Discrete Time Signal Processing
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Lecture – 37
FIR Filter Design by Windows

Next topic is FIR Filter Design. Now, what is advantage of FIR filter? One thing we have already seen earlier that this thing FIR filters can guarantee linear phase.

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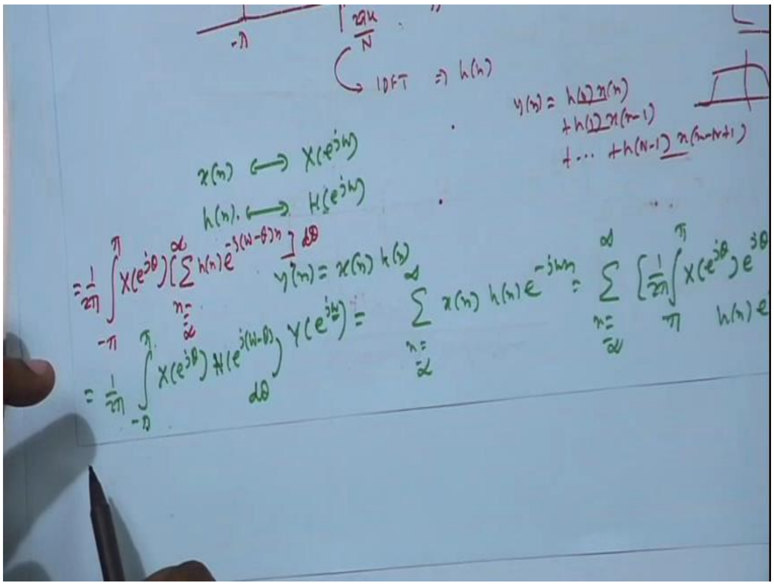

You can design imposing mirror image symmetry condition from the coefficients. So, linear phase conditions can be achieved in FIR, but same cannot be said about IIR filter. That is one advantage. There is another big advantage which is not mentioned in books. Suppose you have got one arbitrary, very arbitrary frequency response given digital frequency response and therefore, it is repetition. This is minus pi like that. Now, suppose you want to design IIR filter, how do we design digital IIR filter? First we have seen we have go to an analogue domain design of prototype analogue filter and then, come back to digital domain either by impulsive variance or by directly finding out such appropriate functions and all that any by that, but H a s whether to have a replica of this. We have to design this in the digital domain taking just a component from minus pi to pi whatever I have as an analogue response and then design it.

So, there we can design some functions like you know I mean filters like Butterworth filter, Chebyshev filter, you know terms and all that, but those filters have got very specific form like these at a form of this kind by or equivalently. Suppose this is omega h omega h by 2 N, now you see if you have got a very arbitrary kind of you know response and you want to design a filter whose frequency will be like that, you can divide match it by you know very nice functions like this. This will be as I told you; it will be a low pass filter like that. Another function could be another type of low pass filter you know by smooth kind of function if you have some much zigzag, so much random fluctuations and all that in your response and still you want to a realise it by a filter. I mean design as such and an analogue filter by one of these forms Butterworth and Chebyshev and etcetera which will be such which will have this kind of you know I mean very random fluctuations, very arbitrary kind of response.

The only way to go about is then directly go for FIR domain. In FIR case suppose I take this sample them very close to each other, I sample them. So, this will be my DFT something prior is 2 pi k by N, N I take to very large, use them n number of coefficients N very large 2 pi k by 2 pi by N is very small. So, they are almost adjacent side by side and then if I take the corresponding in Id f t inverse DFT, I get FIR filter h n, h n response of h n. If these are very close to each other, response of this h n will be almost like this. Only thing is only problem that you face here is that you have got too many coefficients n. So, therefore output y n the convolution will be you know I mean h 0 x n h 1 x n minus 1 plus dot, dot, dot, dot, h.

