Discrete Time Signal Processing

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Lecture - 31
Analog LTI Systems, Fourier and Laplace Transforms

In the last class we started with analog singular system because our purpose is to go for digital filter design and digital IIF filter design and digital IIF filters are designed by first designing equivalent analog filter and then mapping it to digital filter. That is why this important to get back into this analog filter design concepts and I just started giving you some background about the original you know basic analog signal resistance.

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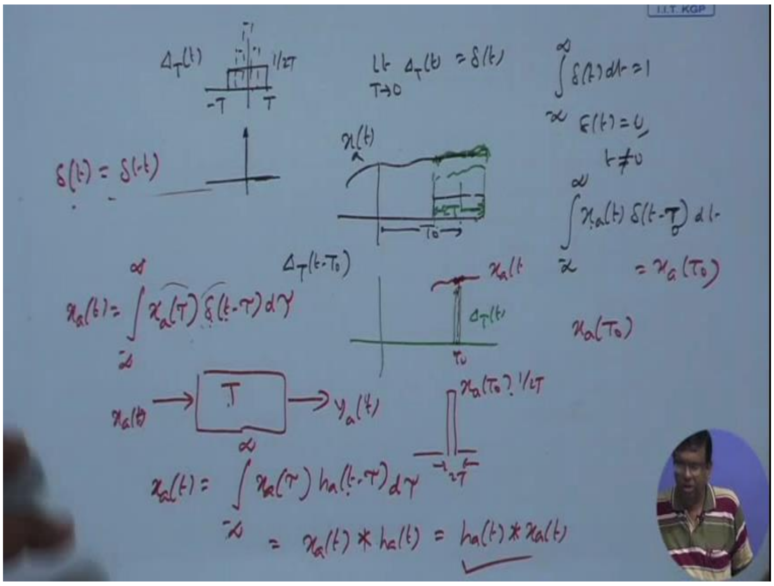

In the last class I discussed convolution, we discussed the delta functions that is you start with one delta T as this, this is a T minus T and area is 1, so 1 by 2 area is 1 and then let limit t goes to 0; this delta T these becomes a function small t (Refer Time: 01:14) these are delta function. That is if t becomes 0 but area is 1, gradually height will go up further go up; in the end it will be something like this up to the function, but it is continuous. So, that is why delta T d t was 1 and delta T was 0 for t knot equal to 0, because delta T is the small t is defined as capital delta subscript capital T small t with t going down to 0. So, that is why this has no value; but T knot equal to 0 that means 0.

Then what I said is these that suppose there is a function x t, x; a t and you want to see x a t delta T minus tau, t minus v. So, T knot t minus T knot d t or instead of t, it will be could be tau (Refer Time: 02:25) any variable what will be these. For this we first start with this function here that is delta T, T minus T knot this much is T knot, this is delta T t minus T knot and multiply x a t by this so this height will be modulated by this part, this part of the analog function that will multiply the height. But slowly what will happen? But an area under this will be (Refer Time: 03:10) carryout the integral area like I mean this will be the height may be going to be like this.

So, under this curve whatever the area that will be it is integral, but then I will go to this small delta from capital delta have to make this height width, width is T 2 t level if you take these 2 go to 0 and this will go up. When it becomes very narrow this function this is your delta T; small t and this is your centre point is T knot and this is your x a t, when you multiply this with this basically this part will multiply this function, but this part if this is the width is so narrow this hardly the variation here. You can approximate this function to some constant at x a of centre point T knot.

So, now it will be a rectangular pulse only height will be x a T knot times 1 by 2 t earlier with 1 by 2 T this is 2 T and then slowly you get capital go down further to two 0 this height will go up, but area under this will be when x a t 0, that is why this was x a t 0. That means, x a t can be written as this way also x a tau, delta T minus tau d tau this instead of very very small t and to a small tau and t minus small tau is same as tau minus 1 delta T minus tau or delta tau minus t they are same. So, small t now is acting like 2 0 there and tau is acting at t and I have just reversed because delta say t and equal to delta minus t.

