Discrete Time Signal Processing
Prof. Mritunjoy Chakraborty
Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture – 32
Pole Zero, and Stability of Analog Filters

(Refer Time: 00:21) from laplace transform will be useful here, but I will not I am not able to prove here.

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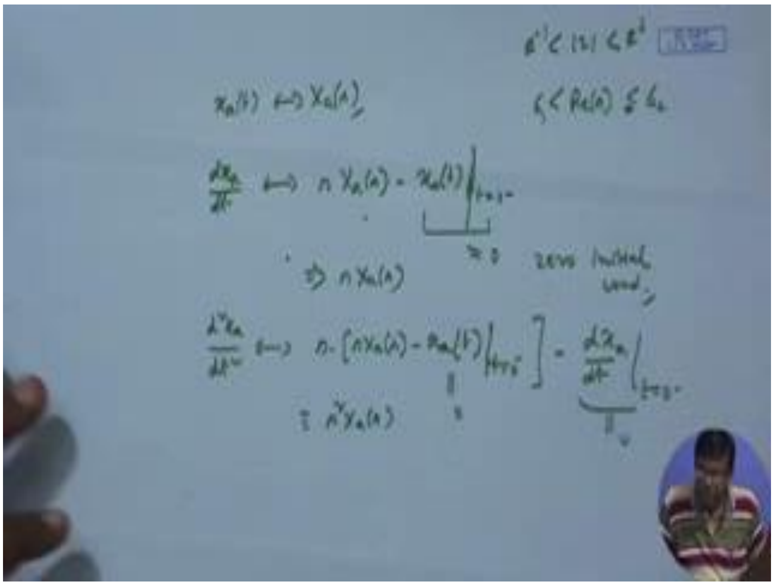

Suppose, x a t has got laplace transform x a a and region of convergence normally do not like if get transform we should write 1 z within sum, here it is real cut of paste that is sigma only matters using some range you know sigma 2, sigma 1 like that this is how you denote, but anyway some r o c I am not writing the r o c some r o c. Then d x a; d t is laplace transform is given by s times x a a's minus this function at p equal to 0 minus. So, this is initial condition, at 0 minus it is initial condition, so I am considering that signal system all will be switched on at t equal to 0, t equal to 0 minus z 0 where z 0 plus t equal to 0 plus signal constant all that.

So, this x a t now if I assume 0 initial conditions that is; if you assume sorry if you assume this to be 0; 0 initial condition then this is equal to just n x a s. Then d square, d x a d t square; this is d d t of of this one. So, it will be s times s x, a s minus that f t equal to 0 minus or I am using the general result, s times the laplace transform of d x d t, the d x d t is this that I am writing minus; this quantity d x by d t at t equal to 0 minus. Under 0 initial condition this will be 0, this also suppose will be 0 then it will be equal to only this is square x a s so on and so forth.

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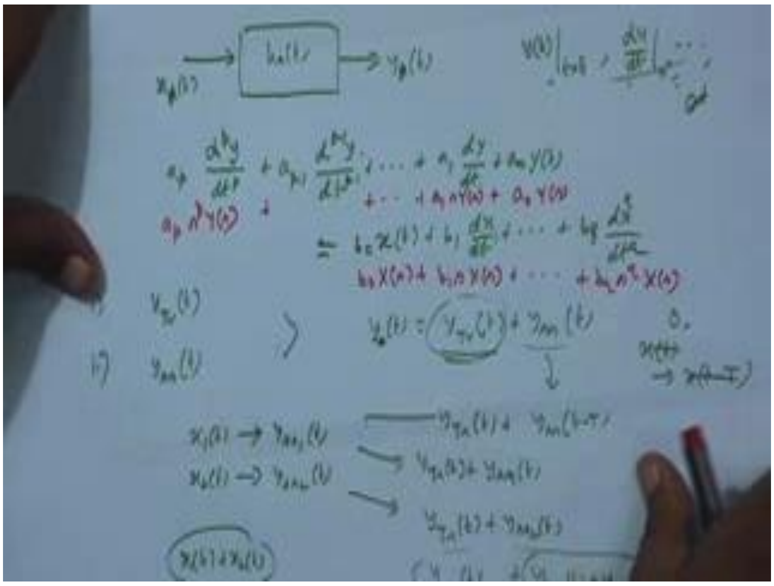

