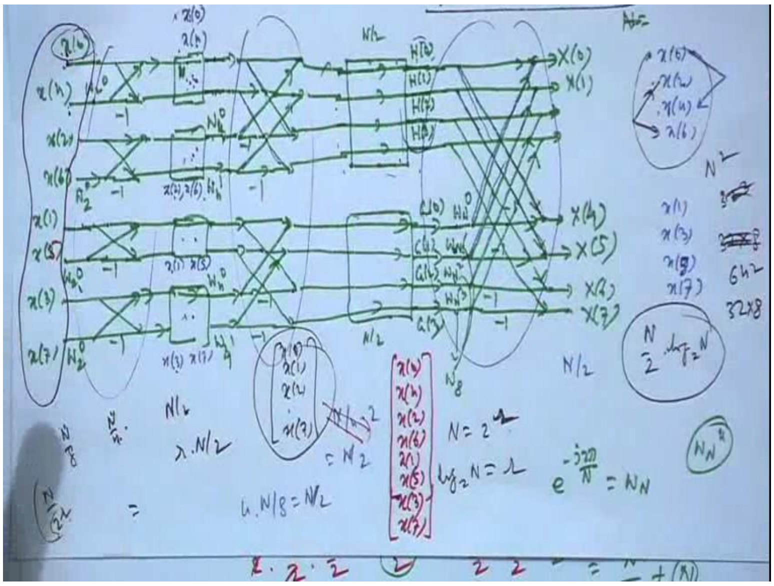
Discrete Time Signal Processing

Prof. Mrityunjoy Chakraborty
Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture – 39
Complexity Analysis of FFT

So, we continue from where we stopped. Now, consider an example.

(Refer Slide Time: 00:29)



Suppose, N equal to 8. So, I have got x 0, x 1, dot, dot, dot up to x7. So, suppose I have got the H; H 0, H 1, H 2, H 3 suppose they are available because N is 8; so N by 2 is 4. If you take the even points X 0, X 2, X 4, X 6, put them, and then, takes their DFT. Now, they are coming out H; and here, suppose we have got this for the odd part, ok. Then, from this formula, you start; you have to find out some output, final output x at 0. So, that will be H 0 plus this factor times G 0. This factor by the way, e to the power, excuse me, we adopted notation, very widely used notation. This factor is called a twiddle factor; we call it W N; and so, this is W N to the power k. So, it was WN to the power k here, this term.

So, start with k equal to 0, and find out capital X 0, X 1, x, upper half. So, suppose, I have to find out X 0; that will be H 0 plus WN whole to the power k, k is 0, W into the power 0 into G 0. So, suppose I multiply it by, I substitute W to the power 0 is 1, but I am still writing. And then, I add these one, two arrows meet at a point, assume there is an adder here; because, if we start drawing adder everywhere, it will be very clumsy and complicated. So, when I write some constant by the side of an H, that means, you are multiplying the, the signal G 0 by this, and they up, and is getting added. So, what you have is H 0, plus W N to the power 0 into G 0; that is your X 0.

On the other hand, if k jumps from 0 to 0 by N by 2, N by 2 here is 4; so, from 0 to 4, a or there was 0, I have found out. Now, 0 plus N by 2, that is, 0 plus 4; I will have same H 0, because H k is periodic over N by 2. So, N by 2 is 4. So, H 0 and H 4, they are same; G 0, G 4, they are same; but only this part will be minus of what it was. So, therefore, what it will be, same H 0, and same G 0; but G 0 now will be multiplied by the same constant, but after that, minus sign; it will be subtracted. So, G 0 multiplied by the same constant; G 0 into this, G 0 into this; but after that, it will be multiplied by minus 1, or, and then, added with the same H 0. You can either multiply by minus 1, but in practice, we will not use a multiplier; we will just use a subtractor; adder we know how to do addition; in binary domain, we can also design a subtractor, if you are using 2's complement number system, you take 2's complement, and add, you get the same thing. But, this will give me X 4.