So, many multiplications, for output at every index, n output will require computation on too many multiplications. In fact, because if the order large this the problem of typically to obtain the same response as that of an analogue filter you need to have FIR filter with large order or to match any kind of arbitrary response, you will have to a large order and in that case I mean when you carry out the convolution, you have to you know have so many multiplications which computation complex too will be high. So, this is one price you have to pay, but then here you have the advantage that you can constitute constant of a filter whose frequency response can match any arbitrary kind of frequency response something that is not guaranteed in the case of IIR filters.

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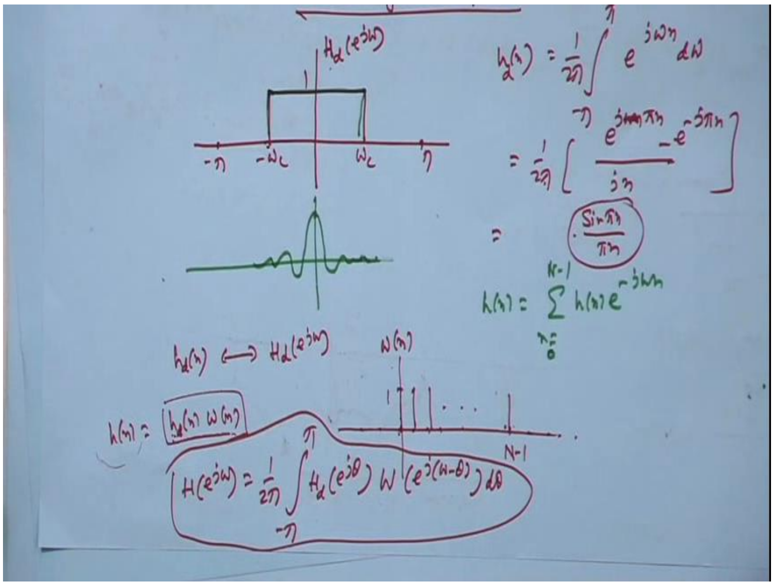

In this class, we consider FIR filter design by a very popular technique called windowing technique. Suppose we start with a very ideal low pass filter. I am considering the magnitude part, only this part. Suppose you are assumed to be 0 or otherwise phase is linear, you can add phase component. I deal with just magnitude, but do not phase to be 0, H d e to the power j omega. This is my desired low pass filter absolutely fine and in the passband, it is flat or outside the pass, it is 0. So, it is appropriately falling to 0 and staying here. If I really want to find out h n, I have to take inverse DTFT. Suppose it is height is 1, so 1 e to the power j omega n d omega, but if you integrate e to the power j n e to the power j pi n minus e to the power minus j pi n, it will be 2 j sin pi n 2 j sin pi n j j cancels 2 2 cancels. So, sin pi n by n by pi n pi is here, n is here sin.

So, basically it is sin pi n by pi n. So, you see this is going in both directions sin function, all right. So, it is a non-causal filter and IIR theoretically going up to infinity minus infinity to infinity that is because if you have got such abrupt discontinuities, you cannot realise it by using just a few filter coefficient, just finite number of filter coefficients and a summation like this. That is N minus 1 because this summation is a finite summation and you see continuous function of omega only if you bring in infinite summation. This can be realised, but if it is a finite summation, we all know it is a continuous function of omega. So, this ideal filter we cannot realise because it is firstly IIR, it is non-causal. But what I can do is, this let me call it h d n d for desired h d n h d n is the inverse transform of. In fact, I have taken this kind of thing, we put then we say that is why a is d to the power j omega. Your desired response corresponding h g n is here and this is IIR. What I will do? I take it only over a zone 0 to may be some N minus 1.