So, you can write this as delta tau minus small t and small t is equal to T knot here tau is equal to t here and you will get this. If this is first two are linear shift invariant system linear time invariant system. This is operator x a t, the output y a t you have to find out y at the particular t of your choice, so t working on this integral, integral is a summation if system is linear t for a summation of signals is same as t over the individual signals and they are to the response and the summation of response. That is first you it will combine the signals and then apply the operator, you will get the same thing. If you first apply the operator individual components and then combine that is the meaning of linearity.

So, t will go into it and this is constant because this is the function of time, this is the function of time this tau is just the (Refer Time: 06:24) integration. You are just taking deltas or infinite range from minus infinity to infinity from on this axis from minus infinity to infinity as controlled by tau. So, for every tau there is a delta function multiplied by a constant x a tau, so the operator t will work on the function of time there is delta T minus tau and then in system is time invariant then if delta T has a response ha t which is called impulse response, the delta T minus tau your response ha t minus tau which give rise to this thing; it was analog convolution and I will not go too much into all these h a, t minus tau d tau, I mean this we will write as x a t convolve with ha t, in the previous class we took t minus tau to may be tau prime, so d tau was minus d tau prime, tau is t minus tau prime and tau prime here minus comes out integral deviates gets reverse because t minus tau is tau prime.

So, when tau is minus infinity, tau prime is plus infinity and when tau is plus infinity tau prime is minus infinity. So, integral is from plus infinity to minus infinity, but outside there is a minus sign, minus sign can be observed in (Refer Time: 07:45) second which means should be from minus infinity to infinity and that will become equal to convolution between h and x this is your x tau, when I always tell you take this form for all practical purpose. Here again the concept of causality and stability come first is the integral can be written as to compound minus infinity to 0 minus there is a approaching 0 from the left hand side; x a tau and from 0 plus to infinity x a tau ha t minus tau d tau.

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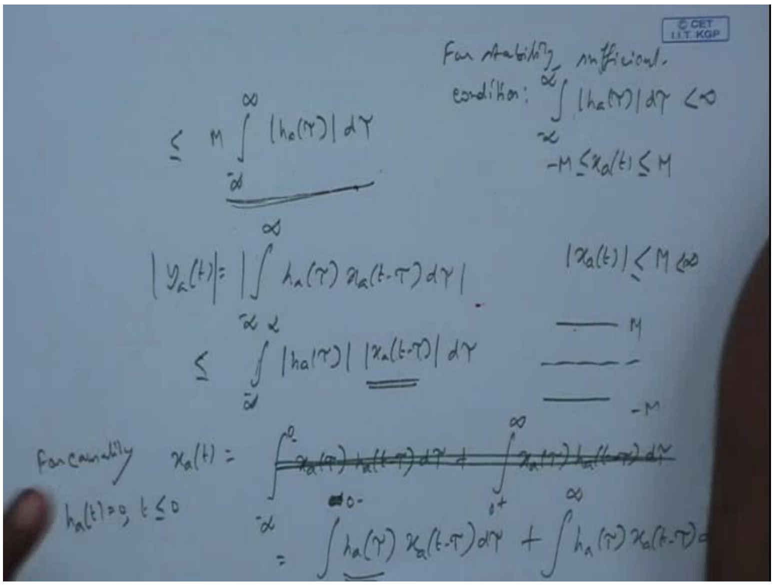

Now, in this part tau is taking negative value, but anyway I will not rather take thing this (Refer Time: 08:39) I will take this form. So I was also making a mistake, let me take that form first h this is we up to 0 minus 0 plus, this remains as it is. Now look at this integral here tau is taking negative value from minus infinity to the 0 minus which means t minus tau, so t plus some quantity because tau is negative so minus, minus plus; which means strengthen at time t, I am looking for future input that is x a t plus something because tau is negative in this integral that makes it non causal. In order to maintain causality this should be 0 there what this interval, here I have no problem tau is positive so t minus tau means going I mean t minus tau means I am going back in the past which is fine because past input is always known.

