Discrete Time Signal Processing
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Lecture – 35
Digital Filter Design from Analog Prototype Filters by s-z Transformations

In the last class, we covered IIR filter design; low pass IIR filter design by impulse in variance, we discussed other in detail.

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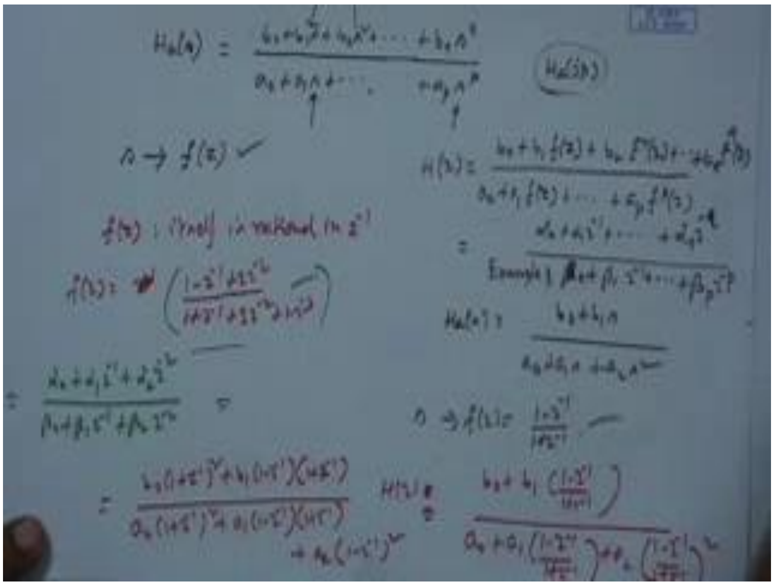

Now, you consider some other approaches also, which is this that we will have some function H a s. Suppose, H a s; suppose, an analog filter has been designed H a s could be Butterworth, could be anything. It is in general rational form (Refer Time: 00:47) b 0 plus b 1 s plus b 2 s square dot, dot, dot b q s to the power q divide by a 0 plus a 1 s plus dot, dot, dot a p s to the power p. Suppose, I, an analog filter has been designed, H a s which is stable, causal, rational. All these (Refer Time: 01:10) frequency response will be what? H a j omega; that is, instead of s, you replace s by just by j omega; should be explained we have the j omega.

Suppose this is given, then what we do? We do the mapping, wherever there is s, you design a function of z, that is, your design. That is where the entire (Refer Time: 01:31) lies, you design a function f z so that wherever there is s, you replace that by f z. See, you will get a function H z now, of this form; b 0 plus b 1, f z plus b 2; f squared z plus dot, dot, plus b q, f to the power q z divided by a 0 plus a 1; f z plus a p; f to the power p z, this is what you will get; so you get 1 form. Now, so basically you get a function of z, I want this also to be a rational function in z; this H z should be a rational function, in terms of j inverse interact, rational function - it should be stable and causal firstly and when it comes to frequency response, some more property but before that let us see one thing. That suppose f z itself is rational in z inverse, like f z could be just z inverse, z inverse means z inverse by 1.

So, numerator polynomial z inverse, denominator 1; it could be 1 by z inverse; it could be z; z means inverse of z inverse, it could be 1 minus z inverse, that means 1 minus z inverse is the numerator; denominator 1; this by 1; 1 minus z inverse by 1 plus z inverse or 1 minus z inverse plus 2 z inverse 2, this is rational. You can have more power in the denominator; 3 z inverse 2; 4 z inverse 3, like that, these are all the rational functions.

Suppose f z is something like this, as an example; so, if I put this kind of form, where f z itself is a rational function of z inverse that is in the ratio of 2 polynomials. 1 numerator polynomial in z inverse in power of z inverse 1; the denominator polynomial in terms of powers of z inverse; if you put; wherever there is s, if you put this f z here, here, here, here, everywhere, so either this will be here or square of this, cube of this will be there. When you simplify, definitely you will get a numerator polynomial in terms of z inverse and power of z inverse and denominator polynomial in terms of z inversion is powers.