Now, consider a differential equation the system, an analog system which takes x a p gives an output y a t a particular system which is governed by or differential equation in the case of sequences. We had different equation it is differential equation, but treatment is same. Differential equation is maybe d p; y for the time being I am broking this subscript a it is all analog, so no point in carrying the subscript a d p y d t p multiplied by a p then a p minus 1, d p minus y, so derivative p dot t minus p dot, dot, dot plus a 1 d y d t plus a 0 y p is given by b 0 x t plus b 1, d x d t dot, dot, dot plus maybe b q, d q x d t cube; this equation will have 2 solutions.

In fact, this equation is given and unit initial conditions; this initial conditions are given that is system is switched on at t equal to 0, input is 0 at t equal to 0 input starts value from 0 plus that 0 minus at the point of which even input is 0 and then it picks up and this conditions are given, this initial conditions this value and 0 minus d y, d t at 0 minus dot, dot, dot, dot up to this these initial conditions are given.

Now, like the difference equation we have considered earlier the additional module here also there will be 2 solutions one is called the steady state solution, one is called the transient solution. For transient solution you take the right hand side to be 0 there is no input is giving and solve the equation with right hand 0 right, solve it and there you plugging all the minimum initial conditions and you get a solution transient. So, y capital t t for transient t r t r for transient t, if you put y t f t on this side that will equal to 0 and that will satisfy all the difference, all this initial conditions.

Another solution is steady state solution y s s, steady state t here you assume the initial conditions to be 0, give the input is 0 at t equal to 0 minus and then we start picking up and then solve it, that solution is given purely by input only no use or effect of initial conditions that is called y s s s t. Then our claim is total solution is a summation of the 2 y a, y t you know a anymore it is the summation of the 2 that is very easily seen like as we have seen in the case of additional systems because if you put this on the left hand side then you take out the y t at t part that further is a 0 y t; at t, a 1; d y t r; d t, a p minus d p minus 1; y t r, d t p minus 1 dot, dot, dot that part will be equal to 0 because that is how the solution was found. The right hand side was made equal to 0, left hand side was solved, initial conditions were used and we got the solution y t at t therefore, if I y t a t is put there on the left hand side, I will get equal to 0. So out of these if we take out y t at t component separately; a 0; y t at t a 1, d; y t at divide by d y t at d t and so on and so.

So, that part will contribute nothing that is be that will be give rise to 0 and if you take out the y s s part; a 0, y s s t; a 1, d y s s d t, a p minus 1 d p minus 1 y s s; d t p minus 1 and dot, dot, dot that will be equal to this because that is how the only solution was obtained by equating left hand side with right side. Therefore, left hand side overall left hand side will be equal to right hand side, component contribution due to y t at t from the left hand side will be equal to 0 because that is also the solution that is obtained by keeping only left hand side and equating only right hand side to 0. If we put that solution, that part gives rise to nothing otherwise you put that y s s t on the left hand side; that will give rise to right hand side, so this equation may be satisfied.

When you go to t 0 minus y s s t that will be 0 because when that time there was no input at 0 minus and when there is no input; there is no output, but y t at t; at t equal to 0 minus this solution, this will satisfy all those initial conditions because that is how the solution was obtained. This solution was obtained by solving the differential equation with left hand in place and right hand side 0 and with all initial conditions given here. So, therefore, y t at t satisfy all those conditions at t equal to 0 minus.

At y s s t; at all this derivative they are 0s, so they will not contribute anything, but overall summation will equate to; will equate to this the initial conditions that also be satisfied. Then as I told you in the case of rationale model because of the presence of these, system will not be linear and shift invariant, time invariant because suppose input is delayed; if input is delayed, if suppose instead of switching on the system if instead of giving the input I switch on the system at t equal to 0, but 0 plus but instead of giving input at that time suppose I give input after t amount of time, then till the time system will be at rest because there is no input alright, there is no input I am talking of y s s t only.