Therefore, for causality ha tau or t or whatever be the way you describe because this is 0 for t less than equal to 0 for causality. So, this is for stability I will take this second form only, I always told you the second form is always useful. Suppose input x a t for any t it is basically it is magnitude is less than equal to sum M that is there is two labials M and minus M this is 0, so input is always limited within this range, how about large m. So, M is finite; that is x a t for any t is less than equal to M greater than equal to minus M it is given. Now mod of these mod of a summation, summation is an integral, so mod of the summation is less than equal to summation of mods, the summation of mod of this product d tau is positive mod of the overall product is mod ha tau mod into mod of these into mod of these, mod of the product is product of mod we have all seen earlier.

This will be ha tau mod of these this is positive so no point in taking mod here and this is for any argument t or tau or t minus tau, this is less than or equal to M that means, this integral is less than equal to M into d tau. Therefore, it is positive, if it is finite it is of course, positive because mod is so less positive non 0, non-negative either 0 or positive and integral the sum so this is nonnegative, but if this is finite then I can said it is finite, so output magnitude is finite so up to the given within a finite limits. Then it will be guaranteed too stable that will be sufficient conditions, though it is not special condition. Even if the this side is infinity, this is less than equal to it could still be less infinity and finite, but if this integral is guaranteed to be final then mod y a t also be finite.

We have already seen in the discrete time space also, therefore for stability sufficient condition that the simple response is absolutely take absolute value sum of that is a integrate integral is finite, this is very much similarly to that, next comes the IN function and IN signal approach.

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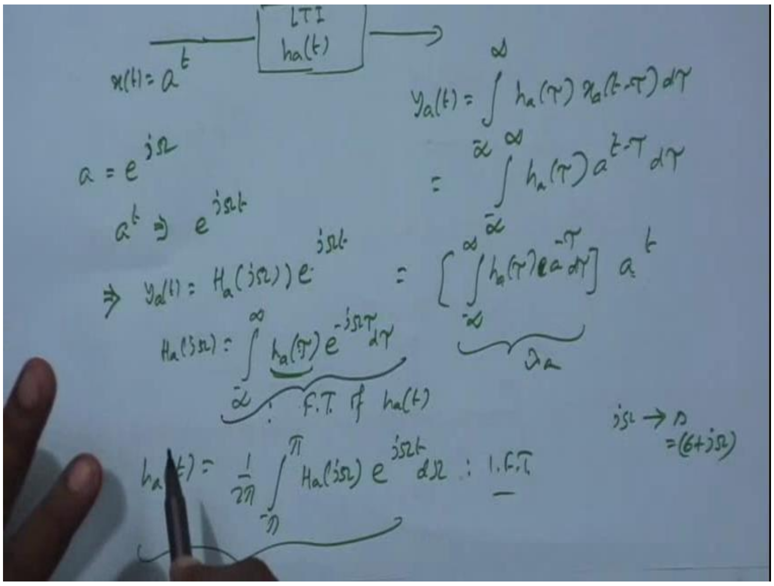

Suppose linear time invariant system so LTI and it is characterised gain by impulse (Refer Time: 13:11) ha t. Also that if you take any function of the form; a to the power of t, but prime t comes in the exp in the power then it is an IN function, because the output in this case is at any time t is; this is a convolution taking the second form of convolution ha tau, x a t minus tau d tau and this is your x t. So, minus infinity to plus infinity x (Refer Time: 13:41) similar same x a t minus tau means a to the power t minus tau d tau a to the power t can come out integral (Refer Time: 13:49) tau only.

So, a to the power t comes out and integral is this, so a to the power minus tau, this is the entire thing after integration you know there is no tau there is just a function of a, so you can call it an Eigen value lambda a. Depending on a, that a to the power t output is lambda into a to the power t. What is the advantage and all you all know is, but this kind of function I mean you do not need to carry out convolution it has straight away write down the output just a to the power t only multiplied by an Eigen value lambda a. A special case is where a is of the form e to the power j capital omega, omega is actually radiant per second because it will be the a to the power t means e to the power j omega t radiant per second in two second is equal to radiant, the entire thing has to be angled e to the power j theta, theta is radiant, so capital omega you need to radiant per second of this form.