Similarly, by the same way, suppose here, you want to find out capital X 1. So, it will be H 1, plus G 1 into W N to the power 1. So, that means, G 1 into W N to the power 1. So, this will be added, to your X 1. And now, instead of capital X 1, if k jumps from 1 to 1 plus N by 2; N by 2 is 4. So, 1 to 5, you will have the same H, same G; there is H 1, H 5 same; G 1, G 5 same; this factor will be same; but this will undergo sign change; it will become minus. So, once again, this will be your X 5; same manner, k equal to 2; k equal to 2 means, H 2 plus W N to the power 2 G 2. And, instead of 2, if k jumps from 2 to 2 plus N by 2, that is 4; that is 2 plus 4 is 6; we will have the same thing, same gain, but there will be undergo a minus sign. It will come from here, and by the same token, lastly, also G 3, we multiplied by W N to the power 3; then, added in this X 1 minus; in the same logic, and then added; that will give you X 7.

But, what is capital H 0, H 1, H 2, H 3 is a DFT of X sequence x 0, x 2, x 3, x. So, this is that, it was computing that N by 2 point DFT, N by 2 point DFT; on what sequence, this was working on this sequence, x 0; then, next event, then, next event, then next event. And, this was working on first odd, next odd, next odd, next odd; this is 5, on this. But, this is a sequence; on this also I can apply the same procedure. I can take out its even component, these 2, the 0th term, second term, and what components these 2, either we subsequence x 0, x 4, and, another subsequence x 2, X 6. By the same token this DFT can be obtained. Now, this box has no meaning; this will be another box. Now, this was 4 point; this is the 2 point; this will give you.

So, I already have used up in H and G, I will not use the name here; it will take this x 0, x 4 here, as a 2 point sequence, and it will give 2 outputs. And, this will take x 3, sorry; this will take x 0, x 4, and this will take x 2, x 6; 0, 0th point, first point, second point, third point; first and third, first and third are the odd ones, H 2, H 6; and 0th and second, there is x 0, x 4, (Refer Time: 08:36) x 0, x 4, I calculated 2 point DFTs. They come out here, and here; here also they come out. There, what I have to do? This will get multiplied by now W; earlier, it was W 8; N was 8; you can, instead of W N, here, you can write as W 8, in all of them. I wrote in a general way; all of them, it will W 8 everywhere.

Because, you have to, the total length was 8. Similarly, here the total length is 4; that is, the length of the sequence is 4. I am trying to find out its DFT. So, it is 4, and I am dividing this into 2 halves; due to two halves, but W formula, the W N, N was coming from this length, final length that is 4; that is why, now this will be W 4 to the power 0, W 4 to the power 1. After that, they will be added, and then, this direction will undergo the minus 1. This direction will undergo the minus 1; this is that we have already seen. So, then, this will be here; this will be here; it is just replica, I mean, same thing, and this will go like this. Same way, x 1 and x 5, they will come under this box; x 1, x 5; and, x 3, x 7 here; N is 4 again. So, W 4; here, W 4 to the power 0, W 4 to the power 1; this will get added. Here, this will not add; mind you, I am telling. Now, these two get added; and in this direction, this minus 1, minus 1, it gets added; these get added. So, it is here; it is here; it is here; all right. So, now, again, this is having 2 point sequence now, x 0 and x 4. This was having x 2, x 6. I am writing horizontally, x 2, x 6. This was having x 1, x 5; and, this is having x 3, x 7, right. So, 2 point.

Again, I break it into its even part, odd part; very simple now. Even part has x 0; odd part has x 4. So, every, I mean, even part is a 1 point sequence; odd part is a 1 point sequence. So, 1 point sequence have only 1 point DFT, and that DFT is same as itself; summation X N, e to the power minus j 2 pi k N by, in this formula the (Refer Time: 11:49) DFT formula, the (Refer Time: 11:52) DFT formula, if capital N is 1, it is n 0 to 0, x 0 into e power 0, so, x 0, capital X k, that is, k equal to 0 only one possibility. So, that is same as the, capital X 0 is same as small x 0. So, DFT is the value of the sequence itself; X, the sample itself, it is a DFT. 1 point sequence, 1 point DFT; sample value and DFT are same. So, I will have one point here, and one point here. So, W, here, it was 4, now W 2; this will be what, x 0, x 4; and, this will get added, and this will get minus 1, and then, added. So, boxes do not do anything now; you can remove the boxes; it was just for explaining.