Suppose initial conditions are not there I am just bothered about y s s t, so there is initial condition system is switched on at t equal to 0 plus, but no input is given at 0 plus input is given after t amount of time. So, till this that point of time system will be at rest because there is no input there is no initial condition, so after the t amount of time I was started getting the same output which I was getting earlier. So, then out the new output will be nothing, but of the same form as the previous output, but separated by a amount of time delayed by an amount of time.

So, it will be that means, it will be shift to time invariant if input is delayed by t output will be delayed by t and it is linear very easily seen because instead of giving x 1 suppose I give x 1 t, I get a solution, I give x 2 t I get another solution it is very simple you should see if I give x 1 t plus x 2 t; what is linear combination c 1 x 1 t plus c 2 x 2 t, the output will be c 1 y 1 t plus c 2 y 2 t because you put that on the left hand side that will equate the right hand side.

So, y s s t alone that is when initial conditions are 0 y s s t alone satisfied linear and time invariants, but this part it is given purely by initial conditions right. Now suppose both are non 0 initial conditions are present, I switch on the system at t equal to 0 plus, but give the input after t amount of plus then what happens contribution due to these, this will be delayed. If x is; if x t is delayed by t this will be delayed; this part this we have already seen this what is obtained by keeping the input initial conditions 0 and solving this and that we have seen this will be delayed, but this will not be delayed because this is not derived from the input this is derived from the initial condition. The moment you switch on a t equal to 0 plus, initial conditions will come here right hand side is 0 this solution will be there, so this will continue to be same as before.

So, this summation is not at a delayed version of this summation, this is delayed by the same amount of t as the input, but this remains as it is. So, overall y at y at here overall output is not a delayed version of the previous overall output by amount of t that is why it is not time invariant then again it is not linear because I give x 1 t as input, I give y t a t plus say y s s some y s s input, I give another input I get the same thing y t at t plus another y s s thing. If I add the two input, I will not get summation of the 2 output because output will be still y t at t plus; it is like this let me explain, suppose I have got x 1 t; x 1 t alone gives rise to y s s 1 plus an listed component that is keeping the certain initial conditions 0 and solving you get this. Suppose I give me x 2 t plus initial component is this when the initial conditions are present and I give x 1 t output will be y t at t as before plus y s s 1 right input is x 1 t, so total this is the output.

This is when I give x 2 t, output is still y t at t plus y s s 2 t, if I just add that two, if I give this as input steady state solution that will be summation of the 2 because that part is linear we have seen, so that will be y s s 1, but this will not be tallied because this is independent of the input, this is given by the initial conditions. So, input is now this total thing, so it is corresponding steady state solution will be summation of the previous 2 steady state solutions, that will get added to just 1 y t at t obtained by delayed from or coming from this driven from this initial conditions. So, this overall solution is not summation of these 2 because if we add the thing, it will become at that 2. It will become twice y t at t, but here I have only 1 y t at t that is why it is not linear, not time invariant, but if I set the initial conditions to be 0 then y t at t is 0 there is no solution is purely y s s t and we have already seen y s s t it satisfies both linearity and time invariants

So, hence forth we will consider this differential equation only with 0 initial conditions and under 0 initial conditions we have seen if I take laplace transform of y it is y s. So, for d y t it will be s y s for d square y discrete square it will be s s square y s dot dot dot. So, I repeat again; I will now consider 0 initial conditions there is y t at t 0 minus equal to 0 d y, d t at 0 minus equal to 0 d square by d t square at 0 minus equal to 0 dot dot dot. Under this we have seen there is no transient solution because that comes only from initial condition, so if initial conditions are 0, this will be 0 because there is no transient solution; solution will only the steady state component governed by the given input and then the system is both linear and time invariant. So, it will be linear and time invariant system and therefore, it will give a dealted the input and record the output and call it impulse response, then another way of obtaining the output will be convolution between these two.

Under this case; system is linear and time invariant because there is no transient solution initial conditions are 0. Under this case, if I apply laplace transform this left hand side and right hand side what will we have; if I apply laplace transform here, a p here it will be a 0; y s, here it will be a 1; s y s; dot, dot, dot, dot, next one if it is a 2 d square by d t square it will be a 2; this is square y s dot, dot, dot, dot, upto here a p; s to the power t y s and here it will be v 0, x s then b 1; s x s then may be b 2 this is square s s, dot, dot, dot b cube s to the power cube x s.

