Discrete Time Signal Processing
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Lecture – 33
Analog Filter Design Example - Butterworth Low pass Filter

So, today our topic is the example of how to design analog, low pass filters remember our task is to digital IIR filter.

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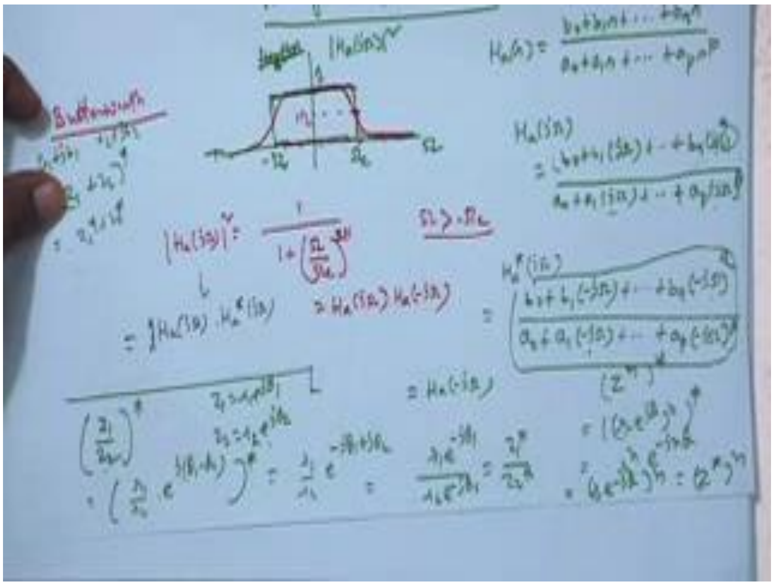

But I told you several times that there are some problems in designing digital IIR filter directly and why, what is the problem is that I will discuss let me not of course, today, but there is a reason we have to go back to an equivalent analog low pass filter. We have to design an equivalent analog low pass filter an appropriate low pass filter, analog low pass filter from that we have to derive how this IIR digital filter IIR low pass filter.

Now, we have discussed the analog signals and systems briefly, but all essential features were discussed. Now, we have we will be considering one rational failure model that is we will design a analog filter which will satisfy certain things that it should be causal and stable, it should be rational that is transfer function will be numerator polynomial divide by denominator polynomial with s because we know that this can be realized by using differential equations though it can constraints be realized using half times you know differential integrator we know so using that.

Then since a low pass filter it is magnitude response should be ideal, maybe it is some say value 1 or it can be value anything, this is omega, this is mod h a forget about a because all are analog. So, I am not I am dropping the index a here or maybe today I require analog and digital both I bring that h that is not bad. So, magnitude out of fourier transform that is transform function fourier transform impulse response till the mod of that either you square it or keep it as it is.

It should be flat which is the first band minus omega c to omega c, omega c is called the cut off sequence. Normally the transform function is given both by magnitude and phase, if it is ideal magnitude should be flat within the first band design first band of the filter and frequency response and phase response should be linear in omega, but you cannot design to satisfy both you know. So, we what do we do we concentrate on satisfying the magnitude base part as much as possible and then we hope that, we know this phase which is that is somewhat close to area not very widely off linear, so we will concentrate on this.

What is our purpose to design and analog rational filter, rational model of this kind; b 0 plus b 1 s plus dot, dot, dot b q s to the power q divide by a 0 plus a 1 s plus dot, dot, dot, dot plus a p s to the power p and it frequency response is which is b 0 plus b 1 j omega plus dot, dot, dot, dot, b q j omega whole to the power cube, divided by just replace this by j omega that is in the explain I am now only j omega axis, vertical axis y axis these h into j omega should be such that is mod square is like this, but you see this is a very ideal function this is completely discontinuous here this is discontinuity here up to this is flat then suddenly it falls immediately to 0.

