Discrete Time Signal Processing
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Lecture – 04
Properties of Discrete Convolution Causal and Stable Systems

So we have been considering convolution, now consider a system.

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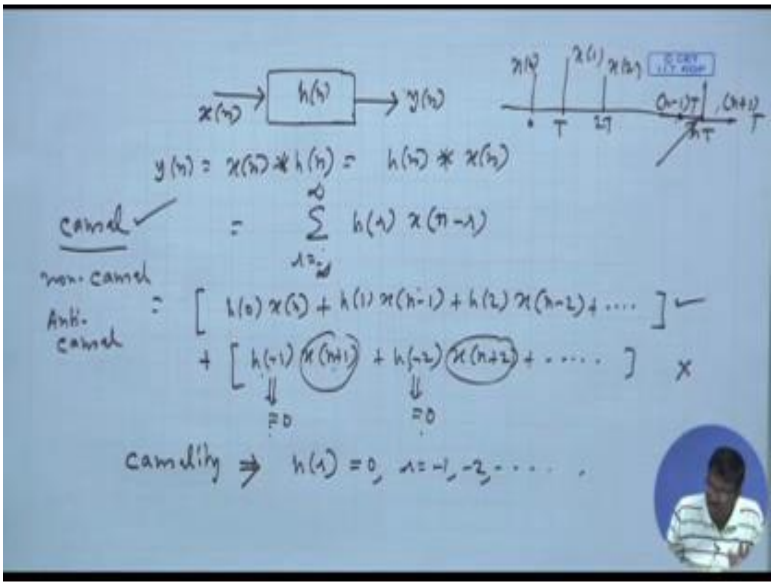

This is linear shift invariant system, and therefore, is characterised by inverse sample response. So, therefore, any input x n output will be a convolution between x and h or equivalently h and x please give in. Now; that means, y n is x n convolved with h n, but as I told you, I always prefer the other form, because then results become easier to get you know. So, this is the rule of thumb I told you. So, I follow this, and this is nothing, but if you write h or next guy comes. I want to find out y at particular n. So, n is fixed that n I put, but r will come with a minus; first guy here, second guy here, and r will be a local index right. This if you expand, for r the summation is from minus infinity to infinity. I divide into two range r equal to 0 1 to up to plus infinity and r equal to minus 1 minus two minus 3 up to minus infinity.

So, one term is from r equal to 0 onwards. This h 0 x n h 1 x n minus 1 h 2 x n minus 2, and then r minus 1 h minus 1 x n plus 1, dot, dot, dot, dot, dot. Now suppose x n is really obtained by sampling and analogue signals. So, there was a time sequence x 0 x 1 and this was 0 this was t x 2, this is 2 t. We wrote in a form of sequence which was just index 0 index one index two like that, but actually it is like this in real time. That means, x n will be the sample at n t; that is my current time. I am standing at this sampling point; this is my current, because I want to find out the current output. So, n is the current index, n t is the current time point. I have to find out output at current time point that is n into t or current index in. Now n minus 1 will be in 1 t backward, so n minus 1 into t. So, that is passed. So, you see here in this summation x n, there is a current index output, that is already known, that is a current input; x n minus 1; that means, intercept time n minus 1 into t.

So, it is passed once something intercepts time it came. So, that is already known, because it is in the past x n minus 2. So, is a past index intercept time n minus 2 into t, there is current index is n t n minus 2 into t means 2 t before. So, 2 t amount of time ago that is sample came. So, these are known information so computing this is no problem. This part is fine but, looks at here n plus 1, we are going to this right hand side n plus 1 into t, but you are standing here. This is my current time, at current time you have no idea about what is the further sample will be. So, in real time operations this is not known to me, because I am standing at current time index; that is time point is n into t index is n. At that time you have no idea about what my further sample will be at n plus 1 into t that time point or index n plus 1. Same for this, same for everyone, for real time operations this is not real eligible. Therefore, in order to make the system real eligible and practical and deterministic and all that this part must vanish.