So, that means if I multiply h d n, I take all the samples as e as it is only over this region outside, it will be all 0s. That means I design one window rectangular window all taking value one up to this. So, h d n which caused by inverse

Fourier transform of what where with the desired response here, it was this it could be anything any desired response. Let us call it as h d n. That is why I am not taking to this form because H d e to the power j omega could be anything else; not just this. So, when I take its inverse transform, it is in general IIR, but IIR filters I am not bothered, neither I am bothered about, neither I am interested in non-causal designs. So, I apply by multiply h d n by a window W n. That means, only this coefficient, only coefficients of h d n at these points will get multiplied by 1. So, they remain s d r, otherwise it will be 0. So, I am approximating whether truncating that original s d n to the right of N minus 1 into the left of origin to 0. So, it is an approximation.

So, obviously response if I call it new h n response of h n will not be same as what the response of h d n was because there is an approximation, but what will be a response of h n, let us see. Now, may be I have done it in the very beginning or I have not done. I am not sure. So, that is why it will be better to look into one basic result that suppose x n is a sequence and it has got DTFT. This h n is a sequence and it has got DTFT this. We know if I convolve x n with h n convolution y n d t f t of y n will be product of the 2 DTFTs, but if I multiply them and call it y n, then DTFT of y n will not be convolution of that 2 DTFTs. So, it will be product of a particular type. It will not be product that is as I told you if x n and h n and convolved and equivalent y n DTFT of y n will be product of these DTFTs, but if they are multiplied, then y n, DTFT of this will not be product of the 2 DTFTs. It will be a convolution in frequency domain between these 2 DTFTs, not product convolution. We can work it out. This is DTFT. Instead of y n, I have between x n and h n is between general minus infinity to infinity.

One of them may be x n. I can write in terms of bit's inverse DTFT formula and you want to find out the output DTFT and a small omega of your choice. So, omega is fixed. So, you cannot bring another omega with within integral as a variable. So, I bring X e to the power j theta naught omega e to the power j theta e to the power j theta n instead of omega, I bring theta d theta is the inverse DTFT of x n. I do not bring omega here because omega is fixed from output. So, you cannot use it as variable within an integral. So, this is your x n inverse DTFT relation for x n and remaining as it is h n e to the power minus j omega n and now integral is a summation, this discrete summation. So, I can interchange that 2 that will give rise to this is independent of n. So, it will remain outside the summation involving n and this summation that goes in n minus infinity to infinity and any quantity which is n which has n will be inside this summation. So, h n e to the power minus j omega n e to the power j theta n will be h n e to the power minus j omega minus theta n this summation and d theta as it is, but this is what this is a DTFT at omega minus theta.

So, then this will give rise to this relation. These are kind of convolution, not exactly convolution a kind of convolution because capital X you write not as a function of omega, write as a function of theta and the other one you first make into function of minus theta and then, put plus omega to that. You flip it and then, shift it by omega and then, multiply and then after multiplication. You are not carrying calculating the area under this from minus infinity to infinity. That is a difference here from minus pi to pi only and multiply by 1 by 2 pi. So, this is a kind of convolution, but there are these changes in remember here the limit is from minus pi to pi and there is a multiplication factor 1 by 2 pi outside, all right.

Now, I am multiplying this h d n original in general IIR sequence which is an inverse transform of a given response. Any given response h d n that I am multiplying by the rectangular window, these are my rectangular window of length N, so that only these samples of h d n are retained and others are 0. So, what will be the DTFT of this product? It will of course not match the ideal one, desired one, but what it will be. So, by the previous result, it is a product of two sequences. So, DTFT of this will be convolution between the 2 DTFTs here, but convolution of this form minus pi to pi naught minus infinity to infinity H d e to the power j theta and we to the power j omega minus theta d theta. This is a formula. What is W? What is this function? What is W e equal to the j omega W j? I know is this, sorry W e to the power j omega will be n equal to 0 to N minus 1 1 1 1 1, so all are 1.