But this is h into omega you can see easily I mean if you take the mod of these; it will be continuous function of omega. You cannot realize a discontinuous function by this; by taking the mod of this I am squaring it up because that will be continuous function of omega. For this expression only you can see you take the mod of numerator by mod of denominator square up and all that, there will be some function of the omega, omega square, omega cube multiplied by some constants and all those things; there will be a continuous function. So, you cannot realize a discontinuous function by a continuous model like this, but you can go close to that you can approximate it by various approximation functions, one of them is called Butterworth and we will be discussing that.

Here, what happens you try to approximate this function the desired function by this; this should be rational because h into j omega is rational. So, if we take mod of h into j omega there is mod of the numerator by mod of denominator and all that and we square up, it will still be rational b 0, b I mean there will be powers of omega and denominator also powers of omega. So, this will be rational but thereafter the function that we are using that is called Butterworth function.

For this h of j omega will be of particular type so that when you take the mod square, it will be 1 plus that mod square means it will be real because you are taking mod so obviously, it will be omega by omega c. Suppose, whole to the power 2 n for n is an integer, what is the meaning we had if omega is less than omega c then omega by omega c is less than 1 less than 1 whole to the power 2 n. Now, if n is large some quantity whole to the power 2 n will be very small as compared to 1. So, you can neglect that it is value will be 1. So, for all omega in this range less than omega c omega here you will see it is value is close I mean it is value is almost 1.

On the other hand, if omega crosses omega c, if omega crosses omega c then this value if omega is less than omega c we know it is value is less than 1, I repeat it. So, that rises to the power 2 which is much more less than 1, so you can ignore that as compared to 1, 1 by 1 on this side if omega is greater than minus omega c. So, omega by omega minus omega c, omega by omega c of this side omega is negative, so there will be this ratio will be negative, but it is magnitude will be less than 1 if this happens if omega is somewhere here, so omega is greater than minus omega c.

So, omega by omega c this ratio will have minus sign, but minus to the power 2 n minus will go this is n it is a even number 2 n minus something whole to the power 2 n. So, minus has no meaning because minus n raised to r given power 2 n, it will become plus and omega by omega c magnitude is less than 1, so again the same thing will happen it is value will be so less as compared to 1 you can ignore it and so 1 by 1 it will be 1, so for up to this maybe in it value will be 1.

Then for omega greater than omega c on this side omega by omega c will be sum of greater than 1 that raise to the power 2 n will be much higher than 1. So, then 1 you can neglect and it will be 1 by omega by omega c whole to the power 2 n and as omega increases this quantity then starts increasing very much and this is the denominator, so 1 by this is greater than much greater than 1, so this ratio becomes much less than 1.

Now, as omega increases finally, it goes to 0 that is first 1 omega crosses omega c, omega by omega c is say somewhat greater than 1. So, that 2 to the power 2 n if n is large will be much higher than 1; so forget 1, 1 by this and this is greater than 1 and raise to the power 2 a much greater than 1, so 1 by something much greater than 1 it will very close to 0 and as omega goes on increasing.

Finally, we will go down to 0 and at omega equal to omega c it is 1, so 1 plus 1 to the power 2 n 1 the power 2 n is 1. So, 1 plus 1 by 1 plus 1 that is 1 by 2 so that means, it will be a function like this as omega increases it will fall, but always for any n, n is called the order of the whatever filter order, for any n it will cross to this that you get from this expression for any n omega equal to omega c this is 1 1 to the power 2 n is 1 1 plus 1 2 1 by 2, so this is 1 by 2.

But as n increases it becomes sharper it goes like this, so this is rational because capital omega it is power is there 2 2 2 n capital omega to the power 2 n their intermediate powers it is like omega to the power 1, omega to the power 2, omega to the power 3 they are absent; that means, the corresponding coefficients are 0, but at least it is a positive real in omega because 1 is there and then omega to the power 2 n that is there, so it is rational.

