**Discrete Time Signal Processing**
**Prof. Mrityunjoy Chakraborty**
**Department of Electronics and Electrical Communication Engineering**
**Indian Institute of Technology, Kharagpur**

**Lecture – 34**
**IIR Filter Design by Impulse Invariance Method**

Now you see we have got two n; number of poles but two n; number of poles are what? Of this function, roots of this function or poles of these overall function.

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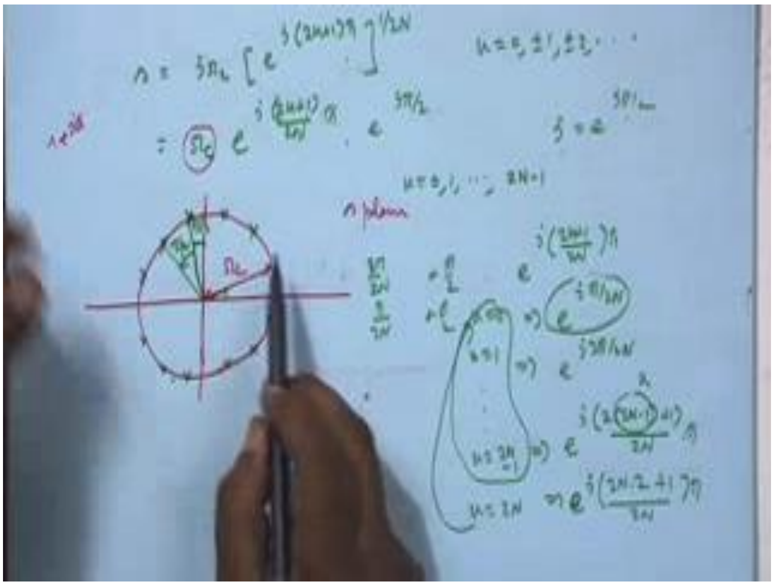

What are the poles? This poles are nothing but the poles of this product H a s into H a minus s, H s is of this form in general form, H a minus is just you know s replaced by minus s that kind of form. There is H a s you can write as N s by D s or N s is this, this is this, and then H a minus s is n minus s divide by D minus s. Now we have seen that if there is a pole, if H a s has a pole at alpha. So, it there will be in the denominator a factor like this s minus alpha, then H a minus s also we have a pole at minus alpha. So, there will be you know term like minus s, minus alpha minus can be taken out.

So that means for this product; if the denominator if there is 1 pole at alpha, there is another polar minus alpha, where these two n poles will be such that if there is 1 pole or so left hand side if you call it say, suppose sum s if you call it system as s 0 then there will be another pole at minus s 0, if s 0 is a, a 0 plus j b 0, a 0 here is say negative they be b 0 is for this much is b 0, j b 0 and a 0 is negative. Minus s 0 means minus a 0 minus j b

0, so minus 0 it will be on this side and this side alright. So, if there is a pole at s 0 there is pole at minus that is why out of the two n poles half this n number of poles on this side and for every poles at any for every pole say at s equal to s 0 or whatever there will be a corresponding pole of the right side and minus of that s 0 or if you call it s 1 minus of this 1 like that.

So, now I know how the poles will be, all the poles are separated by pi this angle pi by n because it from k equal to 1 you got where k equal to 2, increment will be pi by n. So, 2 pi and two n angle two n poles, so 2 pi divide by two n; so pi by n. So, all the poles of the separated uniformly by pi by n angle and there is one more property that I have just discussed that is if there is a pole what any value of case says 0, that there will be another pole at minus s 0 which means c p s 0 is from the left hand side, there is real part negative suppose imaginary part is positive, minus s 0 will be on the right hand side real part positive and imaginary part may be negative and vice versa.