Only way you can do that, is by choosing the system impulse is for unit sample response. So, that this value equal to 0, 0 into this is 0. These values equal to 0. So, that 0 into this is 0, and you add so on and so forth. So, this second term, this second expression does not arrives. A system where current output depends only on the current input and past input is called a causal system. If it depends on both past and future inputs, then it is called non causal. And in the special case where current output depends on, only future inputs, it is called anti causal. Obviously, for our real time operation you want causal. So, for causal system there is for causality, we should have h minus 1 0 h minus 2 0 you should design the system, so that this coefficients of the unit samples responds at 0. There is causality implies h r equal to 0; for r equal to minus 1 minus 2 dot, dot, dot, dot,

dot is causality; u n, u n is causal, because for n less than 0 all values are 0, for n 0 and onwards 1.

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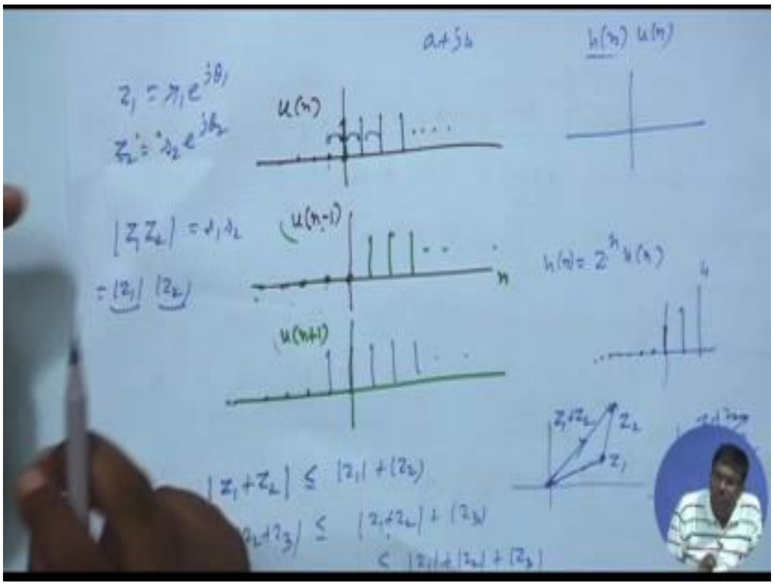

So, if u n is the unit sample response of a system linear (Refer Time: 06:47) invariant system I am happy. That system will be causal. How about u n minus 1? It will be shifted to the right by 1. So, this 0 this one will move here, this one will move here, next one will move here. So, it will be one or like that, but this 0 will 0. So, this will be 0, this will be 0, this will be 0. Again it will be causal there is no problem, because this is 0 to the left of origin. I want this to be 0 to the left of origin. So, this is 0 here 0 here 0 here 0 here, so this causal. This 0 does not matter, only consideration is for minus 1 n minus 1 n minus 2 n minus 3 like that, it should be 0, but how about u n plus 1. Here it is advanced shifted by minus 1. So, this one will come to the left, this one will come to. So, I will have 1, 1, 1, 1, 1, dot, dot, dot and 0. This, if this be the unit sample response of a system, then that system is non causal, because apart from this sample I have about one fellow to the left of origin. So, it will make it non causal, alright. Any impulse response, any h n if you multiply it by u n it will be causal, because u n make your that on the right hand side sample of h n they are multiplied by 1. So, they are as it is. Left hand side samples are of this are multiplied by 0. So, they will become zeros.

So, often to indicate causal system we write like this, these into u n like 2 to the power u n. So, if 2 to the power, u n is not present you will have it is value for 2 to the power positive n or positive negative n also, 2 to the power 1 2 to the power 2, 2 to the power 3 and 2 to the power minus 1 2 to the power minus 2 and. So, both for negative n and positive n you will have values, that will correspond to, if that is h n that will corresponds to a if this is a 2 to the power n no u n, it will corresponds to a non causal system, but the moment I multiplied by u n values of this 2 to the power n will be considered only for positive n, because then u n is 1, 1, 1, 1.

So, the value will be multiplied by 1 you will get as it is. So, 2 to the power 0 it will be 1, 2 to the power 1 it will be 2, 2 to the power 2 it will be 4 as it is, because they are multiplied by 1 from u n, no change. On the left hand side what are will be the value they will be multiplied by 0, because u n is 0 to the left of the origin, 0, 0, 0, 0, 0. So, it is a causal system again alright. So, causal it is very important for real time operation, you should design a system, and all the digital filter we design we ensure that, the they are event causal. Another related concept, very useful concept is stability.