Now how to realize it making sure the corresponding h a is from which this h a omega has been obtained by replacing s by j omega that h s is not only rational, it is causal and stable also and then it is frequency response will be this which almost is approximates this very well and some main purpose alright. For this we see certain things; h a this is mod h a j omega that means, h a j omega sorry into h a star j omega, mod z square is z into z star this equal to this what is h s star j omega.

Now, for that some basic properties you may have forgotten very basic properties of complex numbers that is, if you have got 1 number z 1, another by z 2 and z 1 equal to say r 1 e to the power j theta 1 z 2 is r 2 e to the power j theta 2; the overall thing is conjugated; that means, r 1 this will be replaced z 1 and z 2 by these we have r 1 by r 2 this is the magnitude now, e to the power j theta 1 minus theta 2 conjugate which means r 1 by r 2 2 to the power minus j if you have you have minus j theta 1 plus j theta 2 which means you can write it as r 1 e to the power minus j theta 1 divided by r 2 e to the power minus j theta 2, when it goes up it becomes plus it means z 1 star by z 2 star that is star of a ration of 2 complex number conjugate of ratio of 2 complex number is same as first you take the conjugate of the 2 complex number separately and then take the ratio we will get the same thing that is one property.

Another property is if I have the z and z is raised to the power is raised to some integers power say n and then if you take conjugate z is say r into e to the power j theta. So, r e to the power j theta whole to the power n then conjugate, what is it r to the power n and e to the power j n theta after conjugation this will become minus j n theta which is also same as r e to the power minus j theta whole to the power n r to the power n e to the power minus j theta whole to, but this is z star, so z star whole to the power n.

So, if you raise a complex number to some power n and then take conjugate you will get the same thing if you first take conjugate then raise it to the same power. So, here h is j omega ratio of 2 complex quantities on numerator denominator, so conjugate of overall h s star j omega will be ratio of the conjugate of numerator by conjugate of denominator right because of this fact.

If you call the numerator z 1 call the denominator z 2, z 1 by z 2 whole star it is same as z 1 star there is star conjugation of the numerator, divided by conjugation of the conjugate of the denominator. If you have conjugate of the numerator and then all the coefficients are real b 0, b 1, b q that are real we are assuming a 0 a r a q are real. So, b 0, conjugate of a summation of this also you know z 1 plus z 2 star, z 1 plus z 2 star is same as z 1 star plus z 2 star. How to show that; it is very easy whenever you have plus or minus go for the rectangular form that is z 1 is a 1 plus j b 1, z 2 is a 2 plus j b 2 whenever you have product or division go for polar form.

So, j 1 plus j b 1 plus j b 2 if you add a 1 plus a 2 is a real number real path j e to b 1 plus b 2 it is imaginary part then conjugate it will become minus j within bracket b 1 plus b 2 you will get the same thing j 1 minus j b 1 plus j 2 minus j b 2 you will get the same thing it is very easy. So, conjugate of the numerator divide by conjugate of the denominator, conjugate of the numerator means conjugate of every term because of this conjugate of z 1 plus z 2 is same as conjugate of z 1 plus conjugate of z 2 you can exchange it any number of terms in the summation.

So, numerator conjugation means b 0 conjugation, but b 0 is real, so b 0 remains as it is then b 1 into this conjugation of a product is product of the individual conjugated quantity, b 1 is real. So, b 1 remains as it is s, j omega it is conjugate becomes minus j omega dot dot dot b q; j omega whole to the power q I told you any complex number raise to some power say q and then star it will be same as first 2 conjugative and they raise to the same power.

So, first you conjugate it minus j omega whole to the power cube, divide by same way conjugate of a 0 there is nothing a 0 only conjugate of a 1 j omega this is product of conjugate of j 1 into conjugate of j omega, but a 1 is real. So, a1 remains as it is conjugate of j omega is minus j omega and like this aj p into j omega, conjugate of the product means first you conjugate a p, but a p is real. So, as it is j omega whole to the power p that is a p here; j omega whole to the power p then conjugate is same as minus j omega, so that is first you conjugate then p this is the thing.