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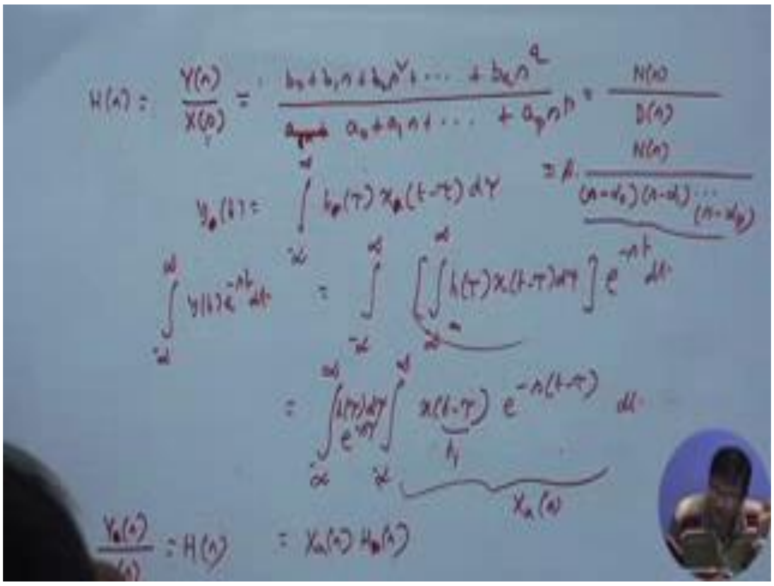

So, y s you take common x s you take common you take the ratio, you have equal to sorry it is 0. So, it is a ratio of 2 polynomial that is why it is a rational form; numerator polynomial denominator polynomial. Now, what is y s by x s we have seen 1 thing that under linear time invariant condition; y a t is a liner convolution between the h and x; h a tau x a it can be taken d d y a forget about the apart all are analog. So, drop a t minus tau d tau t is from outside, t is from outside these are convolution if we take the laplace transform of y it will be y t e to the power minus s t d t. So, minus infinity to infinity, you will have this thing integrated, this is a overall integral is function of t only tau goes out of integration here it is minus infinity to infinity this inner part h tau x t minus tau d tau this is the path here, it is y t and then this into the into that e to the power minus x t this is your y t and then the 2 integrals can be interchanged because 2 summations can be interchanged always that is the next step.

So, outer integral now which is will be replaced with tau d tau at h tau depends only on tau. So, this can go outside it has come on and this remains inside x t minus tau e to the power minus s t; t I can write as t minus tau d t. So, I have brought e to the power minus s to minus tau, but it will require plus s tau I have brought in I have to cancel it by e to the power minus s tau that I can bring because it depends only on tau I can bring that here.

In the integral t minus tau if you call it t 1, x t 1 e to the power minus s t 1 and limits will be same minus because if t goes to minus infinity t, 1 goes to minus infinity if t goes to plus infinity t 1 goes to plus infinity. So, this entire thing is laplace transform from x a s that goes outside and this is again laplace transform of h. So, in time given convolution means laplace transform domain, we are multiplying x a s; h a s which means under linear and time invariant condition; y is output laplace transform by input laplace transform is the forget about the this is h s.

Now, for that differential equation plus model which I called rational model, under 0 initial condition we got this ratio y s by x s is equal to this, but under 0 initial condition also we have seen system becomes linear and time invariant, in that case y s by s is also equal to h s; h s is the laplace transform of the analog impulse system and this is called the transfer function so; that means, under linear for rationale models, under linear time invariant condition, transfer function h s is given by this kind of form; this coefficient actually come from the differential equation that is the ratio of 2 polynomial.

Now, you see this polynomial; numerator polynomial I write n of s and denominator polynomial d of s now n of s let it be as it is, but denominator polynomial I can factorize in first order factors, it is polynomial in powers of s. It could be you can take out a p out a p out, so a 0 by a p a 1 by a p, there will be 1 by a p factor. So, 1 by a p factor it is some constant factor say capital a so that s to the power p and then some constant into s to the power p minus 1, some constant into s to the power p minus 2, a 1 by a p into s x 0 by a p like that.