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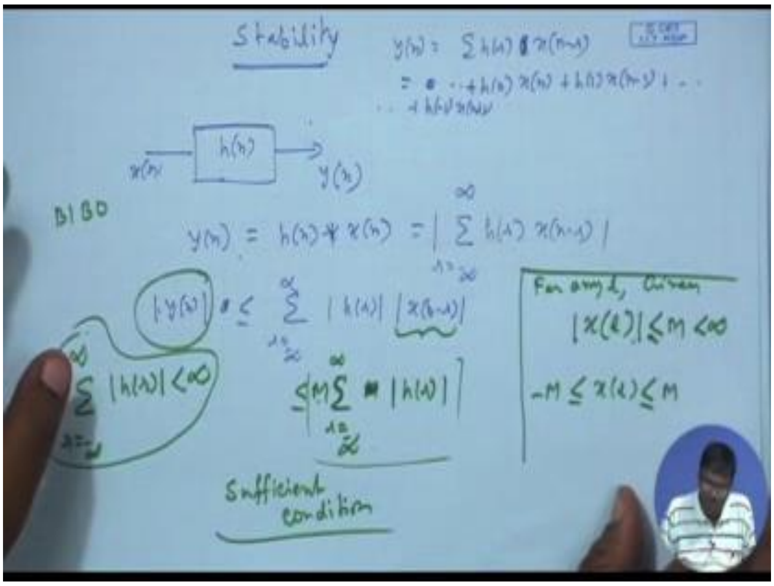

Stability means, very roughly speaking we have seen what a convolution is, we have seen convolution. Just a minute, ok I am not finding it here. So, you have seen convolution is a summation; like y n where I write say h r, sorry x n minus r. So, it is like an infinity sum h 0 x n h 1 x n minus 1 h minus 1 x n plus 1 dot, dot, dot, dot, dot, dot, dot is infinity sum. So, what happens, there is a danger that even if you give an x n as input for every sample is small, is finite value, this is an infinity sum. So, magnitude of the sum should up to infinity, if that happens then the output is no more. In fact, will be it cannot suit up to infinity whatever will be the power supply, it will latch on to that.

So, it will not be a meaning full response of the system to the input, I will see just a line. So, there is inherent danger in this infinity convolution, this because this infinity summation. If that happens I say system is not stable, whether by stable system I want output to be, you know meaning the response of x n; that is should be finite. So, I can see this much it is, it is a response of this system to input x n, but if it is suit's up to infinity and therefore, held up at some supply value, it carries no information for me, it is no good. So, I do not want that to happen, but that can happen, because convolution has an infinity summation, sometimes it can suit up to infinity, so that I have to avoid. So, I have got y n and this linear shift linear system against. So, it is characterised by h n. So, y n is either x convolves with h or h convolves with x, as I told you I always prefer this. So, mod value, positive or negative, or it could be an complex number, sequence not only does not have to be real value sequence, x n can be a complex value sequence h n complex value.

So, mod of this means mod of this summation, but you know there is a triangle inequality; that is mod 2 numbers in general complex, but they could be real mod of z 1 plus z 2, is less than equal to mod of z 1 plus mod of z 2, which follows very easily, because any number complex in general. So, may be z 1 and another z 2. This is by vector sum z 1 plus z 2; mod of z 1 is this length, mod z 2 is this length, and mod of z 1 plus z 2 is this length. We know triangles property that summation of two sides of a triangle is greater than this, you know length summation of the length of; I mean length sum of the length of two sides of triangle is greater than the length of the third side. So, it is greater than it can equal, if they are collinear and pointing in the same direction z 1 and z 2, pointing in the same direction. So, what all is z 1 plus z 2. So, in this case length of z 1 plus z 2 same as length of z 1 plus length of z 2; that is why this is, they are less than or equal to. By triangle law, it is strictly less than when they are not collinear, but if they are collinear and pointing in same direction this equality. So, what we have here, we can generalise it, if I have got z 1 plus z 2 plus z 3, it is less than equal to mod. You can take this two together first plus z 3, there you can take this like that. So, this way you can build up, mod of summation is less than equal to summation of modes.