As an example; suppose example, suppose your H a s was just b 0 plus b 1 is, and it is a 0 plus a 1 is plus a 2 s square and suppose s you are replacing by f z, which is just very simply 1 minus z inverse plus 1 by z inverse, suppose as an example. So, if I put that back here, what will I get? H z will be b 0 plus b 1 into f z. That is 1 minus z inverse by 1 plus z inverse; divide. This function, I am designing but I am saying that wherever there is s in the analog transform function, I replace that with this function. So, I will get one digital transfer function; that is what I am getting; it is not that s equal to this, it is just mapping, wherever there is s, I will replace it by a function of my choice.

So, by this replacement of s by something, some f z, I get another transfer function of a digital system, DSP system and that will be like b 0 plus b 1, instead of this, we have got this a 0 plus a 1, instead of this, we have got this; then, a 2 squared. Now, if you simplify this, you understand denominator and numerator both will be multiplied by 1 plus z inverse whole square and so we will simplify; I am not doing it. You understand there will be numerator polynomial by denominator polynomial of z inverse. If you want to do the silly algebra, it will be like you know I mean because here 1 plus z inverse whole square. So numerator, denominator both will be multiplied by 1 plus z inverse whole square, so b 0 into this b 1 into this by this, 1 plus z inverse cancels.

So, these divided by and now you break it, you get some coefficients; some, if you simplify it, finally you will get, may be some; see, you will go up to what? z inverse square, z inverse 2; here also z inverse 2, see you will get something like, you know may be alpha 0 plus alpha 1; z inverse plus alpha 2 z inverse 2 divided by may be some beta 0 plus beta 1 z inverse plus beta 2 z inverse 2, this is what we will get.

So, one thing is that if you are replacing s by a function of z, and if you take that z to be, f z to be a function, f z itself to be a rational function in z inverse, like this kind of thing or this kind of thing, you know. We are going to substitute that in place of s everywhere and then simplify, you eventually get one rational function in z inverse. That is what numerator polynomial in powers of z inverse divided by denominator polynomial in powers of z inverse; because I want that digital transfer function to be rational, so I get a rational digital transfer function in this way. So, if an analog filter is designed, analog filter is designed and from that digital transfer function is to be derived, we replace s by; wherever there is s, you replace by some function of z which is rational function in z inverse. Then, at least one thing is assured, we get a rational function in terms of H z rational H z.

So, we are actually considering these that there will be a; to start with one analog filter design called prototype filter, which is rational, which is causal, which is stable and from that, we want to derive in this procedure, one digital transfer function by replacing s, wherever s occurs by one function f of z of our choice. But, I want H z that is derived to be rational in z inverse. See, here by this example you see that if f z is a rational function in z inverse, that is a ratio of two polynomials in z inverse, powers of z inverse, then when you substitute s by this and simplify you get a rational H z of this kind of form, so henceforth we will choose f z to be rational; that is one thing here.

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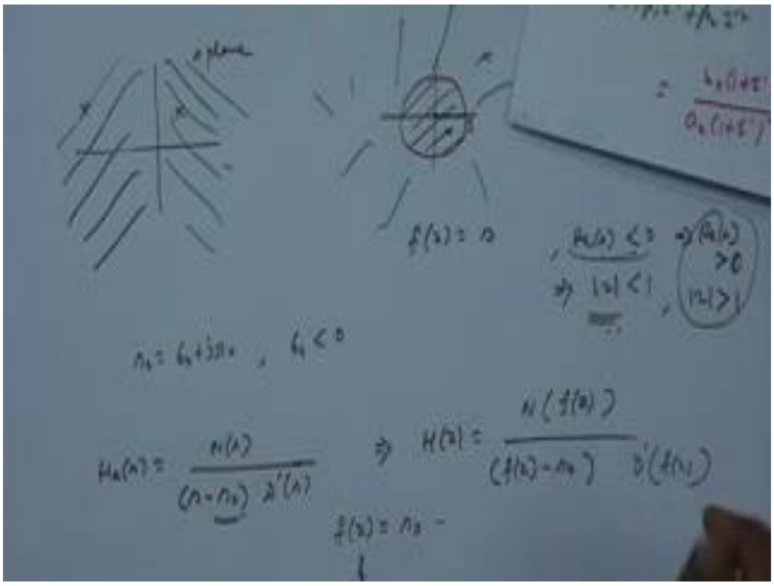

