Discrete Time Signal Processing
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Lecture – 38
FFT: Decimation in Time

We will be considering now the computation of DFT.

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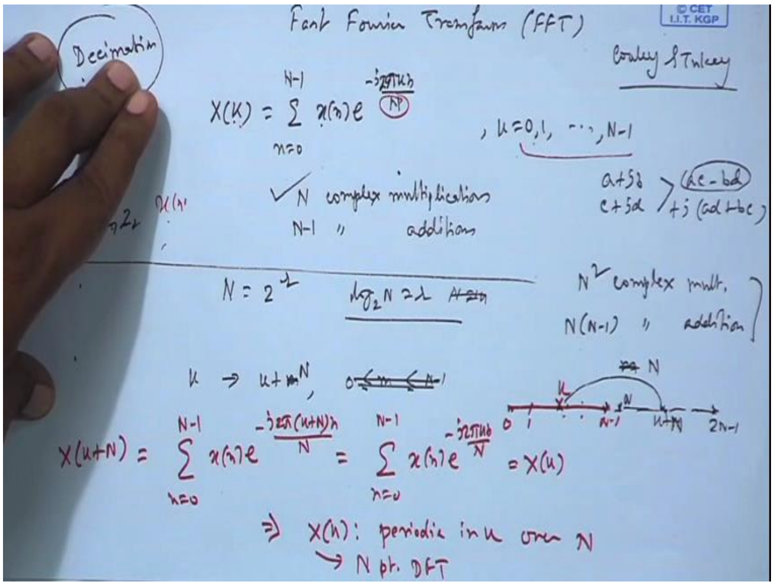

So, you are basically considering discussing today Fast Fourier Transform FFT, though FFT is a big topic. There are many algorithms under FFT. We will consider the basic one which has been by Cooley and Tuckey long back in early 70s when they are working under IBM. FFT now before that suppose you are considering if see the formula of DFT X K is now this you have to calculate for k equal to 0 1 up to again N minus 1. So, k also has the same range, but for each K in this range, you have to carry out the sum to get the corresponding X K. In this summation, you have a product x n into this. So, typically x n could be complex and this is complex x n could be real.

So, let us assume x n is complex. In general, you have got 1 complex number, other complex number Z 1 into Z 2. Suppose you call Z 1. It is Z 2. You have a product of two complex numbers and how many such products N equal to 0 1 case N equal to 1, another case N equal to up to N minus 1 another case. So, total N means you need N number of complex multiplications and then, you are adding them. How many additions? I mean if you are having total N number of terms, you are adding them. Basically you need N minus 1 complex addition.

Now, one complex multiplication in general will be of the form a plus j b, c plus j d. If you multiply, you have ac minus b d plus j, ad plus b c. So, 1 multiplier ac, 1 b d, 2 multiplier and then, add 1 multiplier b c, 1 multiplier. So, you need four multipliers 4 real multipliers. 2 are used for the real part; 2 are used for the imaginary part. In fact, should the total you need four multipliers, but by some tricks which of not showing here you can bring down from 4 to 3. By introducing some additional terms, addition will go up, but whatever this is complex multiplier such that complex multiplication can have 4 or sometimes 3 real multipliers because all your multipliers in hardware will be real multiplier. You know you will putting them in some way, some format and then, we are considering something to be complex and all that, but all multiplication see real hardware will be real multipliers using real digital hardware. So, multiplex take much more complexity than addition.

That is why in our computational analysis; we will be bothered only by multiplication. That is our goal will be to read out the number of multiplication by that process, addition also will go down, but we are not so bothered about addition because addition is going to take much hardware multipliers when your hardware are hungry. They take much more. They have much more complexity getting much more hard work. So, 16 bit into 16 bits multiplier. So, you have got to add you know I mean you have got so many rows shift at shift at shift at and the whole thing will take 16 by 16 array. That will be you know you will be counting lot of hard work. So, you have more complexity or in terms of time if you want to do it, you will take more time. That is why multiplications have multipliers of interest.

Our purpose will be to bring down this computation. Suppose N is large. Suppose N is 1024, then part output part K you need 1024 multiplications, complex multiplication and each multiplication you still have four or sometimes three real multiplications. So, being real multiplication for each k, you will have how many cases, capital N cases. So, N cases each with N complex multiplication. So, you will have N square complex multiplications. You understand if capital N is 1024, it will be 1024 square to compute all the X K. That will be too much of computation. That is why when DFT came 1 1, you know main goal by all the researchers was to bring down the computational complexity and that led to by exerting the inherent, you know mathematical structures symmetry and many other properties and by that process when the fast algorithm came up one basic algorithm which is Cooley-Tukey algorithm called decimation, in time these algorithms will be considering today.