So, there is a summation and mod, there is less than equal to summation over mod, but again two numbers, in general complex could be real also, mod of these is same as mod of z 1 into mod of z 2, why, very simple you write in polar form when you my product of numbers come complex numbers in general, you choose polar form that is z 1 is sum r 1 e to the power j theta 1 j 2 r 2 e to the power j theta 2, whenever product comes, whenever addition comes choose the rectangular form a plus j b kind of form, or replacing a complex number complex number, because there are more general number really part of it. Now z 1 z 2 if you do it will be r 1 r 2 and e to the power some j theta 1 plus theta 2. So, mod of that will be r 1 r 2 they are magnitude part, but r 1 is mod z 1 r 2 is r 1 is mod z 1 r t 2 is mod z 2. So, this satisfies. Therefore, mod of the product is same as product of the mod alright. Now what we are given is that, every input sample is finite; that is any x, may be any x l, l can be any index n r n minus r any index it is mod value lies within some range, mod value magnitude lies within some range, within a bound, mod is positive, so it lies within a bound which is finite, less than infinity.

May be you can put less than equal to so; that means, x of l, for any l any index not only one sample l is sample, for any, for any l. This is given in to us that every sample as magnitude within some range, that mod of the magnitude is bounded magnitude is always positive or 0. If the number is 0 is mod is 0. So, 0 to some positive, but maximal it can be M, not more than that; that means, if you take out the mod it can going the positive direction as m negative direction as minus m, and why m is a positive number less than infinity, if this is giving; that is I am not giving you bad input. I am giving input while behave practical input, lies within some range. So; that means, this fellow is, for any index you can imagine n minus r to be l. So, this is less than equal to m, where there is a mod, so m. So, this summation, m will go out for any index, r is varying. So, index is varying, but mod is less than equal to n.

So, this summation will be further less than equal to, m will go out. Summation mod h r, this fellow will be here summation everybody less than equal to m. So, m might go out of the summation as common, summation of mod h are remains. This is further less than equal to this. Now I want this person not to suit up to infinity. So, if I ensure that this side does not suit up to infinity, and m is finite; that means, if this summation which is an infinity summation those as a risk, that it might suit up to infinity. If I sum out next, at this summation, summation of positive part non-negative numbers, because this mod, this summation does not suit up to infinity. So, there is some, it remains finite, so finite into finite left hand side less than equal to finite, means left hand side is finite. So, then I will call the system is stable, bounded input bounded output BIBO is stable BIBO bounded input bounded output.

So; that means, these are conditions. There is absolute value of every sample of this sequence h n, every sample and summed over all the samples. So, absolute thing should be summable; that is when you sum you should get a finite number. So, this is summable if it is this finite. If this is satisfied, if this is finite, and this is finite, product is finite, then it is guaranteed mod y and also will be finite, it will not suit up to infinity, but this is a sufficient condition. Sufficient condition, because if this happens, then this will be finite definitely, but if this is infinity, it does not necessarily be, this was will be infinity. This cast will be finite, and finite is less than equal to infinity. So, that still may be satisfied. So, if the right hand is finite, I am pretty (Refer Time: 20:58) the left hand side will (Refer Time: 21:00) I mean finite. So, it is a sufficient condition, if this happens is sufficient then this is finite.

But it is not a necessary condition, that if this must happen, then only this will be finite, mod y n will finite, because even it this does not happen. If this is a right hand side, then this summation suit's up to infinity. Still in some cases mod y n can be finite, because still then, even then this less than equal will be satisfied, finite left hand side infinity right hand side. So, left hand side is less than equal to right hand side. So, it is a sufficient condition, but non-necessary condition then, what the list to be always on the same side. We try to enforce this sufficient condition, in our filter designs another processes.

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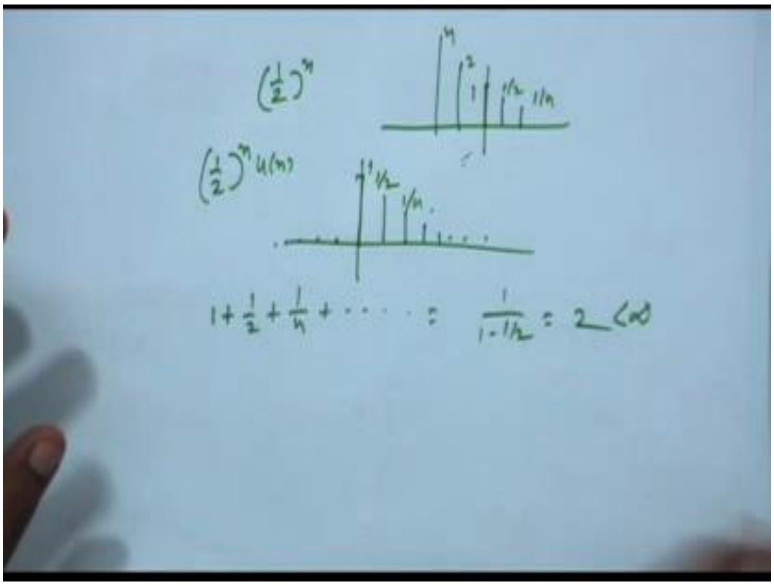