We will see this is same as if in the originally h j omega instead of j omega if I put minus j omega everywhere, j omega minus j omega, j omega minus j omega, j omega minus j omega, j omega minus. So, if the coefficients are real then h s star j omega is same as h a minus j omega which means this be can be written as not only these as h a j omega into h a not only star I can write it h a minus j omega alright.

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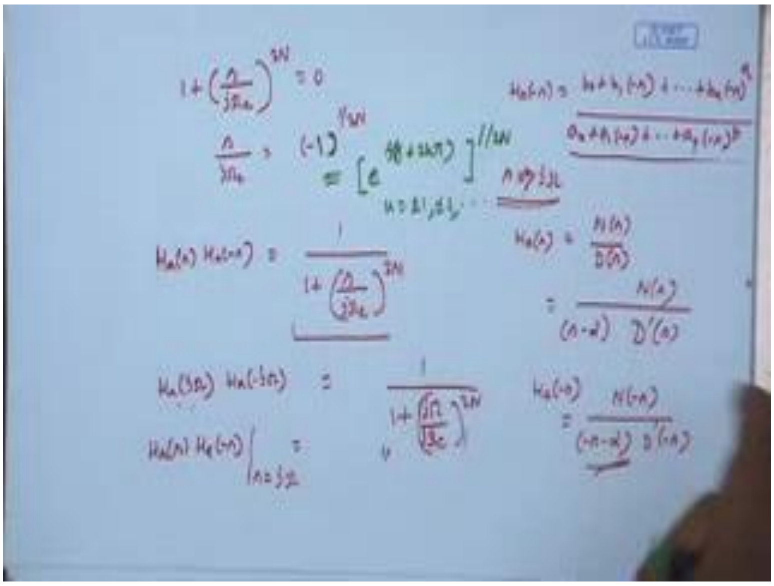

So, I have got is equal to or chosen approximate function or chosen function which approximate that original characteristic nicely this alright, but this I can also write as see h a j omega is nothing, but h a s with is replaced by j omega then h a minus j omega is again h a s say h a minus s; if I put everywhere minus s say let us not complicate. So, let me write h a minus s, so wherever I have s will become minus s b 0 plus b 1 minus s dot, dot, dot b q minus s whole to the power q divide by a 0 plus a 1 minus s right; h a s I had I am just replacing s by minus s this.

Then if I replace s by j omega what I will get; b 0 plus b 1 minus j omega b 0 plus b 1 minus j omega dot, dot, dot, dot b q minus j omega whole to the power q b cube minus j omega whole to the power cube divide by a 0 divide by a 0 plus a 1 minus j omega because this is replaced by j omega a 1 minus j omega a 1 minus j omega dot, dot, dot, dot a p minus j omega whole to the power p a p minus j omega whole to the power p that means, this quantity which is h a minus j omega as well is nothing, but h a minus s then s replaced by j omega. So, h a j omega h a minus j omega this product I can write as h a s and h a minus s in both cases s to be replaced by j omega divide by this and here of course, h s this will be replaces by j omega you get this.

So, that is why you write in the problem minus s, so that for both it is the same thing s equal to j omega. If I write it as h a s then for this it should have been s is equal to minus j omega and for this s is equal to plus j omega, I want s equal to common s equal to plus j omega for both that instead of s I made it minus s. So, this quantity is this for this, but this I can write as j omega by j omega since j and j cancels. So, h a s into h a minus s which is a function of s there if I replace s by j omega I get this.

So, what is this function h a s into h a minus s, so that s replaced by j omega I will get this; that means, instead of j omega I will have I should have add s there only if I replace s by j omega you get this; that is I am going backward what should be h a s into h a minus s. So, that when I put s equal to j omega in this function overall function I get this, so identify where is j omega here is j omega so; that means, I earlier I had s there then there only I replaced s by j omega everything will remain as it is I got this function right it means h a s into h a minus s, I can now say it will be s by j omega c whole to the power 2 n alright because now, if you put s equal to j omega you will get this left hand side, if you put s equal to j omega as here; right hand side also s equal to j omega you will get original Butterworth function.