So, you have got n squared complex multiplications and adding to N into N minus 1 complex additions part K, I have got N minus 1 addition, so N into N number of values for k, k 0 1 up to capital N minus 1. So, total N possibilities of K. So, aging to N minus 1 complex additions and we will be modifying the algorithm, so that this goes down. This computation complexity goes down significantly.

Now, for that first we will assume N to be power of 2, may be 2 to the power r that is N can be 64, 128, 256, 632, 16 like that. Now, you can ask me suppose N is 14, then what to do? Then we will simply apply two zeros to make the link 16. So, we will necessarily 0 to make the link upon nearest power of 2 nearest part of two. So, 2 to the power r or sometimes we can also write log of N to the base 2 is r. This one thing we assume, so the algorithm I present will be valid for this, but then there are many modifications of the algorithms, where N can be any arbitrary number. It may not have to be a power of 2.

There are still algorithms available, but for our purpose we will N to be power of 2 like this. Another property this is X K, k is from 0 1 up to N minus 1. Suppose K will jump from K in this range by another small m, where small m is again between N minus 1 to zero, that means we already had a range 0 1 dot, dot, dot, up to N minus 1 k could be somewhere here K and from here, we are jumping by m. So, it is K plus m and m can be if m is 0, m can be 0. So, you are having K and then, K plus 1, K plus 2, then you cross to this side maximally it can be up to here 2 N minus 1. This is N to N minus 1 because if K is at N minus 1 and then, small m 0 N minus 1 small m 1 n small m 2 N plus 1 small m 3 N plus like that.

So, finally you can go up to this. So, if I take K and suppose m, I do not take arbitrary m. I take N. Now, I am taking a specific N. So, to be K plus N, this will be in this block in the same place from 0 to K minus 1, anywhere you start and by N what will be if I replace K by this thing, K plus capital N in this summation in the right hand side summation what will that be. That will be that summation and if you take that capital N out 2 pi capital N by capital N, that is 2 pi into small n e to the power minus j 2 pi into integer small n that is 1. So, you will get back original summation which is your original X K. That means, if I extend this formula and take K beyond that range, so X K plus N will be same as X a K. K is within this range. If you go from any K within this range by plus N, you will get the same thing and why plus capital N? By the same logic K plus 2 N, K plus 3 N, k plus 4 N, k minus N, k minus 2 N, so that means X K periodic in K over N, where X K is n point. Remember m point DFT that is the original length of the sequence cross capital N x small n was length, capital N, I took its n point DFT that n was coming here.

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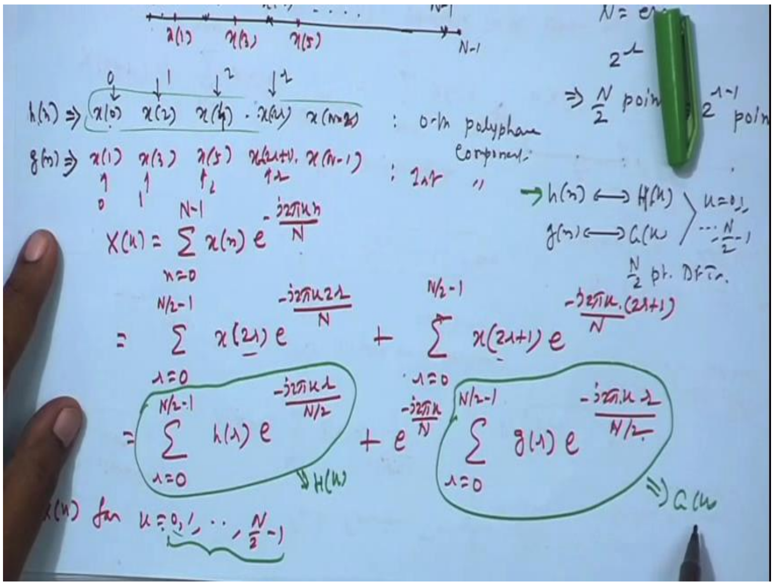