Then what we do is this we take all the left appeared poles may be alpha 0 alpha 1 say there is suppose alpha 0 suppose is alpha 1, alpha 2, alpha 3, alpha 4, alpha 5. So, if it is alpha 1 this is minus alpha 1, if it is alpha 2, it is minus alpha 2, like that. So, we do not consider them you consider the alpha 0 alpha 1, dot, dot, dot say alpha 5, we consider this factors s minus alpha 0 s minus alpha 1, dot, dot, dot up to s minus alpha 5. These pole, these I give to H a s, there is H a s is these by 1, there is I assign this poles on the left hand side of the this plain two H s and then obviously, H minus s, H minus s will be what? It will be a 1 by minus s minus alpha 0 minus s, minus alpha 1 minus s, minus alpha, alpha pi so and so forth and two H s into H minus s (Refer Time: 04:42) all the poles on alpha 0 minus alpha 0 alpha 1 minus alpha 1.

So, out of two n poles is my privilege, my choice which capital n number of poles to assign two H a s and which to minus s, I will assign so that all this left of pre plain left of plain poles come under H a s and obviously, that negative persons which are on the right hand side, they go to H minus s and H a s into H minus s in that case we will be getting to this function, because I found out the roots by equating this to 0 and then two n roots s minus g s minus s minus of alpha 0 minus alpha and dot, dot, dot, dot up to s minus alpha two n there is a product out of which n factors I give to H s taking case so that the corresponding roots or on the side left hand side and there by automatically the roots on the right hand side go to H a minus s, but product remains this which is what I wanted from the frequency response, but what by gain by choosing H a s like this, here all the poles or on the left of plain there is on the left hand side of j omega axis which means the system will be (Refer Time: 05:51) and stable that is what is assumed.

So, by this process you generate the Butterworth function, if it is I mean n if it is nth of if you have got capital N number of I was a two n number of poles capital n goes to this side to H s, at the H s that you get will be 1 by s minus alpha 0 s minus alpha or dot, dot up to s minus alpha capital N. These products that will be nth of the Butterworth function nth or the denotative will give nth Butterworth polynomial and the filter transfer function will be nth of the Butterworth filter. This gets one approximation there is other approximation possible changes you know I mean and there are many others (Refer Time: 06:35) and all that they all maintained stability causality, rationality and they try to approximate these frequency response again by some wood craft alright.

Now, the question is suppose my task is to (Refer Time: 06:46) and IIF filter, so fine IIF low pass filter I proceed with the analogous way that suppose.

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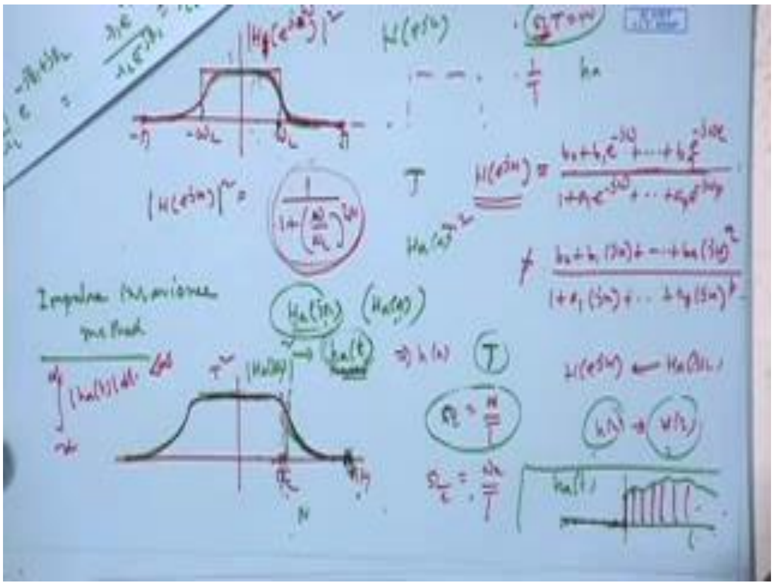

I can since I cannot match both the phase, I cannot match both the phase (Refer Time: 07:00) and frequency phase I concentrate on the magnitude part only is squared. So, this is periodical get from minus pi to pi because the basic function is periodic, so it should be ideally this. So, omega c now the digital frequency pi minus pi and then you can you has got those periodic parts. This is what I want to design; this is a ideal low pass filter but ideal low pass filter means suppose I want to design a Butterworth filter. So, therefore, this will be approximated by a function like this, what is this function? This function will be as before 1 plus it is the height is 1, 1 plus omega in this case just omega by omega c whole to the power say two n, if it is n two nth order function, sorry this is 1 omega, I am very sorry I these not H (Refer Time: 08:06) we have bad habit I was scaring with going on with H a, so H is coming here it is purely digital; so there is no H a s there is no capital omega in this diagram.

