Discrete Time Signal Processing
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Lecture – 28
Properties of Linear Phase Filters

Now we consider the case of N even.

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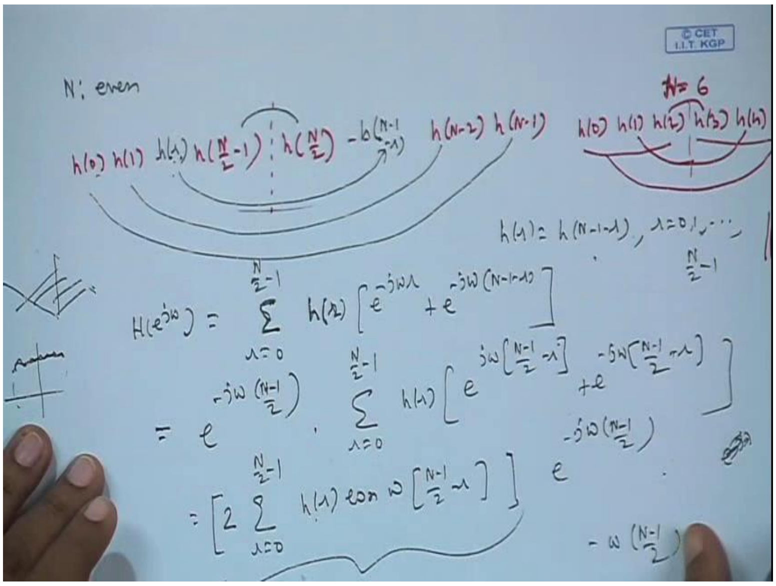

Now if N is even, like suppose N is equal to say 6. So, we have got h 0, h 1, h 2, h 3, h 4, h 5, so total 6. So, I will be defined into just two half, N is 6 to the capital N. One is on this side another on this side and put it middle here and I will in a impose these condition, h 2 equal to h 3, h 1 equal to h 4, h 0 equal to h from like that. So, one half will be N by 2 another half will be N by 2. So, N is 6 N by 2 is 3, so there will be three terms here which is go from 0 1 up to 2.

In general I will have h 0 h 1 dot dot dot total N by 2 terms, so it will go up to N by 2 minus 1, so there 0 1 up to N by 2 minus from this total N by 2, there are mirror here. And then next is N by 2 dot dot dot and I will have this mirror image symmetry. These two will be same, these two will be same, and these two will be same. In general, h r and h, because h 0 so you must say h N minus 1 minus 0, h 1 same as h N minus 1 minus 1 this is h minus 2, h r same as h N minus 1 minus r. That means, h r will be this is my condition in short starting from I equal to 0 1 dot dot dot.

Now, if I take again DTFT and they are same. So, you know there is no independent value at the same time unlike the previous case, because they total number was odd so there is an odd man out in the middle we have to handle it separately. We are there is no such thing or exact illusion into two sides N by 2 on this side N by 2 on this side and you have got a mirror image you estimate. So for the DTFT, h r and h N minus r h N minus 1 minus r I will take them to be under one bracket and h N minus 1 minus r will be simply replaced by h r. So, two terms will be combined together with the coefficient h r and there is r for 0 r for 1 r for all.

That means, DTFT will be r starting from 0 up to N by 2 minus 1, typical r h case h r and again it is h r. But these h r is with respect to we will have e to the power minus j omega r, this h r when is h N minus 1 minus r which is same as h r under this condition that you will have e to the power minus j omega into these index. And as before I take out e to the power minus j omega N minus 1 by 2, then what will happen you very much like in the previous case h r have to bring it back in inside it will be with a positive sin j omega N minus 1 by 2 minus r I will get minus j omega already N minus 1 was present after that N minus 1 by 2 has gone. So, we will be left with another N minus 1 by 2 by r as we did blustering and this is twice was this, twice I put this something a summation r equal to 0 h r cosine omega into this phase part.