X small k also will be over the same range 0 to n minus 1 that n point DFT is periodic over k over a period in k over a period capital N. These two facts I will use. Then what I will do? I divide the sequence into two half that is x 0 x 2 x 4 dot, dot, dot. Now, N is even because N is 2 to the power r. So, this N minus 1 is odd. So, if I am hitting the even point 0, 2, 4 dot, dot, dot, then I will go up to N minus 1. So, these are the points I target. So, how many of them are there half N is exactly even. So, half and N was 2 to the power r. So, now you will have N by 2 points. That means 2 to the power r minus 1 point and so that I take those points out from one set and I take x 1 x 3 x 5 dot, dot, dot. You know these are like polyphase vectors. You are basically doing polyphase decomposition by 2 zeroth. Polyphase component is x 0 x 2 x 4 dot, dot, dot and first polyphase component is x 1 x 3 x 5. Just that since if those good is of FFD polyphased decomposition need not come up. So, they write like even part and odd part, but now you know polyphase decomposition if I have these events x 0, x 2, x 4 dot, dot, dot up to x m minus 2, this is zeroth polyphase component.

How do you get it? If you just decimate by 2, you get it and another component is x 1, x 3, x 5 dot, dot, dot, x n minus 1. Both time how many? This is first. How many terms N by 2 here N by 2 here, there is 2 to the power r minus 1 here 2 to the power r minus 1 here. This sequence we normally call p 0 n polyphase component and then, this is p 1 n, but I will just use another notation may be h n. This is g n. So, h n is N by 2 point sequence. Therefore, if you have to consider this DFT, it will be N by 2 point DFT. If I have to consider this DFT, it will be N by 2 point DFT. If I do not have additional zeros, all that we cannot n by 2 point DFT and h n. Let it have H k and g n, G k both are N by 2 point that is k, here will be 0 1 dot, dot, dot, N by 2 minus 1 because like this sequence, its sequence of length n by 2. So, total number of point N by 2 if you start from 0 and 1, you have to go for N by 2 minus 1 that is h n. It will have capital H K g n capital GK; both are N by 2 point DFT N by 2 point DFTs, fine.

Now, let us come to the original DFT. Original DFT is now here. I take the even part out. Those terms I will write in one place that is x n will be n will be 2 r. So, r can be 0, r can be 1, that is 2 into 1 r can be 2, 2 into 2 like that r here is how much N is a power of 2,s so N by 2 minus 1. So, r 0 that is x 2 into 0 is 0 x 0 and r 1. There is x of 2 into 1 x 2. This term r 2 means x of 2 into 4, this term. So, those terms and how many rs total terms is N by 2. So, it will go up to N by 2 minus 1. So, those terms I write separately and another one also will be here that I am coming to x 2 r e to the power minus j 2 pi k. So, n is replaced by 2r, this is 2r by N was N is taking either 0 or 2 or 4 or 6 or 8 like values. So, if I write n as 2 r that will be 0 1, r is 0 1 2. So, all those 0 1 2 3 is fully up to N minus 1. So, here small n become 2 r here also small n become 2 r. So, that is one part, one half of this summation. Another half of the summation also will be here. What terms x 1, x 3, x 5. So, this is you can write as x 2 r plus 1 if r is again 0. If r is 0, it will be x 1. If r is 1, x 3 if r is 2 x 5 dot dot dot and total how many. Again total number of terms here is N by 2. So, if you start from 0, it can go only up to N by 2 minus 1 and small n is this 2 to the power minus j 2 pi k by N and small n will be 2 r plus 1.

Now, what is x 2? R x 2 r is hr. What is h n? H n is x 2 n. This is the zeroth failure. This is the first failure h 1, h 2, h 2 is x 4. So, hr is x 2 r h 0 x 0 h 1 x 2 into 1 h 2 x 2 into 2 4 dot, dot, dot hr x 2 r. So, this is nothing, but hr is nothing, but just renaming those x samples by h sample x 0 is h 0 x 2 is h 1 x 4 is h 2. So, that way it is hr x 2 r is a hr and this I can write e to the power minus j 2 pi k r by N by 2 N by 2. Now, N is power of 2, 2 to the power of. So, 2 to the power is divisible by 2, N by 2 also is an integer. So, like this and here h x 2 r plus 1. What is x 2 r plus 1? If r is 0, x is 1 that is d 0 d 0 is x 1, then f 1 is x 3 g r is r h zeroth x 1 g 0 is x 1 g 1 is x 3. That is 2 into 1 plus 1 g 2 is g x 5, that is 2 into 2 plus 1 g r 2 into 1 plus 1 that is x 2 r plus 1 is gr. From here you take out e to the power minus j 2 pi k into 1 by N that separate and that does not depend on r. I can write outside and here e 2 the power minus j 2 pi k 2 r by N. Again I write as r by the same manner N by 2.