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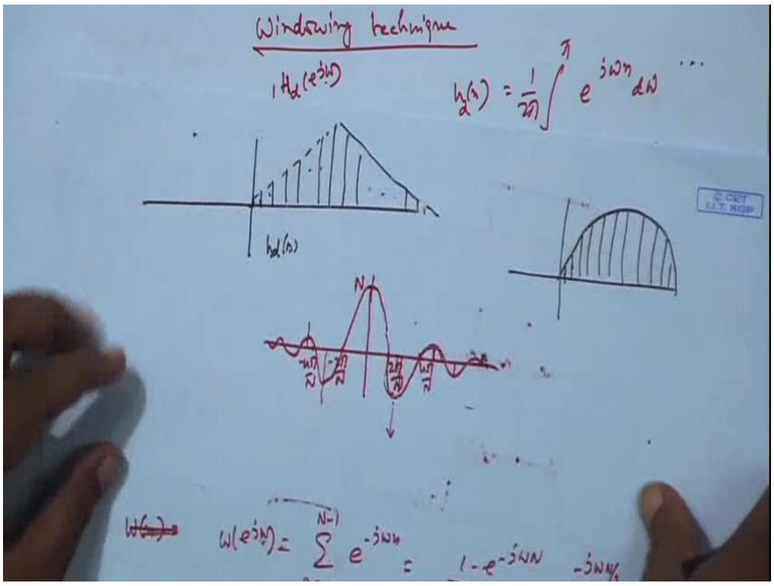

So, no point inputting 1 here just e to the power minus j omega n into 1. If you go on summing e to the power minus (Refer Time: 16:28) series take e to the power minus j omega N by 2 from top and e to the power minus j omega by 2 from bottom. So, it will be e to the power plus j omega N by 2 minus e to the power minus j omega N by 2. So, it will be twice j sin omega N by 2 and it will be twice j sin omega by 2. So, it will be this into this part. Let’s see how it behaves. This will be a phase, all right. So, how it behaves then will put back that in this integral and see at omega equal to 0. It is 0 by 0 sin 0 is 0 sin 0 is 0 so apply in the (Refer Time: 17:43).

So, differentiation of the numerator denominator differentiates with respect to omega. So, it will be N by 2 on top and by 2 here and then, cos here cos here and cos at omega equal to 0 is 1. So, sin N by 2 by 1 by 2. So, it is N. Then, look at the numerator. Only numerator is a periodic function. It will have first 0 crossing when omega N by 2 is at pi or omega is 2 pi by N on this side minus 2 pi by N. So, again 0 crossing at 4 pi by N here minus 4 pi by N dot, dot, dot, dot. So, if the denominator is not present, this will go on where the periodic function of constant amplitude, but you see as omega goes from 0 to pi by 2 0 to pi, it will be from 0 to pi by 2 sin increases is a positive sin theta or theta equal to 0 to pi by 2 because omega will be up to pi you know DTFT from minus pi to pi. So, suppose I am going in the positive directions, so I will go only pi by 2.

So, 0 to pi by 2 first coordinate value sin of an angle that value will goes on increasing from 0 to 1 as angle increases. That means as omega increases, denominator is increasing in magnitude on the positive side on the negative side sin of omega N by 2 omega is negative. So, it will be minus of sin some angle and denominator also minus of sin some angle and so minus minus cancels. So, it was same thing, but denominator is increasing as magnitude of omega starts increasing from 0 to pi as a result amplitude of this numerator oscillation which is as a periodic sin was a pure sin oscillation of constant amplitude, that will continue to get damped. Its amplitude will go down. That is why it will have been like this you know like this like this like this, but that will not happen now it is getting like this amplitude will get them these stuff.

If capital N is large, this will become closer. This height will go up every 2 points. They will come close to each other, but what happens in this case, this peaks of this side loops. These are side loops peaks increase to the area under them remain same that is the property. It can be shown, all right. Now, if I bring it back here, so this is symmetric. So, this function is nothing, but if you carry out the convolution, you have to multiply the two functions and integrate from minus pi to pi 1 by 2. Pi is just a scale factor. So, H d e to the power j theta W e to the power, you can write minus theta minus omega minus within bracket theta minus omega. What does it mean? How to convolve H d e e to the power j theta? So, this function you write as a function of theta.