So, this should be my Butterworth filter, ideally this would be a digital Butterworth filter and point is what is H e to the power j omega this is of the form. So, here lies the big different which analog filter, in the analog filters, a this was polynomial clearly in terms of capital omega in the numerator denominator and therefore, you could approximate this varying function by 1 by a polynomial in capital omega and then you could write this left hand side as H a s H a minus s with basically to j omega and then from j omega equal (Refer Time: 09:19) is j omega by j omega c and replace j omega by s and go on, the way we did.

Because we know H a s is of this form, H s omega is also of this form; it was a polynomial in s, it is a polynomial in omega and I approximate it by this omega by omega c here. So, it is a polynomial in omega, but in this case I want polynomial it is small omega like because it is Butterworth, but my numerator is not in terms of powers of small omega, it is in terms of powers of e to the power j omega that is - this is not equal to b 1 j small omega plus b q, j small omega whole to the power q this is what I would have laughed to have you know - j omega plus, dot, dot, dot, but this is not the case because we know from our discussion so far that discrete time Fourier transform, there is a transform function of a additional system is of this form. We might have is a polynomial, but not in terms of small omega; it is a polynomial in terms of e to the power minus j omega. These also in interval denominator also is a polynomial in terms of e to the power minus j omega, but my function by which I am approximate the ideal filter characteristics it is in terms of omega, omega 2 the power 2 n.

So, I cannot directly use this form to obtain this, this is a big problem, then the question is how to design these filter. So, one technique this is a basic technique called impulse in variance method here first assume that this small omega c is very quiet far from pi at the filter order, this approximation is valid I am trying to bring this approximation, but the order is very large so that this becomes 0 or almost 0 very quickly and when it goes to pi it is almost 0; it is done. Theoretically if there may be very small value that is done, first assume this then what you will assume, next we assume that there is one analog filter with transfer function this, in terms of Laplace transform H a s and this has got this time function. We will assume that there is one such H a j omega which is band limited which is also Butterworth band limited, there is going out to 0 like this and we sample it; sample this h a t above micro state either at micro state or above micro state, so you choose a sampling period T where we choose a sampling period T.

So, in that case from this will construct if it is 1, we construct 1 function Butterworth function, but this would be capital T times this function right. You remember when you sample 1 analog signal and get it discrete time signal, even if you sample that had the micro state what happens; discrete time Fourier transform is replica of the analog Fourier transform, but there are some changes from capital omega frequency we go to small omega frequency by this relation - omega into T small omega and there is a scaling by 1 by capital T that is from analog to digital when you go we have a replica, but there is a scaling by 1 by T. So, here I am go trying to go in the opposite direction and I am trying to construct 1 H j omega, so that mod H is omega square.

Looks like this Butterworth function is a Butterworth function, so here originally it was 1 these basic DTFT suppose plus 1 and square it, it is 1. So, now, it will multiply D T square because DTFT has been a scaled version of this by 1 by t, so if we take mod and square a 1 by T square of that. So, from here if want to go back there it will be T square times, this 1 will be T square and this will be capital omega axis, pi always corresponds to half sampling frequency. So, this will be half sampling frequency pi by T and this will be capital omega c, all frequency omega will be nothing but small omega by T, there is a relations small omega by T radiant per second. So, that is why you map at any small omega whatever the value that will come to corresponding capital omega where capital omega is omega by T, T is your choice. So, there by 1 plot is mapped to another plot and capital omega c will have which is small omega c by T, so you will have a Butterworth function.

This Butterworth function since small omega c is much less than pi, capital omega c will be much is than pi by T because this is omega by T, this is pi by T and therefore, it will go downs up and n is large. So, we will go downs up, so when it reaches the half sampling frequency is almost 0 and therefore, it means that if I really use this T to sample sets the analog this is for a band limited the analog function is now band limited it is almost 0 at half sampling frequency there will be no alias in it. So, if h a t the corresponding capital H a j omega is sampled which is sampling period capital T to generate h n, that h n will have a DTFT which we will look like the original DTFT a analog Fourier transform, there will be no alias in component. Then you take the z transform of this H n, and that will be give raise to your filter whose frequency response will be similar to the analog frequency response. So, therefore, magnitude response squared will be same as the original magnitude response square.