Once again phase is linear this is this with to the negative sin is to causal there is a risk have been. But here this is a constant, so is a linear phase in as before this is the magnitude part and assuming that we are design the coefficient such that at least within the band omega c 2 plus omega c single band these do not change is sin does not become negative with these as positive. So, because if it changes is sin if it becomes negative then immediately it will mean up shift phase shift by e to the power j pi and to the mod of this quantity will come here and they will be e to power j pi coming, which means to the space another pi will get (Refer Time: 05:43).

So, it will like in away phase was falling down and suddenly it will go up by pi then step falling down, so it will learn perfectly linear. We are assuming that design the coefficients such that within this band it does not change it sin does not become negative. And also magnitudes wise also it which oscillates, but does not change merge. So, is kind of constant. Only thing is capital N is even now N minus 1 by 2 is a fraction, that is why this will not be physical a meaningful this delay that is why we prefer N equal to what case there will amount a perfect delay of the signal.

So, this is possible only when the filter is FIER. Now for the FIER for this linear phase system some properties are important.

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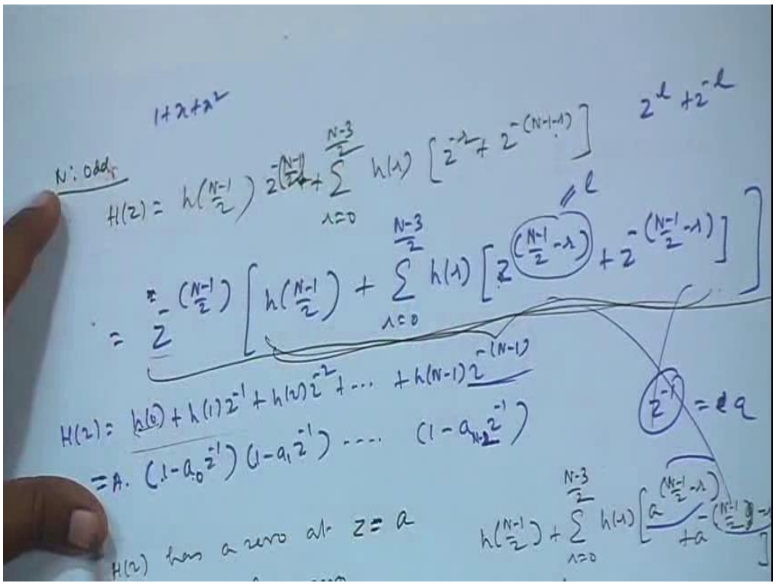

Suppose I instead of discrete time Fourier transform I take the z transform. So, N equal to odd case z transform. So, I go back to that N equal to what case. So, instead of e to the power minus j omega e to the z to the power minus 1, e to the power minus j omega or e to the z to the power minus r. Here H z will be this guy separately, but e to the power minus j omega into this instead of z to the power minus this. Normally z is replaced by e to the power j omega in DTFT, if you want to go the other way e to the power j omega wherever you fine that is z. So, z to the power minus this, z to the power minus this; this is a way to get back the z transform.

And this is summation h r this will be z to the power minus r this will be z to the power minus N minus 1 minus r. As before I will take out; so this is not z to the power sorry this is not z to the power minus 1 is z to the power minus this form N minus 1 by 2. So, from here also you take out this z to the power minus. So these remains as it is, I have taken these out and from here also I take out. What happens, this has gone out I have to cancel it is so this will become z to the power plus N minus 1 by 2 here. So, it will z to the power N minus 1 by 2 minus r and here z to the power minus already N minus 1 was present out of which has N minus 1 by 2 has gone. So, another N minus by 2 if get an r. This was the equation.

Now what is this z transform? H z as such is h 0 before I wrote in this form generalized z of this form right h 1 z inverse h 2 z inverse 2 dot dot dot dot say h. Now z inverse if you take it to be some complex variable say q. So, this h 0 plus h 1 q plus h 2 q square plus dot dot dot plus h N minus 1 2 q to the power N minus 1, so it is a polynomial in q with the powers of q up to degree N minus 1. You can factorize it there will be all master of factors. So, faster of factors will be of this form. Some constant you can take out may be A into 1 minus may be 0 some root a 0 q, q is z inverse. Then 1 minus a 1 z inverse after all to declare simple you can replace all z inverses by q.