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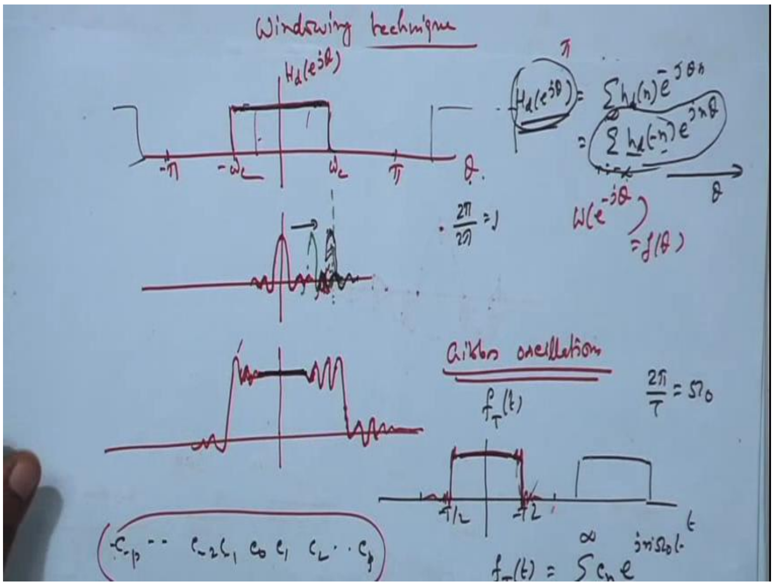

The other one is function of this kind, this function. So, this also you write as a function of theta and then keep it instead of we to the power j theta make it this. That is a function of theta which is nothing, but it is just the flip thing we to the power minus j theta, but you say this is symmetric function. So, if it is that is left hand side goes to right hand side, right hand side will go to left hand side, you will get same flipping. It remains same and then, I am making in narrow now. Then in general will be shifting it by omega if omega is 0 I have got e to the power j minus theta which is f of theta is now theta minus omega and so, f of theta minus omega. So, this will be shifted to the right or to the left by omega. So, first start with omega equal to 0. So, you want to find out the output at 0 omega that time you multiply this function with this function and whatever be the value get this much, you get now you keeps shifting to the right as long as this main part remains under this umbrella constant high area after the product will remain. So, you continue to get some constant figure, but after a while what will happen may be a positive part will come out, positive part of this will come out like this.

So, this positive part comes out the form the previous total value, some positive amount because it has positive part into this height that contribution will go out. So, total value will go down. So, it will be this value will go down after that at negative part will move out after that a negative part will move out, but negative part means as you have seen here this height is larger than this height, this height is larger than this height. So, amount of contribution then goes out, some positive part goes out the negative part. So, larger part goes up negative part. So, what all earlier positive part went up out, overall contribution, overall value came down. Now, a negative part goes further slide to the right. So, these parts also go to the right and some negative contribution goes out. So, it will go up and then, again further slide to the right, but larger positive part will go up this part. So, it will again go down. So, it will be having oscillation like this. After sometime, this will come here. This this is going in and this direction, right. So, this will come here the moment this will be like this.

The moment this goes to the right and starts sliding, a huge positive chunk will go out. So, value will come down rapidly. So, it will fall and then, again for this side may be a negative part will go down further. So, it will go up and then a positive part like this because of this part also some first negative parts go up. So, value goes up and then, positive part goes up, value goes down, something like this. Same on this side drawing may not be good. For correct drawing you see the book, but it will be an oscillation like this. So, it will not be exactly this, but there will be these oscillations. This oscillation is called Gibbs oscillation. Gibbs oscillation comes from Fourier series.