In this case this quantity there is y a t is this quantity, this quantity you write instead of lambda, we use the notation it is a function of omega a is nothing; a is basically a function of omega. So, it is basically function of input this here is function of omega e to the power t, where ha j is omega is this integral only ha tau; e to the power minus tau gives e to the power minus j omega tau d tau. This quantity is called Analog Fourier transform of any signal ha t, you can have t here tau here anything any integral variable no problem you continue (Refer Time: 15:54), this is called Fourier transform; there is a Fourier transform.

Now, it can be shown that you can get back this ha time domain function from this frequent situation function in this way where the inverse transform, ha t for a chosen t the t is fixed one by 2 pi where integral is over omega. So, basically all frequencies upto n for any capital omega, this is a carrier e to the power j omega t multiplied by an amplitude function and this your integrating and you get a ha t is called inverse Fourier transform this Fourier transform, this IFT inverse Fourier transform both are analog Fourier transform.

Now, from this we can generalize we can go to a more general transform which is called Laplace transform and that we will do from here in fact this one. Here you see e to the power minus j omega, but j omega is just purely imaginary, a more general complex number will have both complex number will have real part and imaginary part. So, from j omega we will go for a more general complex number. Suppose I go sigma plus j omega, an integral of this form will be considered and not that will be not be this; make sure that will not be this, but this kind of form, this form I integral but instead of just one minus and just one imaginary part only j capital omega, I will take a more general complex value which will both real and imaginary part; I will assume that integral will be existing and all that is underneath.

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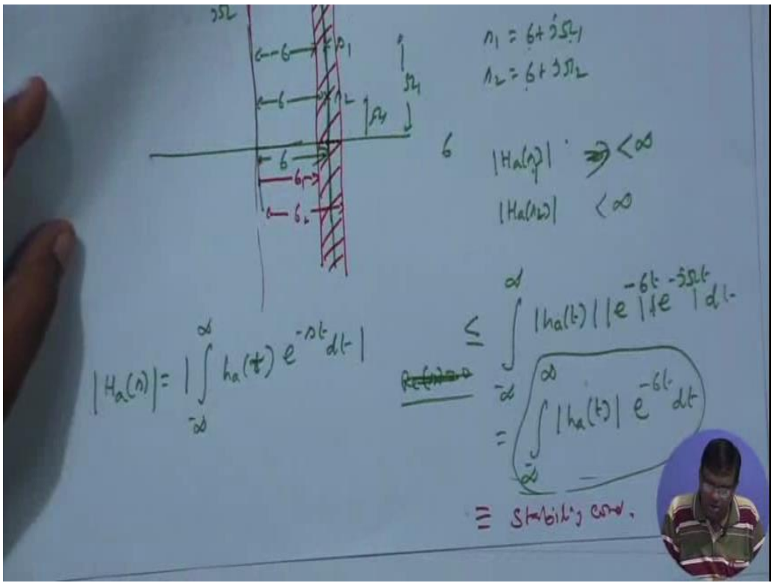

In that case the general integral will be of this form, this is not this I repeat again; this is not this after to generalize this form here. I saw there is only one imaginary part j omega just imaginary part (Refer Time: 18:03) what y knot having more general complex (Refer Time: 18:04) both real and the imaginary part something of this kind is; you can make it t also. This is more general integral and we will call it Laplace transform of ha is and just to get back this Fourier transform, what you have to take you have to make sigma equal to 0 that is real part of phase if you take 0 this integral will give rise to Fourier transform.

This is more general than that and I am assuming this integral will exist this is called Laplace transform, but here s is a more general complex number. Now for h as to exist this should be finite, that means this should the finite. The mode of this is less than equal to mod of e to the power minus sigma t e to the power minus j omega t, s is sigma plus j omega. So, break it like that e to the power sigma t and mod of the product is mod of these into d t. But mod of e to the power minus j omega t is 1, so this is equal to at e to the power anything is positive; so mod of that what will you know mod is required on that.