So, it will be q this square q q of a q to the power N minus 1. It a polynomial q up to the degree q to the power N minus 1 it is polynomial can always be factorize into first other factors, the factors are in this form. I am forcibly putting a 1 here, so the multiples of factors of 1 comes that 1 times a, which will be giving this term h 0. 1 minus a 0 z inverse 1 minus 1 z inverse like that dot dot dot dot up to 1 minus a. How many terms? 0 1 N minus 1, so N terms, so it should be a minus 2 because 0 1 up to N minus 2 there is total capital N minus 1. So z inverse, z inverse, z inverse, you multiply up to capital N minus time 0 leading power gets. This is just like that, so you can factorize like this.

Now what are this factors that A 0 is a root that is if you take z equal to a 0, a 0 into a 0 inverse there is 1 1 minus 1 0. So, H z becomes 0 that is why h 0 has a 0 at z equal to a a 0, a 0 has a 0 at z equal to a 1 a 0 has a 0 at z equal to a N minus 2 dot dot dot, because this is a general equation. Now, in this case for this filter this is by z transform H z I am re writing like this. Suppose it has a 0, suppose H z this special H z not this general case, H z has a 0 at suppose z equal to say a. That if I put a here and a here obviously this factor will become 0. This cannot be 0.

There are consider a equal to infinity because if h 0 is not 0 this filter is then I get transform if you put z equal to infinity. These parts go to 0, but h 0 remains. So, there is a rigid, we cannot have z equal to infinity here as this. This factor will become 0 at z equal to infinity, but over all things will not be 0 because of course this is nothing but this and here if you put z equal to infinity this part goes to 0, but this remains.

Therefore, if I have a 0 if this whole thing has a 0 at z equal to a that means, this inner part will become 0 at z equal to a. That is a to the power plus a to the power minus sorry, minus r this equal to 0. Then my claim is H z also has a 0 at z equal to 1 by a, because if you put 1 by a here inside what will happen? 1 by a means a to the power minus this and z equal to 1 by means a to the power plus this, I do not have to about write it. Suppose instead of a, we have put a here a to the power this part we call it N minus 1 by 2 minus r if you call it l, z to the power l plus z to the power minus l minus the same thing l. That means, a to the power l and a to the power minus l, this is becoming 0. This summation here it is becoming 0.

If I do not replace z by 1 by a, this will become 1 by a whole to the power these there is, 1 by a whole to the power l means a to the power minus l. So, a to the power minus l will come and if z is 1 by a, so a to the power minus and then minus 1 by a whole to the power minus this 1 by a means a to the power minus 1, whole to power minus this. So, minus minus plus a to the power plus this, this will come. So, this will go here this will go here and summation did not say should will become 0.

So, this proves the claim that at it has a 0 at z equal to a, then if it has a 0 at z equal to 1 by a. So the reciprocal appear zeros of a linear phase system or a reciprocal appear. Alternatively if I have got two factors; one is having 0 at a, you have got one factor so z equal to a means a into a inverse there is 1 1 minus 1 0. Another is at 1 by a times z inverse.

So, if I construct a system second order system by multiplying these two factors only. One having root at a 0 at a 1 having at 1 by a. So, there is a just a small system, second order system for other sub system, by taking out two factors out of them one at one a 1 at one having root at a, another being root at 1 by a. If I multiply these two we will have 1 minus a plus 1 by a into z inverse and then a into 1 by a so that is 1 minus minus plus, so z inverse 2. So, if this is a transfer function of a sub system corresponding impulse response will be one from here then minus a plus 1 by a the next coefficient is 1. And you see you satisfy it is that becoming symmetry. It is a filter now of oblige 3; h 0, h 1, h 2, and this is where bigger will be if H z, so these are the rod men guy so let it be what it is. On this side you have got 1, on this side I have to 1. So, phase of this will be linear.