Now, as an example if I give you sequence, half to the power n u n. If I give half to the power n just, then you see, it will be non causal, it will be go on both side. So, half to the power 0 is 1 then half to the power 1 is half, half to the power 2 is 1 4 like that, but half to the power minus 1 is 2 half to the power minus 2 is 4. So, 1 2 4 half, or 1 by 4 like that, all positive numbers, if I add their mod values mod is same as this because all are positive you see it is (Refer Time: 22:35) on this side, but it is going on left hand side. So, if you add all of them, summation will not converge it will diverge, is going to infinity, which means this condition will not be satisfied. So, it will not be BIBOS stable alright.

On the other hand if I give you half to the power n u n, then there is no problem. It will be 1, it will go on the right hand side it will be causal also, because left hand side is 0 dot, dot, dot, dot, so 1 half 1 by 4 dot, dot, dot, dot. So, if you take the mod value there is these values only, because they are positive and add it will be this dot, dot, dot, and you know what this sum is, it converges to 1 by 1 minus half which is 2, which is a finite number.

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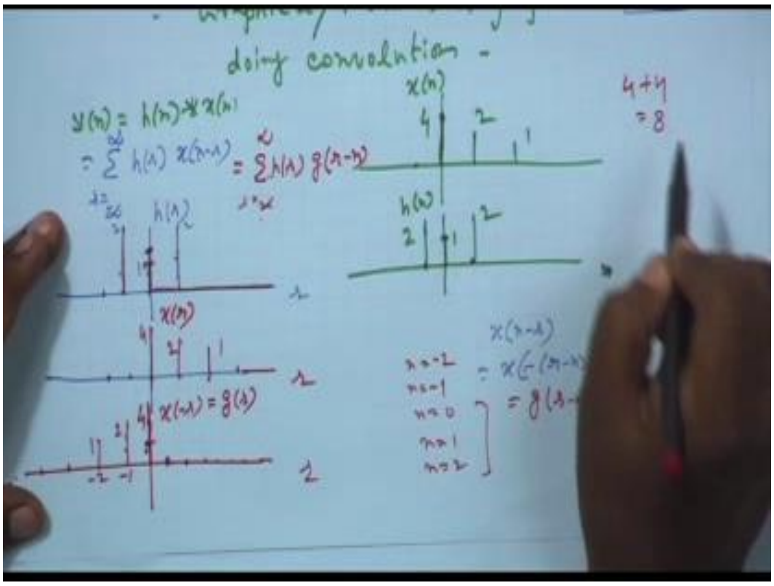

So, in this case it will be positive. So, we want to enforce these steps, which are both, we have to design systems which are both causal and stable, for real life operations alright. Next is convolution how to carry out, convolution graphical or manual. Remember convolution means it may be y n is given to be h n x n, convolve with x n or vice versa And both the sequences are giving to you how to carry out convolution. So, you can write either has h convolved with x or x convolves with h, suppose you write like this, h r x n minus r, this you have to find out. Sometimes you can do it graphically very easily.

Suppose you are given two sequences x n as this 4 2 1 and h is given as, is a non causal sequence and minus 1 this 2, then 0 it is 1 and at 1 again 2, you have to convolve them. Now 1 to convolve them, first thing is I have got h r. So, instead of writing as h n you write it as h r, as r this 1 2 2 x n minus r you have right, x n minus r in this summation I can write as x minus r minus n, n is your choose n you have fixed up you want to find out the convolution at particular n. So, this n is constant. So, like h x also I write as a function of r, then I write it 4 2 1 as a function of r, but here I have what x minus some index, say you write x minus r. This will be interesting what is x minus r. If you are confused you give it a name g r, g r is x minus r. So, you are plotting g r verses r. So, what is g g 0 g g 0 means x minus 0, x minus 0 is same as x 0. So, g 0 is x 0 no problem, x 0 is 4. So, g g 0 remains 4, what is g 1 g 1 is x minus 1, but x minus 1 was 0.