Next, we can also write H a s as numerator polynomial N s divided by denominator polynomial. Denominator polynomial is this; this you can factorize into first order factors, one factorize separate out. One factor may be s minus s knot and remaining ones because we will study only one pole at a time; s equals to s 0 is a pole. Remaining all factors are put into one in or combined to form one another polynomial, may be D prime s is decrease one less; because one factor has gone out.

So, here in this form if I write H z; it will be N, In place of s, everywhere it would be f z. So, it is in the polynomial, this polynomial in s, powers of s, wherever there is s, replace it by f z, so you get whatever you got there earlier. In the case of denominator also D prime, what about s? You replace s, a f z; f z is rational in z, z inverse. So, this a N of f z is rational D prime, f z will be rational and everything will be rational. But, here f z, this is in z minus s knot, so earlier in terms of s, there was a pole at s knot. So, in terms of f z for what value of z, f z will be s knot? There will be a pole in terms of f z. Then, if you solve this equation and find out z that will be a pole in terms of z because for that z, if you take that z, put that in f z, that f z is equal to s knot; because that is how that z is obtained, you can call the solution to be z knot; so f of z knot is s knot.

So, H of z knot if I want to find out, I have f of z knot but there is s knot. But that is how z knot is obtained and s knot minus s knot 0; which means you solve this equation and find the solution and this solution will be your new pole. Now I want my original pole, my original system was causal and stable. Therefore, pole is s 0, if s 0 had sigma 0 plus j omega 0, then I know sigma 0 is negative; that is the pole should be on the left hand side of the j omega axis, so sigma 0 should be negative.

So, if it is given that sigma 0 is negative, then for this H z to be causal and stable, z 0, the new pole should be such that mod z 0 is less than 1; that is, z 0 lies within unit circle. Where that means, if s 0 is such its real part is negative, then lies in the left half plane in the s plane, then that is H a s is causal and stable then, f z equal to s 0. If I solve when I get a pole, new pole z 0, in terms of z, corresponding mod z 0 also should be less than 1; that means, the pole in terms of z should be in unit, within unit circle. See, all the poles are within unit circle, we know that the system will be causal and stable.

So, the f z should be such, so there is another condition on the f z. One condition was that f z should be rational like this or like this. Now, another condition that if you solve f z equal to some s, and if real part of the s is less than 0, then this should lead to mod z less than 1; which means, if real part of s greater than 1, then the system is not stable, then here also it should be greater than 1. That is, that means if the in pole; original analog system is causal and stable, so that pole if instead of s knot, I keep more (Refer Time: 13:03).

If the pole is within in the left of plane, then the corresponding pole in the z domain is within unit circle because mod is less than 1. But, at the same time if the system is not stable; analog system, that is real part of that s is greater than 0, sorry greater than 0 then it should not also lead to the same mod z less than 1. That is, my H z should be stable and causal, if and only if analog system is a stable and causal; not when analog system is not, unstable and not causal but still I am getting a range H z with this causal; no causal and stable - if it is causal and stable in s domain, this analog domain should be causal and stable in digital domain and if it is, but not otherwise. That is, if real part of this greater than 0, analog system is, I mean it does not satisfy causal and stability both and therefore, in that case digital system also should not satisfy causal and stability both, the mod z should be greater than 1.

So, this is one condition; that means all left half plane poles should map within, I mean should map two poles within unit circle; which means, if this s plane and if it is z plane unit circle; this unit circle and this much is 1. Any pole here should map to some pole here, not outside; which means, entire zone should map inside and entire zone outside, any pole here should map to pole outside from this part; which means, right up plane should map to outside unit circle is a z plane. This is another property of f z; you should choose f z carefully, so that this is also satisfied and when you solve f z equals to s, if real part of s is less than 0, corresponding z should be such its mod is less than 1. If real part of s is greater than 0, corresponding solution z should be such mod is greater than 1; these 2 conditions.