You see if this quantity, if this is finite then this is finite the Laplace transform (Refer Time: 19:50), so this to be finite. But, that has a beautiful meaning that suppose I have got this complex s plane, this is a real part sigma this is j omega or you have got suppose one line vertical line in vertical line be real part is fixed and suppose I have s 1 here, s 2 here; s 1 is sigma plus say j omega 1, s 2 sigma plus j omega 2. But this condition the h, for this to exist and for this what is the condition that for this to exist the condition is absolute condition sufficient condition is you see this to be finite and there is no omega here. For this also for this to exist this, for this to be finite for any s 1, s 1 is sigma plus j omega it was for (Refer Time: 21:00).

Now sigma plus j omega 1 is one I am taking, this is a condition this to be final but you see there is no omega 1 here, it is only sigma that matters and for ha that is why for ha is 2; mod of that also be to be finite; since a sigma is common the same condition. So, this integral if it is finite then if I am moving in this line any point you take they all have the same sigma, only omega differs, this one is omega 1; this much is omega 2. But you see this exist a condition it has got no omega because only to mod of e to the power minus j omega t become 1. So, as long as sigma (Refer Time: 21:41) is fixed you move anywhere on this line; this integral will remain same because there is no omega, it is only varying on sigma and sigma s is constant on this line.

If you move on this line, if you take any s on this line for any s if you want to carry out the integral, you will only pick up the sigma from the case not the omega. So therefore, if h and s 1 suppose you are here and mod h s 1 is given to be infinity that this is to be finite. The same holds for ha s 2 also because for this also the sigma is same which means here convert the s this is on line vertical lines and not only vertical lines, if it is converting for one sigma then from continuity this is not that (Refer Time: 22:35) only at sigma. So, I immediate left of sigma to be immediate right of sigma it will divert this will become infinity this is not possible, because of the continuity there will be at least some strips however, small the strip there will be some strip.

So, this much is sigma, this much may be sigma 1 this much may be sigma 2, so for a zone not for the specific sigma, but because of sigma continuity that if you put a sigma here if it is finite if you slightly increase sigma still be finite because of a continuity or if you slightly decrease sigma still be a finite. So, around sigma there will be a small band over which this integral is finite and this always depends only on sigma that is why I am not bothered about the vertical axis. That means, this kind of (Refer Time: 23:32) region of convergence that is (Refer Time: 23:37) and they explain for which the Laplace transform magnitude will be finite it will exist, so degree of convergence will be vertical strips.

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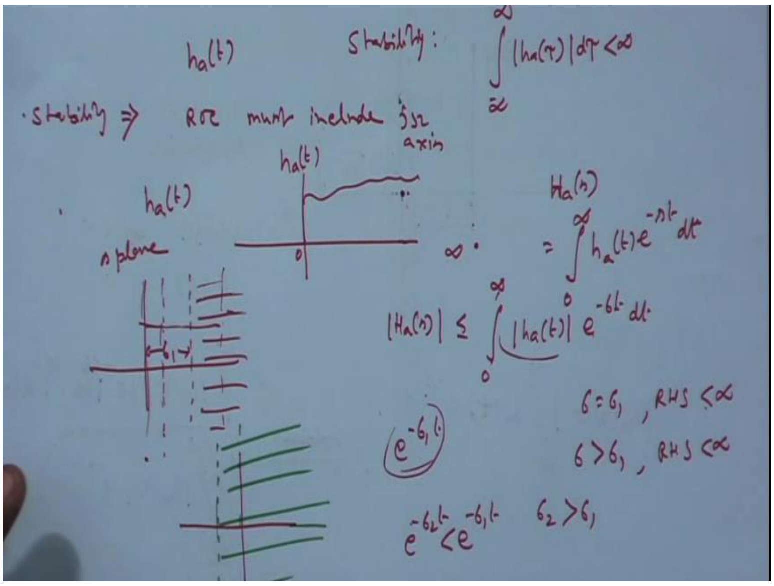

Now, if suppose the (Refer Time: 23:52) function is to be the signal there is a sequence not sequence; signals ha t suppose there is impulse response h a, I want it to be causal and stable. If it is causal or stable first the stability means stability we have already seen means that is stability, if you see this if you take sigma equal to 0 then this integral, this condition that summation minus infinity to infinity mod ha t d t less than infinity; that is also the stability conditions. Therefore, if mod ha exists for sigma equal to 0 sigma equal to 0 means on this line or this is y axis that is if ha s exists that is this mod value is finite on this y axis means j omega axis for which sigma equal to 0 if you put sigma equal to 0 e to the power 0 1. So, you get that summation integral minus infinity to infinity mod ha t d t less than infinity that is the condition. But that is also equal equivalent to stability condition, is not it; that is what equivalent to stability conditions.