So, you can take out one sub system out of with phase will be linear. That is as the roots occurred in a reciprocal pair you can take out those two roots, those two factors and make a sub system just of this transfer of product this is the second order filter. This will get a here linear phase filter, because it is impulse response is what h 0 this is h 1 into z inverse and this is h 2 in to z inverse h 2 is 1, h 1 is minus a plus 1 by j h 0 is 1. And you see this filter impulse response total length is 3, so there is an odd man out at the mirror position, and I need mirror image symmetry if you go to left hand side by if a 1 if you go to the right hand side by a 1.

One more thing or you factorize this is again the general transfer function (Refer Time: 19:03) here when you factorize it is not guaranteed that all the co efficient will be real, because if you have a 1 plus x plus x square you can factorize into first order factors, but this root will be complex. But if the original h 0, h 1, h 2, up h N minus 1 of they are real h 0 real h 1 real h 2 real up to h N minus real then my came is if there is one root which is complex then there must be another root which is a conjugate of it.

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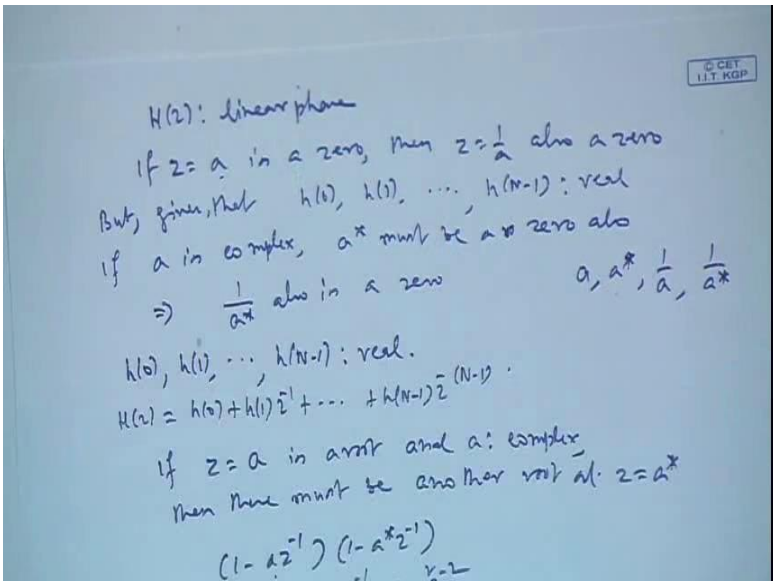

That is if there is a root, suppose h 0, h 1, dot dot dot h N minus 1 there all real. Then H z is h 0 plus h 1 z inverse dot dot dot dot, there all real. Now if z equal to a is a root and a is complex then there must be another root at z equal to a star, the conjugate of a. And then you will have the product a z inverse another is a star z inverse, because then if you multiply just these to what will happen is 1 minus a plus a star z inverse and a into a star that is mod a square z inverse two. And you see a plus a star is real and mod is square is real, so you get a polynomial is real coefficients.

So, this way whenever there we get complex root there will be another root conjugate of it you can take the two factors, product will be a second order term will be a second order polynomial with all real in coefficients. And there will be many search second order polynomials where all real coefficients for product is H z then H z will be have real coefficients. But if I do not have a star with a then there is no problem there is a whenever there is a complex root it is conjugate will be there.

Now, suppose I have got a linear phase system and with real coefficients suppose I have got a linear phase system with real coefficients. So, I now if it has a root at a from linear phase property it will have a root at 1 by a. But since coefficients are real if a is complex that is which is giving H z linear phase. Means, if z equal to a is a root is a 0 then z equal to 1 by a also a 0. But given that h 0, h 1, dot dot dot dot h N minus 1 they are all real.

In that case if a is complex you have seen a star also must be if a is a root a b 0 earlier have seen 1 by is a 0, but again under this condition that the coefficients are real then if a is complex then a star must be a root also, must be a 0 also. But since the filter is linear phase this is also implies 1 by a star also is a 0. First for any root a N 0 a 1 by z is a 0. And separately the filter coefficients are real if a is complex a star is a 0 and therefore 1 by a star also 0 because filter is linear phase. So together we have got a, a star 1 by a 1 by a star; four roots together. It is called conjugate reciprocal pair, a and this is reciprocal 1 by a a conjugate a star and it is reciprocal 1 by a star.