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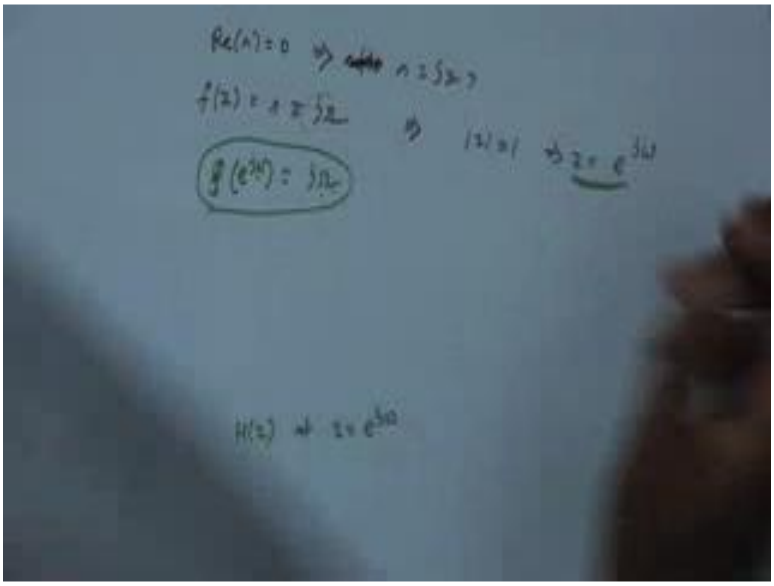

How about real part of s equal to 0? This part of z, we have not shown. This will now consider suppose real part of s is 0; that means s is just j omega; s equal to j omega; which means the analog system will be based on the, I mean, I will have the frequency response; that is, s is replaced by j omega. In that case, when you solve f z equal to s, which is equivalent to just j omega, no real part; real part is 0, only this. Then, I say that mod z should be; and, why I will explain? Mod z should be 1, that is z should be in magnitude 1 and some angle may be omega, why because; suppose you take this substitution, you get an H z final in terms of, you know I mean alpha 0 plus alpha 1 z inverse and this way dot, dot, dot, may be alpha q z inverse q.

I am taking the same order q can be larger also; does not matter, it can be q prime divided by a 0; not a 0, beta 0 beta 1, z inverse plus dot, dot, dot plus beta p z inverse p. I am taking same order p and q, but depend on f z. If f z has higher order terms, then numerator degrees will go up, remember that. But, suppose I have substituted s by that; simplified, I got this. I want to find out is d t f t at; so I want to find out H z at z equals to say, this is e to the power j omega; that means, here I have to simply replace z by e to the power j omega; e to the power minus j omega, e to the power minus j omega q and dot, dot, dot, that is the d t f t.

But, I can also use this expression and then it is b 0 plus b 1 f; e to the power j omega plus b 2 f square e to the power j omega plus b q f to the power q; e to the power j omega. But, what is f e to the power j omega? Here, I am saying that if f z is such that when you solve f z equal to j capital omega, just pure imaginary part, you would get z equal e to the power j omega. That is, magnitude is 1; that means, f e to the power j omega will be just j omega, j capital omega, isn't it. So that means, if I want to find out the d t f t, the transfer function at e to the power j omega, that is alpha 0 plus alpha 1 e to the power minus j omega plus dot, dot, plus alpha q e to the power minus j omega q by dot, dot, dot.

This, you can also write equivalent as b 0 plus b 1 f of e to the power j omega plus b 2 f square e to the power j omega, dot, dot, dot. But, f of e to the power j omega is j capital omega; that means b 0 plus b 1 j capital omega plus b 2 j capital omega square, dot, dot, dot, which is analog transfer function at j omega, at omega; that means, digital. If you solve this, you get small omega in terms of capital omega or you get capital omega in terms of small omega. That means, if you want to find out the transfer function, the frequency response at digital frequency omega.