Similarly, here x 2, x 6, 1 point sequence, 1 point sequence. So, its DFT itself, x 6 is DFT itself; H k, G k, like that. So, this we combine, this I multiply by again by W 2, to the power 0, because W N, you go from 0, 1 up to N, N by 2 minus one. So, N here is 2. So, 2 by 2 minus 1 is 0. So, you can go from the 0 to 0 only. And then, you add, and here minus 1, like this. Then again, by the same philosophy, sequence is x 1, x 3. Now, W 2 to the power 0 is 1, I agree, but still, I will be writing like that; last one is x 3, x 7. So, this is the entire diagram, you see. Now, let us see what computational advantage we get. We suppose have, here N is 8, but suppose, a general N; for general N, I have got this box. This is, suppose generality, this is by (Refer Time: 14:27) I have got H k, for k equal to 0 1 up to N by 2 minus 1; and, G k also. So, for this butterfly, these are butterflies.

How many are there? For this structure, this multiplication is common; for the next butterfly also, this multiplication is common. This butterfly, this is common, this is common. How many butterflies? N by 2 butterflies; like this one, and this one, they together form one. Then, this one, and this one, they form one, totally, N by 2 butterflies; and for per butterfly, only one multiplication, only one multiplication. So, you have got N by 2 complex multiplications. Of course, I can say that, W into the power 0 is 1. So, no need to multiply. So, it will be N by 2 minus 1; all those things I am not doing; very rough calculation, N by 2 multiplications, right.

Then again, I assume that, these are somehow generated. Now, how to generate them? This is again generated by this. That suppose, these 2 DFT s are available, then, you have got how many? This was N by 2; N by 2. So, now, N by 4, N by 4, N by 4, N by 4 here, N by 4, N by 4; how many butterflies? N by 4 butterflies here, it was N by 2, N by 2; how many butterflies? N by 2; N by 2, because, one from here, and here, and another from here, or here; one butterfly means, you take one output, and one output, you have combine this way after multiplying, and then, again, going in this direction, multiplying by minus 1, and in this direction, there is 1 butterfly. So, obviously, if we use N by 2, here, N by 2, here I have totally N by 2 butterfly. Similarly, if it is N by 4, N by 4, you have N by 4 butterflies. Like, N was 8 in the previous example. So, 8 by 4 is 2.

So, you have got one butterfly; this, another butterfly this. 8 by 4 butterflies; or again, what butterfly, how many multiplier? 2 multipliers, N by 4; how many butterflies I have got, N by 4; per butterfly one multiplier. So, N by 4 into 1; but, I have got N by 4 here, another N by 4 here. So, again, N by 4 into 2, so N by 2; next time, N by 4, if N by 4 will give rise to N by 2, sorry, N by 8, N by 8, total N by 8 butterflies; per butterfly 1 multiplier, N by 8 multiplier, but, how many N by 8s? Because, this is giving rise to N by 8, N by 8; this is giving rise to N by 8, N by 8; this is giving rise to N by eight N by 8; this is giving rise to N by 8 N by 8. So, how many? 4. So, it will be again N by 2; I repeat, here I, what I had? These 2 blocks, like, these 2 blocks, give rise to, I mean, because, I was computing this DFT. So, these 2 blocks, they give rise 2 butterflies. In general, it was N by 2, N by 4, N by 4. So, this block, these 2 blocks, give rise to N by 4 butterflies; N by 4 into one multiplication, because, per butterfly 1, and 1 more case here, 2 into N by 4.

Now, this was N by 2, N by 4. Now, this N by 4, this is giving rise to N by 8, N by 8; N by 8 butterflies; per butterfly, one multiplier; N by 8 into 1. Again here also N by 8 into 1, here also, N by 8 into 1, N by 8 into 1; so 4 into N by 8; again, N by 2, so every stage, this is one stage; this is one stage; this is one stage. Every stage, your length of the DFT reduces, but number of such parallel butterflies, you know, parallel, this increases. So, total multiplier, again same. So, if you have got, say k number of stages, l number of stages, l into N by 2; that will be the total multiplications. Now, how many stages you have? N by 2; divisible by N by 2.