I mean in this cases under for a linear phase is stable if there is a root complex root then there will force has roots together you know it is a (Refer Time: 23:44). If a is complex here a star minus must be present because filter coefficients are real, as since they are linear phase both 1 by a and 1 by a star must be present which means if the complex plain it will look like this.

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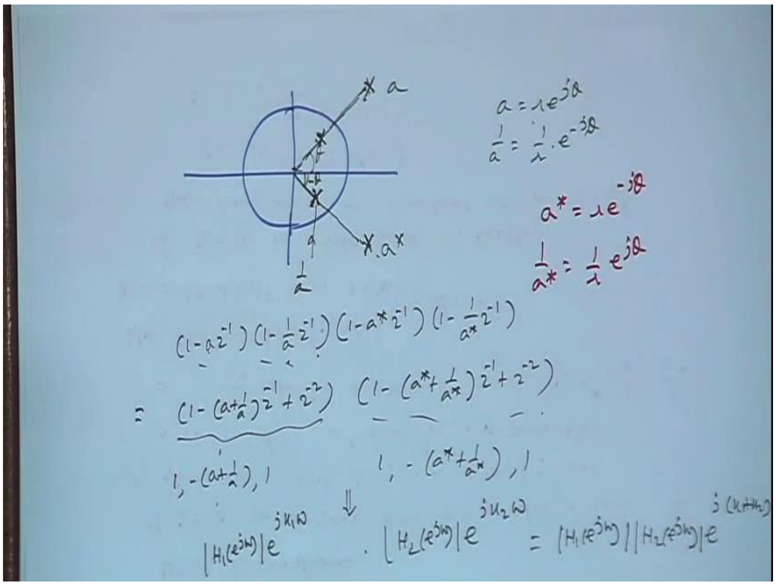

If there is one root here say a, this is a, suppose is r e to the power j theta then about the 1 by a 1 by r into e to the power minus j theta. So, here r is the phase r is the magnitude so much is r so much is theta, so much is r and this angle is theta. Now r is greater than 1 means 1 by r will be less than 1 and angle is the minus theta in this direction. So, here will be one root, this is minus theta, this is your 1 by a.

On the other hand a star is another root which you have same magnitude only phase is opposite. So, angle will be minus theta, but same radius r. So, here is one root here is one, same radius r but angle minus theta. This will continue it will come here, this is a star. And 1 by a star another root there will be 1 by r e to the power j theta. So, angle is again plus theta, but magnitude is 1 by r which is less than 1, so this much. These are the four root and they for the (Refer Time: 25:31).

So this is conjugate of this, this is conjugate of this, this is this is reciprocal of this, this is reciprocal of this. If you take out this four factors now; 1 minus a z inverse 1 minus 1 by a z inverse and then 1 minus a star z inverse 1 minus 1 by star z inverse you will see is a linear phase system. Obviously, if you multiply them coefficient will be real, because if there is one look at a there is one look at a star. If you multiply them we will have a second order polynomial real coefficient. If got 1 by a 1 by a star, so 1 by is conjugate is 1 by a star. If you multiply these two you will have a second order polynomial with real coefficients and they multiply the second order polynomial so real coefficients you get a (Refer Time: 26:24) polynomial in real coefficients.

So, the product will guaranteed to have real coefficients that are done. But now phase wise if I take up these two together it is again, you have seen earlier and it is just z inverse two and it is. This is one sub system, this is one sub system and they are cascading of that two what is a impulse response here 1 minus and 1, and here 1 minus a star plus 1 by a star 1. The second order the total term is 3, length N length is odd so there is a mirror here this is a odd man out and from the mirror if you go to the left you have one to the right one, so it is symmetry. So this will be linear phase, this will be linear phase by the same way.