That is, you replace z by e to the power j omega, what will be the value of this? You can easily find out by what? Solve for this e to the power j omega f of e to the power j omega; find out, it will be j capital omega only because of these assumptions and this transfer function then transfer to be nothing but H s, what is replaced by j capital omega because f of e to the power j omega is j capital omega. So, it will be b 0 plus b 1 j capital omega, that is b 1 j capital omega plus b 2 j capital omega whole square, that is b 2 j capital omega whole square, dot, dot, dot, as though it is H a s, where s is replaced by j omega.

That means, I can find out the digital frequency response at particular frequency small omega. How? I just put; in this equation put f e to the power j omega, I get a purely complex, purely imaginary number j capital omega and then the digital frequency response will be analog frequency response at that capital omega. So, I can map one response to other, so digital response, digital filtered response at the digital frequency small omega will be same as analog frequency response at s equal to j capital omega; analog frequency capital omega, where capital omega, small omega related by these; where f is a function of your choice, these are the 3 main important properties.

So, by the last property you can transform one figure, one frequency response figure into another; that is, I want to find out suppose this H a s omega plot is given to me, I have to find out digital filter frequency response plot. So, I started with any small omega, I have to find out what is the digital free, what is H capital e to the power j omega. So, the small omega, I want to find out the frequency response of a digital frequency response of small omega.

What I have to do? Simply, I put in this equation f e to the power j omega, find out, find the left hand side out. This will be equal to; by this assumption, it will be equal to just pure imaginary number j into some angle, capital omega j into some frequency and by this theory, then this digital filter frequency response at small omega will be same as analog filter frequency response at that capital omega, that is b 0 plus b 1, j capital omega plus b 2, within bracket j capital omega whole square and dot, dot, dot. That is because the moment you put small omega here, I instead of taking these form h 0, I can write in this form also, they are equivalent, here b 1 f e to the power j omega; f e to the power j Omega is j capital omega; b 2, j capital omega whole square and dot, dot, dot. As the b 0 plus b 1 into j capital omega plus b 2 into j capital omega whole square dot, dot, dot, that is this analog frequency response at j capital omega.

So therefore, small f should be such that find real part of s is 0 and s is just j capital omega. Then, the solution of this equation when z equal to s should give raise to a solution, whose magnitude is 1 and z should be e to the power j omega, which gives f of e to the power j omega, so it will be a pure imaginary number or if you are solving f z equal to pure imaginary number, corresponding z should be having mod z 1, it will be just e to the power j and angle. So, this three properties; by this, we can transform one, any analog filter into digital filter.

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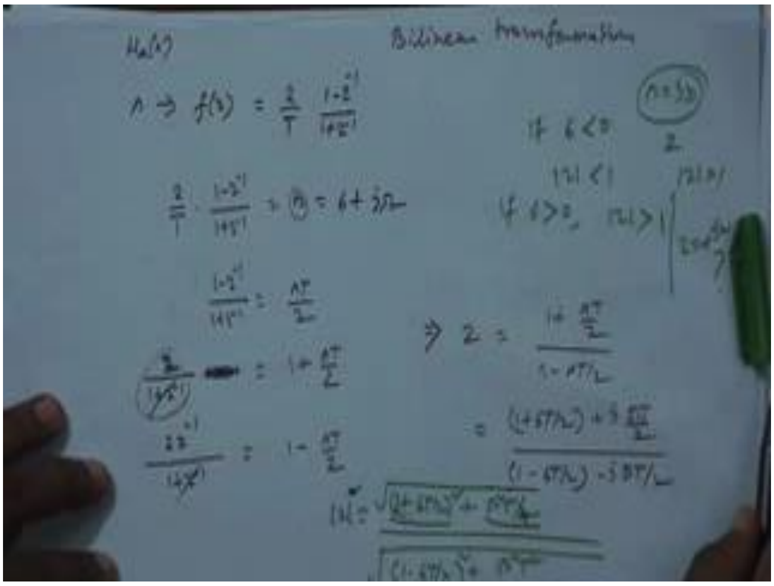

One example is the famous example called bilinear transformation; I do not think I have time to cover it. But, I can just give you a formula; s to be replaced by f z is an analog rational function, rational transfer function which is rational, causal and stable. In this, I know they have replaced s by some f z, which should be rational in z inverse. To satisfy the other two properties, that is if s real part is negative then the corresponding solution f z equal to s, it should give raise to a solution z, whose mod is less than 1 so that pole on the left half plane, maps to pole within the unit circle in the digital plane, where digital and so and so and one more property which we discussed.