S to the power has it come same to 1 and then if you factorize all of them will be have s only there is coefficient with case will be 1 s minus may be you know some pole alpha 0 is minus alpha 1 dot, dot, dot, dot, s minus alpha p. I took a p out 1 by a p if call capital a and therefore, s to the power p has a coefficient 1, that is why when you multiply this factors s s s s; s to the power p will have with coefficient 1 whether; every factor gives rise to 1 pole suppose s equal to alpha 0; at s equal to alpha 0, alpha 0 could be complex could be linear complex alpha 1 could be liner complex; at s equal to alpha 0 this factor is 0 and therefore, this is division by 0s will be this transform function does not exist. So, this transform function does not exist this function at this pole therefore, r o c of such a function cannot contain this poles, therefore r o c could be like this, if you draw the poles giving such a rational function.

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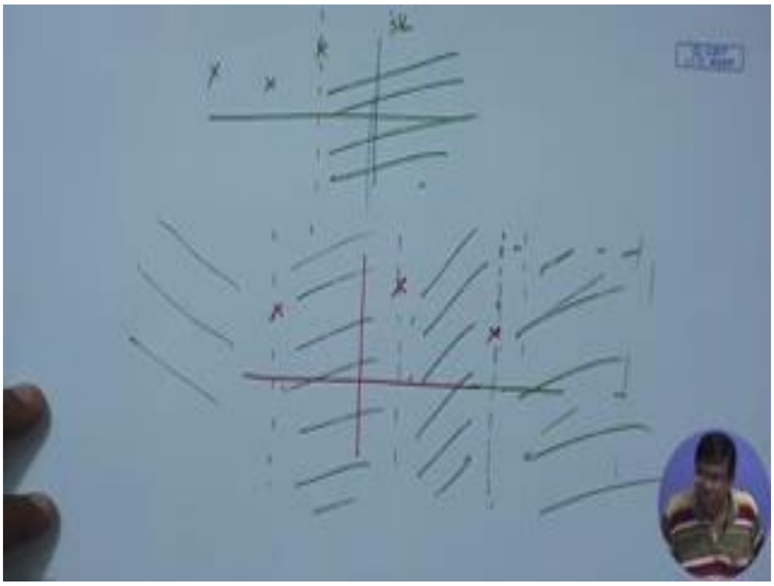

Suppose you draw the poles; if there is plane if there is; suppose here is a pole here, there is a pole here, there is a pole here; you draw vertical lines. So, these poles cannot be part of r o c at the same time we have seen that for linear time invariant systems r o c of a laplace transform will be governed by what I mean strips, so those strips; vertical strips like there could be vertical strip, this vertical strip should not contain any pole.

Suppose I start from this line, just to the right of the line and if I consider any strip and go further wide in the strip, go on up to infinity since there is no pole, this would this entire region up to infinity can be 1 r o c or then I consider the line between these two, these joint between these two lines; this is 1 strip, this strip does not contain any pole because I am not including the lines that could be 1 r o c; r o c cannot contain the poles because the poles function does not exist or it could be this trip without including the lines that could be 1 r o c or that could be another one, going up to minus infinity that is an infinite strip.

Now, for the system to be causal and stable I have seen or we have told we have already explained that r o c should start from a vertical line to the left of the denominator axis and go entirely right towards. That is possible only if already in the poles, suppose there are three poles, all the poles from the left hand side of the j omega axis then I and then I consider the one, which is closest to the j omega axis draw a vertical line and then consider this.

This will not have any pole because all the poles are assumed to be on the left hand side of j omega axis and I am taking the 1 closest to the j omega axis, drawing a vertical line through this and going right toward. So, these joint contains j omega axis and it is going to the right side of a vertical line should be both causal and stable because it means all the pole must be on the left hand side of the j omega axis that side is called l h p left of plane in this plane.

So, all poles, but the linear and time invariant for a causal and stable rational system; all its poles must be on the left of plane and you take the one closest to the j omega axis and start draw a vertical line through it and then go to the right side from that it will include j omega axis and it is going to the right side of a vertical line, so it will both causal and stable.