So, this is 0 what is g 2 x minus 2, x minus 2 here 0 value. So, 0 and so on and so forth zeros. What is g minus 1 g minus 1 means x minus minus 1, so x plus 1 x plus 1 was 2. So, g minus 1 at this point minus 1 it is 2, what is g minus 2 x minus minus 2 that is x 2, and x 2 is 1 at point number 2 it is 1. So, this will go become 1, and then g minus 3 x minus minus plus. So, x 3, x plus 3 is 0. So, it will be 0 and 0 onwards; this, here x minus r alright. I want to carry out this convolution at various points. I want to carry out this convolution at various points; n equal to 0 and find out n equal to 1 n equal to 2 dot, dot, dot, dot. If I do it start with n equal then n equal to minus 1 n equal to minus 2 like that. Start with n equal to 0, n equal to 0 means. This will be if you write x n minus r is x minus of r minus n; that means, g of r minus n, because what is g r.

g r is x minus r, g of l x minus l g of k x minus k that is x minus k g k x minus r g r x minus m g m x minus some index g of that index. So, x minus r minus n means g of r minus n. So, I can write this as g r minus n, and r is varying as we go alright. So, now, we have to find out see this convolution at to start with at n equal to 0. So, h r g r you have to multiply, what is g r, this. So, h r into g r for every r r equal to 0 e equal to 1 r equal to 2 r equal to minus 1 like that I have to find out, multiply. So, h 0 this guy g 0, this is g r g 0 we multiply, we get 1 into 4 then I have 0 here non 0. So, product is 0, sorry, product is 0.

This side is 0, so 0 into 1 0 into this fellow, but there is 0 next is 0 into whatever we do. So, right hand side nothing I have to carry out this summation for all the rs. I have to do this sample wise product h into g h of sum r and g of sum r, because I am taking n equal to 0 case; h r g r. So, g r is here, this we form, and I will do sample wise multiply r equal to 0 multiply this with this, r equal to 1 multiply this with this, r equal to 2 multiply this with this, r equal to minus 1 multiply this with this, and then add. Then you see only two terms are coming, this 4 into 1, so 4 and this 2 into 2 another 4. Here I have got minus 1, but here I have got 0.

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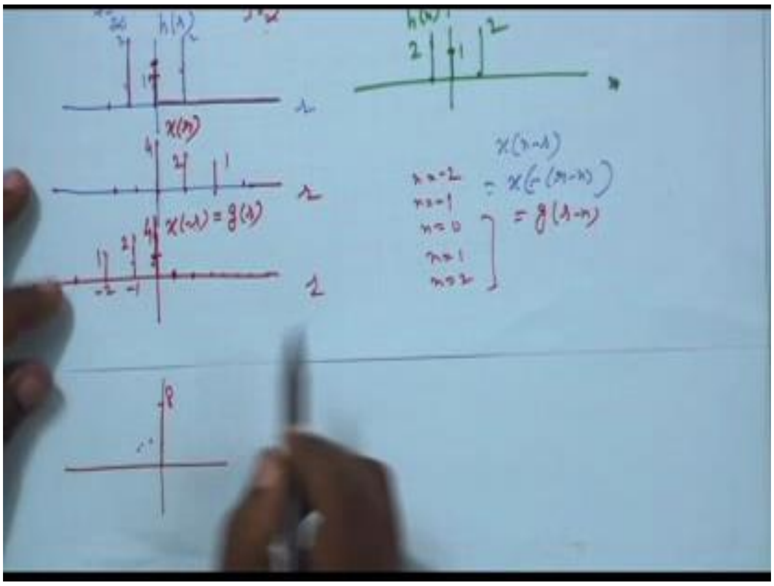

So, reality is 0, then 0 is here 0 is here they do not contribute anything. On the right hand side we have got all 0. So, they multiply the samples from here 1 0 and other 0, they do not contribute anything, so only this into this plus this into this. So, I have got 4 into 1 2 into 2 4 plus 4 8. So, output will be 8. So, similarly we can carry out the other one, I mean I will continue this exercise in the next half.

Thank you.