Now, one thing we will see from here suppose h a s which is a numerator polynomial divided by denominator polynomial. In short I write as n s by d s; n for numerator d for denominator, I will factorize the denominator polynomial suppose I get 1 factor separately s minus something can be alpha and then remaining path I multiply and it is a polynomial again. So, I call this d prime s 1 degree less as compared to d s because this 1 has alpha 1 particular factor has been taken, now s remains as it is. So, s equal to alpha is a pole, so 1 pole I am examining; so I am taking 1 factor out remaining 1s are in d prime s. So here it is a pole at s equal to r equal, how about h a minus s if in this function I replace s by a minus s, so n minus s divide by minus s, minus alpha here and d prime minus s.

So this factor now, earlier it was giving a pole at s equal to alpha; now it will give a pole at s equal to minus alpha. So, which means if h a s has a pole at s equal to say any value, so alpha h a minus s will have pole at minus alpha. Now let us consider this denominator, it is a 2 n a product polynomial it is a 2 order polynomial, what are the roots because we find out the roots and from the roots only we get; if there are there will be 2 n number of roots say alpha 0, alpha 1, alpha 2, alpha 3. So, we will have s minus alpha 0 s minus alpha 1 s minus alpha 2; those will be the factors, so what will be the roots? So, if you find the roots; you take this equate to 0 which means minus 1 whole to the power 1 by 2 a 1 goes to this side minus 1, 1 by 2 a.

Now, minus 1 you can write as e to the power j pi and then e to the power j pi plus 2 pi; e to the power j your j pi plus 2 pi; pi plus 3 pi; pi plus 4 pi like that, pi plus 2 k pi, you will get the same thing; e to the power j pi is minus 1 then e to the power j pi plus 2 pi it is 3 pi this is minus 1 e to the power j pi plus 4 pi; 2 into 2 that is minus 1 dot, dot, dot, dot. So, it similarly it can be minus 1 also pi minus 2 pi that is z pi minus z pi minus 1 then pi minus 4 pi; e to the power j minus 3 pi that is minus 1 dot, dot, dot, dot. So, k can be plus minus 1, plus minus 2 dot, dot, dot, dot, dot; that is minus 1 and then you have got 1 by 2 n.

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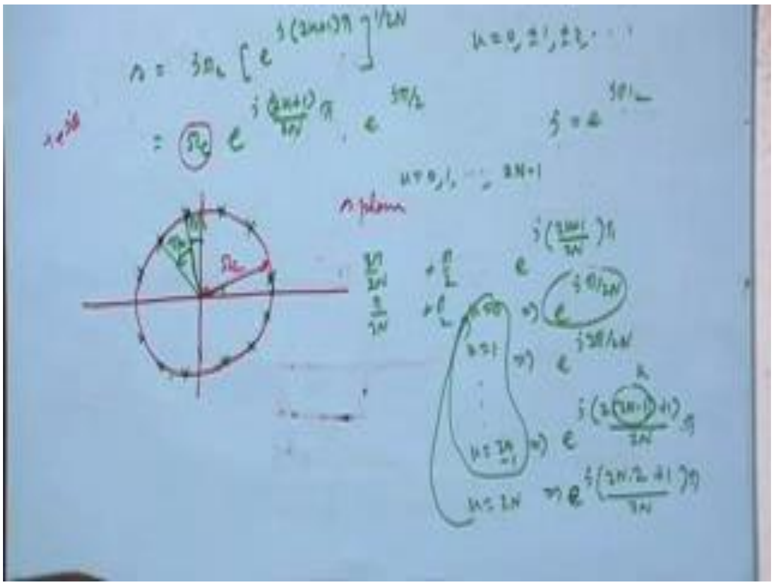