This f z in this example - example is bi linear; is 2 by T; T is your choice because you can view it as a sampling period 2 by T, you can take it 1 also - 1 minus z inverse plus 1 by z inverse; take this form. Firstly, you satisfy this condition one that is rational; because there is the numerator polynomial in z inverse, denominator polynomial z inverse, so first condition is satisfied. Then 2 by T 1 minus z inverse by 1 plus z inverse equal to s, s as sigma plus j capital omega; we will see if sigma is negative, corresponding z should be such from this equation plus, that is there is a pole which is on the left of plane in the s plane, where sigma is negative. Then if you solve this equation, it will be a inertia pole in the z plane, whose mod is less than 1, that is it will be within unit circle, we will see this.

How to obtain this? So, 1 minus z inverse by 1 plus z inverse is s; I am writing in the form of s; now, s T by 2; if you add 1, add 1 to it. You get 1 plus z inverse plus 1 minus z inverse; 2 by 1 plus z inverse is 1 plus s T by 2 by; if you add 1 to the left hand side, 1 to the right hand side, what happens? 1 plus z inverse, plus 1 minus z inverse, the z inverse cancels; 2 by 1 plus z inverse, that is, 1 plus s T by 2 and if you subtract it from 1, 1 minus this, then what happens? 1 plus z inverse minus 1 plus z inverse, it will be 2 z inverse by 1 plus z inverse and subtracting right hand side from 1 means, 1 minus s T by 2. If you divide these by these (Refer Time: 25:50) these by these; which means z; this will cancel and 2 by 2 z inverse means z; 2, 2 cancels; 1 by z inverse; that is z. That will be 1 plus s T by 2 by 1 minus s T by 2 and now 1 plus sigma T by 2 plus j omega T by 2 divided by 1 minus.

So, what is mod z? Mod z will be mod z, I mean, mod z square otherwise you are putting a square root a plus (Refer Time: 26:41). It is mod is square root of a square plus b square; I do not know the square root, so putting a square, mod z will be what? Mod of numerator divided by mod of denominator, we have already said last time. So, I am taking this term mod square, so square root now will not be required. Or, if you want, you can put a square root; does not matter, this is real part s square and this is (Refer Time: 27:07) b square.

Now suppose sigma is negative, if sigma is negative, 1; it will be 1 minus some quantity and it is 1 plus the same quantity. This is same as this, so this does not; this is common but if it is 1 minus a quantity and 1 plus the same quantity, then this will be less than this; which means then, so if sigma is less than 0, obviously mod z is less than 1 because numerator is less, denominator is more, higher; because it is actually minus likewise sigma is minus 2.

So, it will be 1 minus; so sigma is minus 1; 1 minus T by 2, it is 1 plus T by 2. This part is common in both; numerator and denominator, it does not change anything. So, numerator becomes less than denominator; that will be 1 z less than 1. On the other hand, if sigma greater than 0, mod z greater than 1, so this satisfies the condition number 2. That is, any s on the left of plane; that means any s following sigma less than 0, corresponding z will be such if mod z is less than 1, that is it will be within unit circle and if it is on the right half plane, in the s plane, the corresponding z will be outside the unit circle; which means entire left half of plane. Thus, the entire s plane to the left of j omega axis will be mapped within unit circle in the z plane and entire area to the right of the j omega axis in the s plane will be mapped to the outside of the unit circle in the z plane.

The third condition when s is just equal to j omega, what happens to z? Does it have magnitude equal to 1 and therefore z equal to e to the power j omega, this we will verify in the next class.

Thank you very much.