, that if I bring here in all that this can be I will get x n by this result, because this is true for any sequence h n, not basically particular h n which has system into response. So, these, x n I am giving. The system is also repeated by linear operator linear shift in linear operator t. So, my output y n is t working on x n, there is t working on this. Since t is linear this constant will come out t working on summation, summation of this sub sequences. Here omega is continuously varying that is a, we do not have discrete sum, omega is continuous is varying from minus pi to pi we have a continuous sum.

But t is linear means summation of this equation says e to the power j omega n with my amplitude X e to the j omega, this is the meaning of this, sequence e to the power j omega n amplitude is so much X e to the power j omega and not only one sequence because omega for various omega. So, we have got various sequences, but omega is continuously varying from minus pi to pi not discretely varying not omega 1 omega 2 omega 3, its continuously varying, but t over the summation of this sequences, the response will be same as summation of individual responses, due to its sequence with this t can be moved inside individual response then summation. So, it is same as this comes from linear t minus pi to pi t will work on this, but t will work on this, this part does not depend on n these like the constant c 1 c 2. So, it will be outside t, will work only on the sequence part, but it is e to the j omega n is an Eigen sequence. So, its output

I know, output of I give e to the power j omega n, we have walked out output will be h e to the power j omega into same sequence.

If I give e to the power j omega n as input, we have already work it out output will be the same form multiplied by Eigen value, which we called named as DTFT. So, this part will be same as all right. So, output, you can directly obtain by multiplying the 2 DTFTs, and then carrying out this inverse DTFT has this product. You can also see y n if it has we right leg this if it has its own DTFT is, then I know what is why these, but y n can be re represented by a given similar formula like this 1 by 2 pi minus pi to pi. You take these y n again take these, this also y n. So, if you compare these two, you will see this is same as this e to the power j omega and pi n e to the power j omega and pi n d omega minus pi to pi integral one by 2 pi both (Refer Time: 30:37).

So, from here you get that output DTFT, is a product of input DTFT into the DTFT of the input impulse response or unit sample response, which equal transform function, it occurs analog signals we have seen earlier, that output Fourier transform of analog system linear shift in time invariant system, is product of the input Fourier transform and the Fourier transform of the impulse response, which is called transform function all right which is actually convolution in time domain means multiplication in frequency domain. The same thing here, these and this in time domain it was small x n small h n they are we are getting small y n. In the frequency domain there is in these case discrete time Fourier transform, there is digital frequency domain they are individual DTFTs are multiplied to get output DTFT.

I am stopping here. I will continue from here in the next class.