This means there will be various poles for each value of k, there is a root k can be 0 also because if it is 0 still it is minus 1. So, we have got the roots; the roots are s equal to j omega c times; e to the power j, 2 k plus 1 pi; whole to the power 1 by 2 n where k was 0 or plus minus 1, plus minus 2 dot, dot, dot, dot. This j is nothing, but e to the power j pi by 2 this also you can write omega c; e to the power j, 2 k plus 1 by 2 n pi into e to the power j pi by 2, this just an offset this is independent of k, so whatever phase you get every phase for every k gives you pi by 2, this will be pi by 2 additional rotation of k z.

So, here you see k will have 0; if you take k positive case, k will have 0 1 dot, dot, dot, up to what; it will be up to 2 n minus 1 because the moment it is 2 n, you will get back the same thing 2 a I mean if k is 0 whatever k you had for k equal to 0 that will come up what is your k equal to 0; it is your j pi by 2 n, then e to the power j like you know if you consider this factor e to the power j, 2 j plus 1 by 2 n into pi.

So, k equal to 0 how much it is; e to the power j a pi by 2 n, k equal to 1 how much it is; e to the power j 3 pi by 2 n dot, dot, dot, dot, how far it can go to; it can go up to k equal to 2 n minus 1; e to the power j if it is 2 n minus 1, how much will it be 2 into 2 n minus 1, 2 into 2 n minus 1 plus 1 by 2 n into pi this is k. So, 2 into 2 n minus 1 plus 1 by pi why because up to this because the moment you go to next one; 2 n it will be e to the power j, e to the power j; 2 into 2 n, 2 n into 2 plus 1 by 2 n pi and you see 2 into 2 n, 2 n cancels; e to the power j 2 pi that is 1 you are left with what j pi by 2 n which is this.

So, you get back k equal to 0 case, it is nothing but this case; that is why you get this distinct these many distinct roots, how many roots; 2 n roots obviously, 0 1 up to 2 n minus 1 and obviously because your polynomial had a power of 2 n right this polynomial to the power of 2 n, this is a polynomial intermediate powers are missing; s to the power 1, s to the power 2 they are missing. So, s to the power 2 n and this j omega c will give raise to a coefficient only, so you set 2 a kind of a polynomial where inter all intermediate terms are missing only s to the power 2 n is there and if you factorize it, you will have how many factors; 2 n factors that is if you take the roots, there will be 2 n number of roots, so these are the roots.

How are the roots located; this is s plane, you see all the roots are a typical complex number of the form r into d to the power j theta, r is the this part omega c and s is e to the power j into some angle plus into e to the power another angle. So, e to the power j some angle, all the roots are on a circle in the s plane and circular radius, this radius is omega c what are the roots you start with k equal to 0. So, e to the power j pi by 2 n plus pi by 2, pi by 2 n maybe this 1 this much, so this much and then pi by 2, it will be here. Next it will be k k goes up by 1, so it will be 3 pi by n, 3 pi by 2 n; pi by 2 n plus pi by 2 was here, next one if I jump into how much will be the gap earlier I had pi by 2 n.

Now, for k equal to 1 2 into 1; 2, 2 pi by 2 n, so pi by n; there will be a gap of pi by n are you following me; first start with k equal to 0, so pi by 2 n, this is smallest amount and then added you at pi by 2; so instead of here, you go here. Then next pole what will be the angular separation if you go for k equal to 1, so it will be 3 pi by 2 n plus pi by 2 and earlier I had pi by 2 n plus pi by 2. So, how much is the gap 2 pi by 2 n that is pi by n alright, so there will be next pole will be here; pi by n this much is pi by 2 n and there will be poles like this on this side, on this side also. There is some interesting properties of this poles; that we will discuss in the next class, that will give raise to the Butterworth function.

Thank you.