In classical Fourier series, suppose you have got a function, a purely function f of T at t so small t is minus T by 2 to T by 2 to function t, you can break it into fundamental first thermionic, second thermionic and all those things. So, typically a period is capital T. You define the fundamental frequency 2 pi by T as omega naught in classical Fourier series, you break it as c n e to the power j n times, the carrier in a thermionic. This n goes from minus infinite. This classical Fourier series that is n equal to 0, then n equal to 1, n equal to minus 1 n equal to 2 say you got first thermionic second thermionic minus plus minus second like that.

If you make call of them, add all of them, you get back your stuff, but suppose you take c 0, c 1, c minus 1, c 2 c minus 2 dot, dot may be up to some c p, c minus p larger 1 and then do not take any more, what will happen if you add them? The resulting thing will match; this part because these are low frequency part. There is no oscillation. So, I am considering only the low frequency first n equal to 0. So, that is d c n equal to 1 just e to the power j omega naught frequency omega naught and then, 2 omega naught 3 omega naught like this. So, they are low frequency part, but as high frequency part is present here, what this appropriate change from high to low immediately. So, high frequency components are located here. So, this part will be largely mismatched by a summation taking only from minus p to p because those high frequency components will not be there. Therefore, here we will have deviation from this. In fact, we see oscillations. If you do not take this additional component, they will be oscillation. This is called Gibbs oscillations.

Here what is the connection DTFT as such suppose this is the thing DTFT original H d e to the power j theta, this is some h d n e to the power j theta n, but H d e to the power j theta is a function of theta, but we know it is a periodic function. DTFT is periodic for a period 2 pi or over a period 2 pi. So, it was a function of t and now it is a function of theta. So, what is the fundamental frequency that time it was 2 p by T and now it is 2 pi by 2 pi. This period is 2 pi and that time it was t. It was 2 pi 2 pi by T, here 2 pi by 2 pi here. So, omega 0 in fundamental is 1 and n times you can write this way. Also, h d you replace n by minus n it is j n theta. So, n you replace by minus n. So, it should be some from plus infinity to minus infinity, but it does not matter where you start from plus infinity to minus infinity or minus infinity to plus infinity. Number of points you cover in the discrete sum is same whether you go from left to right or right to left. So, we can write in this. So, this is actually the Fourier series expansion of this. So, discrete type Fourier transform DTFT expansion is nothing, but a classical Fourier series expansion of this periodic function of theta, where you have Fourier series coefficient for n times, the fundamental e to the power j n into 1 1 was the fundamental.

So, n times fundamental, there is any thermionic theta is the function this axis is theta like this axis any thermionic. It is corresponding coefficient is h d minus n you can give me h d prime n whatever and it goes from minus infinity. Now, in this filter design from this desired thing h d 0, h d 1, h d 2 and all that you know h d minus what I am selecting only a few h d 0 and then, h d minus 1 h d minus 2 like that. So, h d minus 1 means minus 2 like that. So, few of them and I am throwing others as a result. When you add them on the right hand side as long as the low frequency part is concerned verses theta, the oscillation may variation verses theta the low frequency part is concerned. This will be matched, but there is an abrupt transition from high to low that reflects the products of high frequency components. There will have large oscillation. There is mismatch and that is what you see here this gives oscillation.

This is to minimise this Gibbs oscillation. What they do then instead of taking a rectangular window, they take a window which is not so abrupt. This rectangular window was an abrupt either 0 or 1 1 1 1 1 and then, 0. They make it smooth like you know may be in a triangular, it is triangularly modular rate may be this. So, it goes like this, like this and this one it could be one half of this. So, here you multiply h d n samples h d 0 by this h d 1 by this h d 2 by this like this. Here h d 0 by this h d 1 by this h d 2 by this like this and you get better performance in terms of this. So, that is all for the windows. This various windows have been revised. If you see open and others you will see various window functions are giving butler window, hamming window, hanging window, Kaiser Window and so on so forth. They are very famous windows. So, this is for FIR filter design technique.

Thank you very much.