This is a (Refer Time: 16:00) now let me repeat again, we have seen the difficulty in the IIF filter in the same as we did in the analog case because finite that we approximate this ideal filter, I have this kind of function Butterworth function, but the denominator is the polynomial omega, then only if I put positive value omega then only I can say if omega is greater than omega c, it will be greater than 1 and this whole thing is raise to the power 2 n you must list the it will have then 1, so 1 we ignore; 1 by this quantity this much higher than 1. So, it will be most 0, so it will be fall down the moment x cross omega c this function will go down, so start for ignore rapidly at become 0.

On the other hand, if omega is less than omega c this is less than 1, so raise to the power 2 n which means it will be much less than 1. So, you can ignore it as compared to 1 and 1 by 1 is 1, so to the flat. So, all these business I could do because there is a direct polynomial, polynomial directly in terms of small omega, but and that I wanted to have is the magnitude response, but H in j omega is of this form, b 0 plus b 1; e to the power minus j omega plus b 2; e to the power minus 2 omega plus dot, dot b q e to the power minus j q omega and numerator also that is let us say ratio of two polynomials, but polynomial are in terms of powers of e to the power minus j omega not in terms of omega that is the program that is the polynomial.

So, therefore the question arises as to then why how can you design at the IIF digital filter, then we say first we start with this function I will then construct a prototype analog low pass filter this way that, I assume that there may be, there was 1 analog Fourier transform, analog transfer function H a j omega I did corresponding inverse transfer that is time function is h a t, that H a j omega is also is Butterworth type and I assume that the corresponding time function will be sampled by period T.

Now, here if it is Butterworth whether you go for constructing that analog low pass filter, first I will do these things. These will be a frequency transformation or this is will be the frequency transformation at any small omega whatever value we had here that will go to corresponding capital omega by this relation. So, omega c will become capital omegas like this, pi will become pi by T and so on so forth.

So, I construct this function, I construct one giving this frequency response, magnitude response I construct one analog magnitude response in this way. I choose one sampling period T then using this relation capital omega is small omega by T, I transform this figure to this and multiply these by T square and T is anything T can be 1 also. So, then if it is pi, it is pi by T; if it is small omega c, it is omega c by T it is capital omega c and this is Butterworth. So, functionally it will be same because shape of this form; shape of this response will be same as shape of this response.

There is only frequencies scaling, but since small omega c was quite far from pi and the order was large. It was going down to 0 before going to pi, here also capital omega c will be quite less than pi by T and order is large. So, it will go down to 0 before going to pi by T, pi by T is the half sampling frequency this means the signal is band limited and it become 0 before it is half sampling frequency which means sampling at the sampling period T by this sampling of this time corresponding time function by this sampling period T, if I generate a sequence h n that h n will limit frequency response which will be replica of these kind, then which will corresponding replica of this kind without alias in pi by 2 will then become pi capital (Refer Time: 20:11) will become small omega c, height will become 1 like that.

So, then by that process there is by first constructing this analog prototype filter then I will start with these analog start with this (Refer Time: 20:24) filter then T comma any I mean just choose any capital T which is (Refer Time: 20:30) take to be 1. Then for that capital T, using that capital T and using this frequency transformation; you transfer this magnitude response to, and equivalent analog description where if it is small omega we will give raise to capital omega and also multiplied these by T square, if T is 1 this T square is 1.

So, this function will be a prototype magnitude response of an analog low pass filter, with cut off frequency capital omega c, but here this was pi - this is pi by T, this is small omega c, capital omega c. So, it is omega c was quite less than pi and order was large this function became 0 before going to pi same applies here capital omega c is less than pi by T because capital omega c is omega c by T and this pi by T, if omega c is small omega c is less than pi must less than pi omega c by T also must less than pi by T and the order is large, so it will fall down to 0 which means when I reach the half sampling frequency, the signal is already band limited this is already become 0 I mean a frequency response; magnitude response has already become 0 which means if I take this H j omega and take the h a t how to get the h a t omega I know it is now analog Butterworth of nth order.