The next time, N by4; next time, N by 8; finally, it will be 1 point. So, that will happen only if it is N by N. So, how many stages I have? It is N by 2; then, N by 4; N by what? So, Actually, your (Refer Time: 19:20) by N was 2 to the power r. So, basically, you have r stages, r into N by 2. You have r stages; like, N was 8 here. So, 2 to the power 3, one stage, one stage, one stage; per stage I have got? N by 2 computation; N is 8; you can put 8 here, or write N by 2, 3 into N by 2. Why r, because, N by 2; then N by 4 then, N by 8. N by 2; if the basic block was, I am dividing into N by 4, then, N by 8. So, 1 stages, another stage, another stage. So, how many cases? It was 2 to the power 1, 2 to the power 2, 2 to the power 3, up to 2 to the power r; then only, N by N, 1. So, I will get 1 point. Here, I am taking N point sequence, and breaking into N by 4, N by 4; into N by 2, N by 2; total was length N, even point here, odd point here; that is, corresponding to these DFTs, N by 2, N by 2.

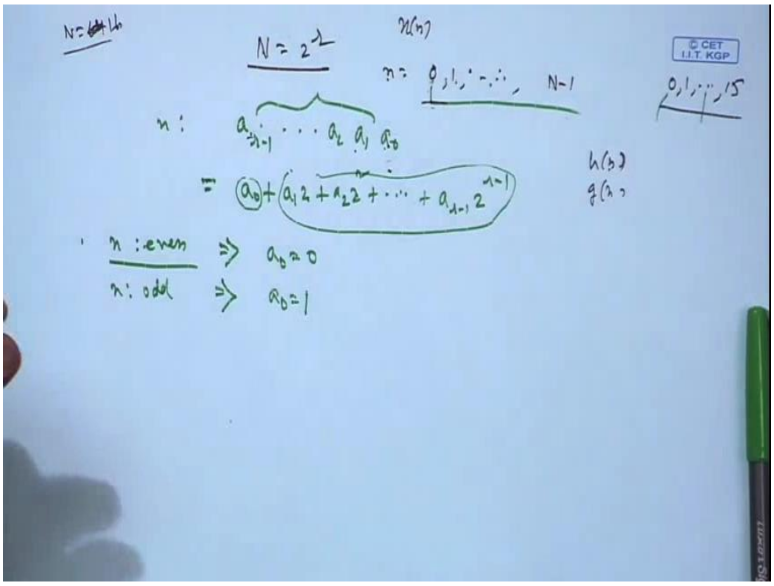
Then again, each N by 2 was broken down into further half. So, N by 4, N by 4; N by 4, N by 4; they are corresponding to this sequence, x 0, x 4, x 2, x 6; x 1, x 3, x 5, x 7. Then again, this was broken into two; N by 8, N by 8, N by 8, N by 8, N by 8, N by 8, N by 8, N by8; till the point I get 1 point sequence, x 0, x 4; it become x 0, 1 point, that is the even point of this 2, length 2 sequence; because, length 2 x0, x 4, even point or odd point. So, I got x 0 is the only point, even point, x 4 the only point, odd point. So, it is a 1 point sequence. So, then, then only I, can stop.

So, when do I reach 1 point? I started with N by 2, stage 1; then, N by 4, 2 to the power 2, stage 2; N by 8, stage 3; three stages; N by 2 to the power r, to the power is N; then only, N by N is equal to 1. So, after r stages, sequences then break into sub sequence, then break into sub sequence, I get 1 point sequences. There is here, here, there is 4 point, 4 point; then, 4 points were divided into 2 point, 2 point, 2 point, 2 point; then, 2 point was divided into 1 point, 1 point, 1 point, 1 point. So, sequence length becomes 1 point; that happens only when I have got r stages. I repeat again; start with N by 2, N by 2 point; N by 2 point; then, you go for another stage, everybody is N by 4 point; not still 1 point; then N by 8 point. So, when will I have 1 point sequence, like x 0 is a 1 point sequence; x 4 is a 1 point sequence; when? When? In the denominator, I have N; N is 2 to the power r. So, I start it with 2 to the power 1, one stage; if I worked N by 4, 2 stages, 2 to the power 2; if I go for N by 8, 3 stages, 2 to the power 3. So, if I go up 2 to the power, I mean, 2 to the power r, r stages, then only, I get 1 point sequence; I stop this business. So, r stages; per stage, total computation is N by 2 complex multiplications. So, total will be r into N by 2, which also you write as r was, what was r, this. So, log 2 N, that is 2 N is your r, right. So, there is your famous formula comes, N by 2 log 2 N.