So we can correlate the two, we can say that each Laplace transform exists that is if the mod value of ha s is finite on the y axis that is j omega axis for which sigma equal to 0. That means, that this sufficient condition is satisfied for sigma equal to 0 that is e to the power 0 is 1, so what condition is satisfied, integral minus infinity to infinity mod ha t d t less than infinity, but that condition is same as stability condition. So therefore, if Laplace if the sufficient condition for the existence of Laplace transform on j omega axis exists this is the sufficient condition. That is sufficient condition for mod of ha s mod of ha s to be finite for h on the j omega axis if sufficient condition for that exists is satisfied that will be equivalent to stability.

So, then the system will be stable, because on the j omega axis sigma is 0 sigma 0 is 1 and what about you get out of it you get the stability condition. Therefore, what the system to be stable, stability means ROC (Refer Time: 26:43) convert must include j omega axis. That is it may be a strip, it may be a strip around this but it should contain the j omega axis, so that this condition with this is equal to 1 and this mod ha t integral minus infinity to infinity d t is less than infinity that is satisfied. What you see on j omega axis sigma is 0, so condition transferred to that which is against stability condition.

On other hand ha t is causal then what happens for (Refer Time: 27:22) is one thing suppose ha t it starts from here. I will not give a full (Refer Time: 27:36) suppose this function like this, in this case ha s is 0 to infinity because integral function does not exist in this side, so no point on carrying out the integration on that side; ha t e to the power minus s t d t and therefore mod ha s is less than equal to mod of the summation is less than equal to summation of mod; ha t has reported mod e to the power minus s t (Refer Time: 28:18) it is just e to the power minus sigma t d t.

Suppose for some sigma is equal to sigma 1 this is finite, RHS finite then for sigma greater than sigma 1 also my claim RHS should be finite, because as sigma for any particular t in this range, suppose we had e to the power minus sigma 1 t, so this was multiplying by a damping function e to the power minus sigma 1 t. For the same t you are not choosing mod value and the sigma higher than sigma 1, so value will go down value will be because of the negative sign is not it e to the power minus sigma 1 t if I choose sigma 2 it is greater than sigma 1; I know sigma 2 T less than e to the power sigma 1 t because of minus sign.

So, earlier was multiplying by e to the power minus sigma 1 t, but still this integral was finite that is our assumption. Now if I sum multiply by e to the power sigma 2 T then these are multiplied by lesser quantity and then integrate it obviously it will be less than previous integral, so obviously previous integral was finite this also be finite. That means, if the ROC they explain if you have got some sigma 1 then all (Refer Time: 29:47) or we should draw vertical line sigma of sigma (Refer Time: 29:52) will be greater than sigma 1.

So, all the region to the right of this line will be part of ROC and then you start sinking sigma 1, it can be this going down to 0 or it can be remain I mean you go on sinking it till the moment you arrive you reach a situation where if you sink further integral (Refer Time: 30:15) you have to stop there, so ROC will be the start from that line go to the entire right that will be our ROC. That means, for causal system ROC starts from one vertical line some vertical line and then goes to the right. If I want to have stability and causality that stability requires ROC to contain vertical this j omega axis, causality requires ROC to be the right of some line.

Therefore, for stability and causality means ROC should start at a line to the left of j omega axis and go rightward. Since you are starting on a line and then going rightward, it is causal and since it is starting on a line which is to the left of j omega axis, this region will contact the j omega axis and they produce stable. So, ROC if there is left of plane in ROC should be like this, but a causal and stable system you should start from a vertical line to the left of origin and go to the entire right. From this we will consider arrive at the concept of stimuli 0 for these systems; rational systems here, and then we will talk about analog filters.

Thank you very much.