Now, remember I am finding out this DFT of original sequence. So, k range was 0 1 dot, dot, dot, N by 2 minus 1, then N by 2, then N by 2 plus 1 dot, dot, dot, up to N minus 1 basically 0 1 up to N minus 1. I have written this intermediate point because original sequences that much of length capital N. So, k also will go from 0 to capital N minus 1 because small n was from 0 to capital N minus 1, but these I am writing in this way and I am taking the upper half or maybe you consider the lower half up to this 0 to N by 2 minus 1. That is 1 half and that is another half 0 1 up to N by 2 minus 1, 1 half other N by 2, N by 2 plus 1 N by 2 plus 3 dot, dot, dot, up to N minus 1. That is another half. Both halves have n by 2 terms.

Suppose k is having for this entire range from 0 to N minus 1, so I have k equal to 0 from this summation. Find out if k equal to 1. Do this again. Find out the x 1 dot dot dot. Suppose for time being I am finding out X K for this upper half k equal to 0 1, only up to

N by 2 minus 1. This is very important. So, I get the lower half. The upper half part I will do later. So, k takes these values. So, look at these summations. Now summation hr, hr is a sequence. How many points N by 2 points h 0. There is h 0 h 1, that is x 2 h 2, that is x 4 h 3 that is x 5 x, sorry h 0. There is x 0 h 1 x 2 h 2 x 4 dot, dot, dot, hr x 2 r like that. So, this is hr sequence e to the power minus j 2 pi k into r and k also is taking to be over the same range as that of r. So, h was finitely sequence of length N by 2 h 0 h 1 h r that is x 2 r up to h N by 2 minus 1 which is this. For that sequence if I carry out this DFT N by 2 point DFT, this one what will that be that h k is nothing, but this summation where k will be over this half upon the same range 0 1 2 up to N by 2 minus 1. That is what I am finding out because I am taking k for the time being only over this lower half. So, this will be nothing, but H K that is from this sequence if I take its N by 2 point DFT, k will be from what this range only because r is over this range time index r is over this range. So, k is also with the same range. So, this summation will be H K.

Similarly this summation gr, gr is a sequence g 0 g 1 g 2 dot, dot, dot. So, gr is this sequence and what is the length, what is the range r from 0 to N by 2 minus 1, what is range of k same range and instead of a, I have been here N by 2. So, is basically if you want to carry out G K, this will be G K because length of this time sequence is gr is N by 2. That is why it is N by 2 coming e 2 the power minus j 2 pi k into 2 r r is the summation index time, but k should vary over this part only 0 to 0 1 up to N by 2 minus 1 because r time index also varies over this range only. K also have the same range, must be having the same range and that is d1. So, that means for that k this will be G K. So, H K plus e to the power minus j 2 pi k by n into G K up to this if you see we are gaining anything see for k equal to 0, you carry out this summation for k equal to 1. You carry out this summation every time. How many multiplications? It is N by 2 because the summation has N by 2 terms 0 1 2. Every time you multiply the h 1 complex value with another complex value, now N by 2 for each k and here again for each k N by 2.

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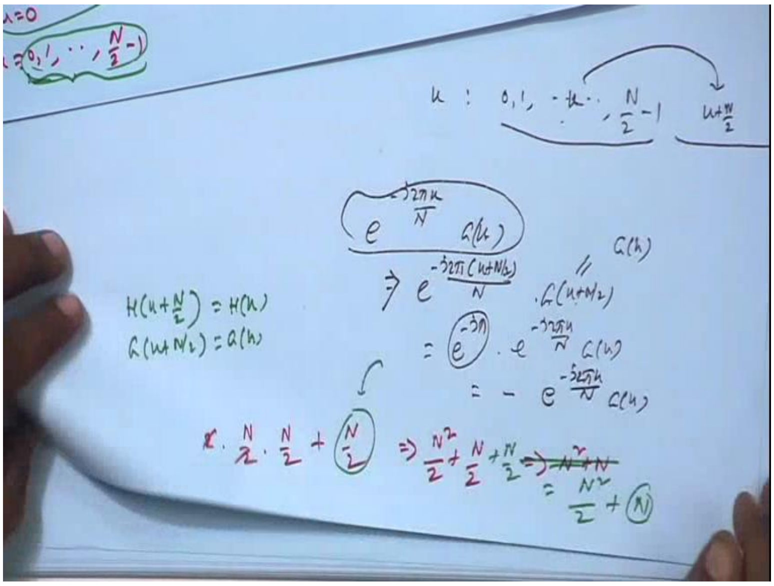