So, I know H a s; how to construct H a s, from H a s I can get H j omega by inverse Laplace transform, by invert or by inverse analog Fourier transform I can get this time function small h a t and then I sample it at the sampling period T to generate h n; this h n it is discrete time Fourier transform because of no alias in we will replica of these magnitude response if we take then we take the magnitude part of square, it will be replica of this where pi by 2 will become pi, capital omega c becomes small omega c, height will become 1 this is I am going the reverse way, that is it will give raise to the desired function.

So, then what is the root? I repeat again from the analog giving a digital you know this approximation in the DTFT in the magnitude response, you construct from the that is desired magnitude response of the digital filter which is of the Butterworth type in this case as a example. You construct at analog prototype magnitude response, analog low pass filter with magnitude response, replica of that with this transfer frequency transformation. That is choosing a sampling period capital T, you deduct capital omega small omega by T, so by this frequency mapping you just map it here, pi becomes pi by T omega c because capital omega c; it was going down to 0 before reaching pi here also it will go down to 0 height will be T square and if it is of this shape, the shape will be shape. So, this will be Butterworth again, so you have to (Refer Time: 23:18) now and analog Butterworth filter cut off frequency capital omega c and order n.

So, you can design this H a s from this either by in inverse Laplace transform or from H a j omega there is replaces by j omega and by inverse analog Fourier transform, get the analog time function then sample it you will get h n; this h n I know it is DTFT if you take that H e to the power j omega it is magnitude square will be replica of this because of no alias in because this function has become 0 before pi by T and pi by T is half sampling frequency. This h n if you take z transform in z that will give raise to this filter be at filter alright, and it will be causal and stable easily because h a t, h a t is how h a t is obtained? By inverse analog Fourier transform analog in inverse Laplace transform of H a s, but if H a s is stable and causal therefore, h a t is 0 on this side if it is (Refer Time: 24:25) 0 on this side here it can be anything and this inverse sampling, you get h n when h n is 0 on this side and h n is having on 0 values on this side of the origin right.

So, h n also causal and if h a t stable that is if this is finite then if you take samples take the magnitude and add that obviously, will be finite because you are not been taking the intermediate values surely finite, so it will be of after the summable, so it will be causal and stable we can also say it will be resonal.

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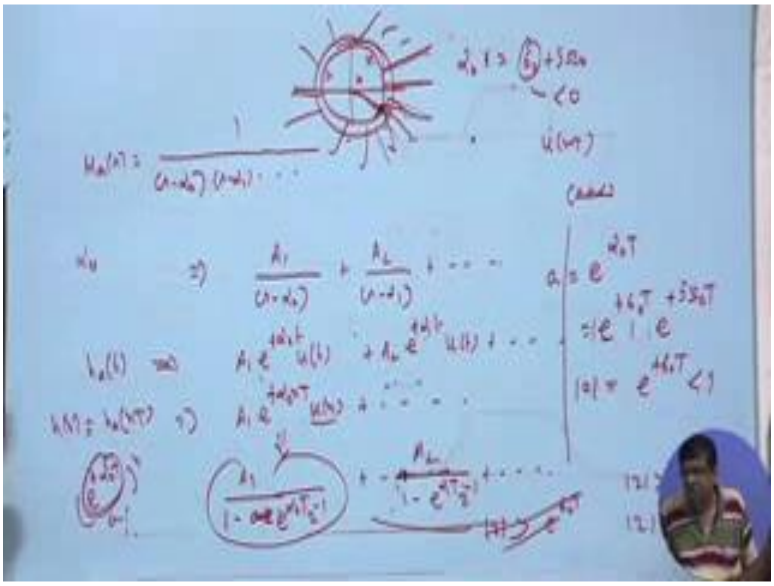

It would be resonal because H a s we will have this kind of form right; s minus alpha 0 s minus alpha 1, dot, dot, dot, dot, so if by suppose 1 it will give raise to some partial effect spectrum like you know s minus if you go for partial effects are these plus a 2 s minus alpha 1 dot, dot, dot, dot if you take the analog; inverse Laplace transform you sample it at capital T. So, it will be a 1 e to the power minus alpha 0 T and since it is causal and stable these plus a 2, a to the power minus alpha 1 T, u d t and dot, dot, dot, dot.