So, it is a huge saving. Originally, it was N square. Suppose, N is 64, 64 square and it is 32 into If N is 64, it is 8 log 2, 64 is 8, 8 into 32. So, you have 32 square on one hand, and then, is just 32 into 8. See, the kind of Not even 32; 6 into 8; sorry, 32. If you have N equal to 64, sorry, if is N equal to 64, originally you had a 64 square; or here, 64 by 2, that is 32, and log 264, that is 8. See the huge difference. This is 64 square. So, 32 into 2 into 64; that is, 32 into 128; and here it is only 32 into 8, huge difference; 16 times less. And, if the capital N is large, I mean, your gain will be more and more. Lastly, you see, our original sequence was in this order, x 0, x 1; like, if you write a vector, it will be, if you it write in a vector form, it will be, you know, this kind of form x 0, x 1, x 2, dot, dot, dot; x 7. Write the vector form.

This is the sequence, but look at this order. This is, I am writing here only, this is different. This is x 0, then x 4, then x 2, then X 6, then x 1, then x, it cannot be 3, it is 5; x 5; then x 3, x 7. So, the whole order has got disrupted. So, question is what is this new order? If I give a vector, and that vector is long, there is no point in going back calculating like this, and you know, forming the new vector. There is a way to get this new vector.

(Refer Slide Time: 25:24)



How to get this new vector? Now, suppose your N is 2 to the power r. So, I have got number 0, 1, x 0, so, index n, x n; n will be 0, 1, dot, dot, dot, up to N minus 1. Suppose, there is an example N is 64; that is 2 to the power 8. So, it will be 0, 1, up to 63. Or, say N equal to 16. So, it will be 0, 1 up to 15. Now, all this decimal numbers, integers, 0, 1, 2, 3, up to 15, if I want to represent them by binary words, how many bits will you require? 4 bits here; 2 to the power 4; 16, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 2, like that; in general, if it is 2 to the power r, I will require r bits. So, every index value n, 0, 1, 2, all the index values, up to capital N minus 1, like 15 here, or, if capital N is 64, it is 63 here, that will require r number of bits; if N is 64, 2 to the power 6. So, we will go up to 0, 1, up to 63, 0, 1, up to 63.

These, all these indices can require if you want to represent all of them, you require 6 bits. If N is 16, there is 0, 1, up to 15, there is index set, the small n goes about this. Then, if you want to represent each value of small n, time index, I will require 4 bits; 2 to the power 4 is 16. So, in general, I have got, suppose a 0, l s b, a 1, a 2, dot, dot, dot, a, how many, r minus 1. So, how many bits? 0th bit, first bit, second bit, up to r minus 1 bit, total r. This is how my n is, and what is the binary value? a 0, because, the decimal value a 0, plus a 1 into 2, a 2 into 2 to the power and, these are all positive. I am not representing a negative number here; negative is impossible; this is up to range. So, a to the power 2, to the power 2, dot, dot, dot, 2 to the power r minus 1.

Now, suppose n is of even, n is even, in that case, if n is even, then, it is divisible by 2; that means, a 0 cannot be 1; a 0 has to be 0, because if a 0 1, this remaining part, you see, they are all 2, 2 to the power 2, 2 to the power 3, up to 2 to the power r minus 1. So, they are the, that is divisible by 2; but, if it is 1, and if you divide the entire thing by 2, there will be remainder 1. So, it is not be divisible by 2. So, n is even means, a 0 must be 0, and, n odd means, a 0 must be 1. Now, I am dividing this sequence into even component and odd component. So, what I will do, I will basically 0, 2, 4, 8, they will come under 1 umbrella; and, 1, 3, 5, that will come under 1 sub sequence; that is you remember h n here, and g n. h n, g n, 2 length N by 2 sequences.

But here, indices will be 0 to, in terms of x, x 0, x 2, that is, h n is x 2 n; g n is x 2 n plus 1. So, it is, in terms of x, x 0, x 2, x 4, x 6, like that. So, n will be taking even numbers; that means I will take the N; it is their binary representation of this form for various Ns; whenever a 0 is 0, and it is even. So, I will put them in one half, and whenever a 0 is 1, it is odd, I will put them in another half; while a 0 is 0, I will consider 0, 0, 0, 0, 0, then, 0, 0, 0, 0, 1, 0; that is decimal value 2; 0, 0, 0, 1, 0, 0, that is decimal value 4. Then, 0, 0, 0, 0, 1, 1, 0, decimal value 6, like them. So, in that order, I will put them; same for the odd part. I think time is off for this class, I will continue from here, in the next.

Thank you very much.