Therefore in d. I. d analog filters you make sure that you design these coefficients a p, a p minus a 1, a 0; not only you have some desired frequency response, but also to make sure; the resulting poles are all on the left of planes. Keep these two in mind and another thing is as I told you that if there is a rational system this is h s and these coefficients are coming from the difference equation, but differential equation; the differential equation number going to infinity. Suppose you are drawing to designing them how will you designing them; one condition is that it should give rise to pole which are all within the left hand side of this, that is to the left of j omega axis that is must.

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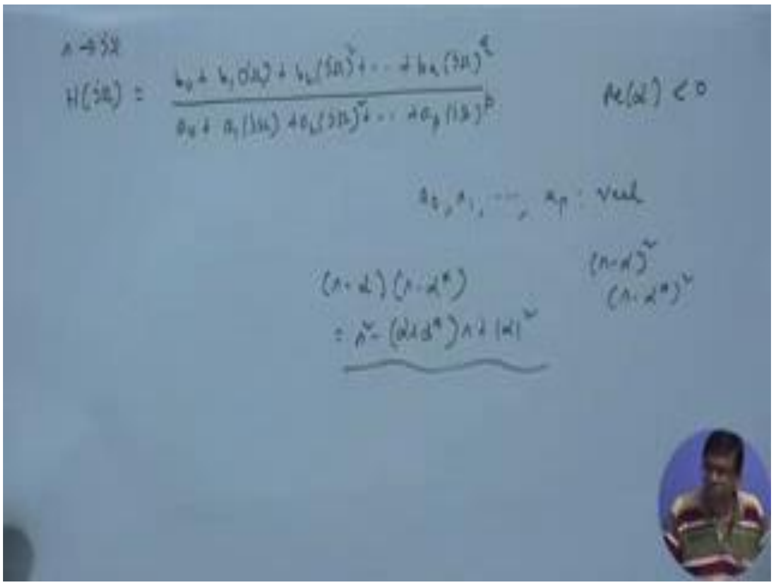

Other considerations are these, if there are other consideration is what will be the frequency response. Frequency response means s will be replaced by j capital omega that is sigma is 0 which means in the s plane I am on the j omega axis for which sigma is 0. So, I am not moving on the entire s plane, I am moving only on the 1 line that is j omega axis y axis and there if I take s that is s is equal to j omega replace this you have b 0 plus b 1 j omega b 2 j omega square so on.

This is a function of omega, so this will be your transform function; this is a complex number in general any fourier transform is a complex number, this is a fourier transform of the analog fourier transform of the analog impulse response. So, these were the mod general transform laplace transform, this is a fourier transform you take mod of these and square to the so that the magnitude square you take the phase of these all those things are there.

So, this will give you the frequency; I mean if you take the mod square of these this will give you the magnitude response if you want the good filter you have base specified or you have a liner able frequency response try to design the coefficients. So, that if you plot this mod square of these versus the omega, you will go close whatever the desirable that is one consideration and make sure you choice of a 0, a 1, a 2, a p are such the poles are all on the left of planes, that is real part of any pole alpha should be less than 0 and one more thing, if suppose this coefficients a 0, a 1, a 2, a p they are all real, suppose a 0 a 1 dot, dot, dot a p all are real then either there will be a real pole or there will be a complex conjugate pole that is if there is a pole at alpha, there must be at alpha star. So, that we have s square minus alpha plus alpha star s plus mod alpha square this is real and this is real alpha plus alpha star this twice the real path and mod square.

So, that is another thing that if we have coefficients that are real then either we will have real pole or we will have conjugate pole. If suppose we have got second load of pole here s minus alpha square where alpha is complex then it is conjugate also must be second order. So, that we have got s minus alpha s minus alpha star, you get a polynomial another s minus alpha; s minus alpha star another polynomial both second order and both have real coefficients.

With these we will consider analog filter designing particular in whatever filter design, how to derive that expression all those things there and then we will try to move to the digital filter design, we will show the similar treatment cannot be carried out there because of sub problem and then we will show how to design this IIR filters digital filters by taking recourse to analog filter first that is all for today.

Thank you very much.