Now, if I sample it and the chosen sampling period T, so h n will be h n T means alpha 0 n T; u n because u n T means same as un, u n T original u analog u raise to the function at n T will be 1 if n is positive, if n is negative it will be 0. So, you get same as u n, so this and dot, dot, dot. If you take z transform of these, so because of u n it will be a 1, so e to the power minus alpha 0 T, whole to the power n right and this will have magnitude less than 1, why? Because alpha 0 is on the sorry, it will plus if it is s plus alpha if it is your minus alpha t, so it is minus alpha 0 e to the power plus alpha 0 T plus alpha 1 T like that.

Now, it will give raise to, now one thing you see alpha 0 is what? Is 1 pole what is this lying on the left hand side of this length. So, it is real part may be sigma 0 it is negative plus j omega 0; this is negative because it is lying on the left hand side of the j omega axis. So, e to the power alpha 0 T means e to the power minus sigma 0 T into e to the power minus j omega 0 T alright, this is the quantity. If you call this quantity a then what is mod a? Mod a will be mod of these into mod of these, where mod of this is 1 e to the power minus j anything, mod of this is 1. So, it will be and this is positive; mod of this is this itself, because it is always these, but sigma 0 sorry this is plus because alpha 0 is sigma 0 plus alpha 0, so plus. So, if you sigma 0 t, but sigma 0 is negative, so e to the power some negative into T that is less than 1 which means this quantity if you call it a, a to the power at a; a is less than 1 in magnitude mod a is less than 1.

So that means, if you take the z transform z transform will exist; it will be this at a is e to the power alpha 0 t. So, alpha 0 T into z inverse plus again similar things and for this r s you will be; for this part r s you will be greater than mod a, and mod a is which is less than 1. So, this is will give raise to a pole at z equal to e to the power alpha 0 T, but e to the power alpha 0 T which you call a, it is magnitude is less than 1. So, it will give raise to a pole within unit circle similarly all the poles will be unit circle and since it is causal all r s will be if there is unit circles rejected there is one pole here, one pole here, one pole here; you did the pole nearest to the origin, nearest to the unit circle and then go out one side, it is causal and we have already seen. So, there is with order (Refer Time: 29:55) because you just outside a circle and (Refer Time: 29:57) including the pole. So, go to the pole which is nearest to the previous circle, all poles are within unit circle because mod of a, there is mod of every pole is less than 1, so you take this and go out.

So, unit circle is contained and it is outside a circle, so it is both causal and stable and resonal because this will be of the form a 1 by 1 minus e to the power alpha 0 2 z inverse. Next one will be 1 minus e to the power alpha 1 T z inverse, dot, dot, dot, if you add all of them, there will be polynomial in the denominator in terms of powers of z inverse, numerator also will be if you add all of them simple I there was said, numerator also will be a polynomial in terms of power of powers of z, so you get a additional form.

Here mod z will be greater than e to the power alpha 0 T that is sorry mod z will be greater than sigma 0 T which is less than 1, also found these it will be e to the power sigma 1 T. These also less than 1; only these are within a unit circle that is why said see go to the near 1 which is farthest from the origin, that is mod is largest upon the pole and draw in the circles through that and then r s will be outside that circle and that will not continue any pole; that continues unit circle it will be stable because all the poles are within the circle. So, if you start from this pole which is farthest from the origin; draw a circle through that and then go outside will continue with the unit circle because all the pole service is unit circle.

So, your r s will continue the unit circle which may is stable and it is going out, it is already causal we have seen, so it will causal and stable and (Refer Time: 31:44) channel because if you have these kind of factors we add them, did you matter by simply algebra will be a polynomial in terms of powers of z inverse, new matters will be a power of polynomial after we add everything, it will be polynomial in terms of powers of z inverse; so we will get a additional form, that is way to design by impulse invariance method. There are another method called by linear transformation sub general method which we will discuss in the next class.

Thank you very much.