So, N by 2 times N by 2 for this half of the k and again N by 2 times N by 2. So, twice this and plus this term into this term into this means for k equal to 0 1 multiplication for k equal to 1 1 multiplication. How many? It is k N by 2 k. So, another N by 2, so n square by 2 plus N by 2 all right N square by 2 plus N by 2. Originally I had N square; originally I had N square complex multiplication, right. Now, it is appearing to be. So, there is for the lower half of the k, but I want to find out the same thing for upper half of k. So, if you go for upper half of k, there is k equal to you, put these n by 2 plus 1 dot, dot, dot, n minus 1. So, apparently for that half also if you carry out, if you put the various values of k from that upper half, again same thing have to do. I mean the summation for every k you choose, there will be one complex number times. Another complex number h r, I am taking to be general. If it is real, so I am considering generality.

So, complex into complex, complex into complex, complex again another N by 2 because upper half also has length N by 2 and for each k, there summation again N by 2 terms. So, N by 2 into N by 2, same thing plus N by 2 into N by 2, the same thing you know and this also will give raise to another N by 2. So, it will rise to N square plus N. So, we are not getting anything because if you forget about this N, N is very large. So, N square is much larger than N. It is n square originally N square. So, what the hell about what out of this you know by this breaking and all that I got anything? Answer is yes. When k goes out of this range, k H K is periodic in k, but about how much for the range of k is N by 2. So, H K is periodic over n by 2 that is H K plus N by 2 is same as H K. Similarly g k plus n by 2 is G K.

The moment I go from the lower half to upper half, what is N by 2 N by 2 is 0 plus N by 2. What is N by 2 plus 1? It is 1 plus N by 2 plus 1. What is N by 2 plus 2? It is 2 plus N by 2. What is N by 2 plus k? It is k plus N by 2 plus k. See you have to find out H at these values N by 2 plus k. I do not have to do any calculation because I can just go back by N by 2. Whatever value I got earlier for H K, I can simply substitute that. So, I do not have to do any calculation. So, for the upper half, this does not require any calculation for the upper half. This multiplication you know this you have to carry out because here it is capital N.

So, if it is periodic, it is periodic over k equal to 0 to N minus 1 that is for a period of k over equal to capital N, not over n by 2. So, k equal to 0 1 value, k equal to 1 on value will go up to k equal to N minus 1. We will get different values and then only it will be repeated. So, that means when I go for lower half, I had these many terms coming from this. When I go to upper half again another N by 2, so n square by 2 plus N by 2 plus N by 2, it will be N square by 2 plus N. Originally I had N square. So, N square becoming N square by 2 just plus N, it is coming down by a factor of 2. If N is large, forget about N as compared to N square by 2 N square from N square, originally I am getting N square by 2 and one more thing, this term e to the power minus j 2 pi k by N you are multiplying something right this into G K.

Suppose you have k from this, then 0 1 dot, dot, dot, N by 2 minus 1. Suppose here is k if find there is product and then, k goes up to k plus N by 2 in the upper half, this was the lower half, upper half for k plus N by 2, it will be e to the power minus j 2 pi k plus N by 2 by N into G K plus N by 2, but G is k is periodic over N by 2 because G K is N by 2 point DFT. So, G K plus N by 2 as I told earlier, G K plus N by 2 is same as G K because G K is periodic over N by 2 g. From any k if you jump by N by 2, you get the same thing. What about this quantity? When I jump, k jumps for the lower half to upper half if you take out this N by 2 terms out, e to the power minus j 2 pi into N by 2 by N. So, N, N cancels, 2, 2 cancels here. I have got e to the power minus j pi into the same thing 2 pi k by N into G K. Originally I had this k has jump. Now, k has jump by N by 2 instead of k i had k plus N by 2 instead of k, k plus N by 2 G K plus n by 2 is G K and this is this. So, I replace G K and this part is out, but what is e to the power minus j pi that is minus 1. So, it is minus of what I had earlier.

So, you see when i k moves to the upper half, if in these product I do not have to calculate again whatever be the product was, this product G K into this was if k jumps from k to k plus N by 2, then the same thing will prevail. Only sign you will undergo a change from plus to the minus. So, it will not require any multiplication. It will only be minus. So, what kind of structure, it gives the external. I will do in the next session.

Thank you.