So, in terms of in the DTFT domains this might be have one h 1 e to the power j omega which will have magnitude part into phase will be some constant k one omega, and this will be giving raise to one h 2 e to the power j omega which will have a magnitude part these phase some k 2 omega if you multiply that 2 because there cascaded same you know this magnitude will be product of the two and phase will be added. If going to be continuity to be linear phase. So, if I have to very large order linear phase if I had filter a real coefficients I can factorized the transform function in to first order factors and then take out this what do gets and form sub systems which will be themselves in linear phase.

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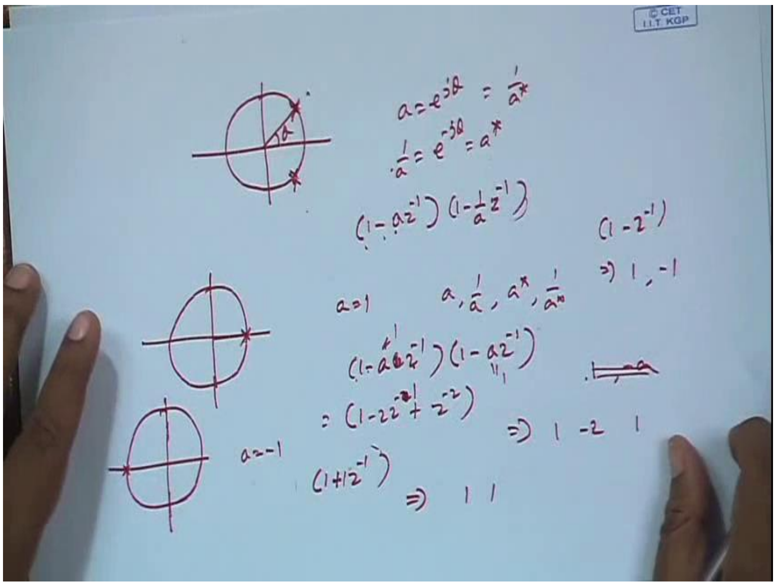

Now suppose this a is having magnitude 1, in that case what will happen suppose a is having magnitude 1 here. So, a is just e to the power j theta but this is theta, then a 1 by a e to the power minus j theta which is same as a star and 1 by a star e to the power j theta, so this is again this. So, a is same as 1 by a star 1 by a same as a star. In this case you see it is enough if you have just. First if you multiply a two since 1 by a is a star 1 minus a z inverse 1 minus a star z inverse, so we have to will be a second order polynomial with real coefficients because it is a it is a star 1 by a star. And earlier we have seen that you have a and 1 by a then the reality filter will be linear phase.

So, you can here take out just two. In fact, is pick out just these two if there will be two poles only, because whatever is conjugate there is also reciprocal so there is no four points. If this is one pole it is reciprocal as well as conjugate will be these and vice versa, so will have just to two roots not poles to 0s not 4 when they are lying on a unit circle. If you take the two make a cascade it will be made having both linear phase property and also since 1 by is a star it will give raise to a filter of transfer function that has all real coefficients.

Some most special case (Refer Time: 30:23) a special cases suppose you are here that is a is just one in this case a is 1 means a 1 by there is 1 there same a star because this is real and 1 by a star they are all same. So you have only one term, 1 minus a e to the power a z inverse. Now, this is first order but this is as a real coefficient, that part is satisfied, but is it linear phase answer is no. Because it is a length two filter; 1 and minus a, and 1 is not same as minus a. Length 2 even this would have been same they are not same. Therefore, in this case if a is here then there must be you see there will be one more term that is this 1 minus a z inverse should repeat, because then what happens you will have a is 1. So you will have 1, I mean a is 1 right this equal to 1 this equal to 1. So, 1 minus z inverse 1 minus z inverse 1 minus 2 z inverse 2 sorry, 1 minus 2 z inverse 1 plus z inverse 3. So, this filter now is having again real coefficients and impulse response 1 minus 2 1.

So length 3 because a mirror here this minus 2 is a independent guy to the left you have 1 to the right you have 1, then it is linear phase. So, in this case if the 0 is here you cannot have just this 1 minus z inverse come alone there must be 1 minus 1 by z a z inverse. So, that the product is linear. On the other hand if you have here then a is minus 1. So, earlier what we had 1 minus a z inverse a is 1 so 1 minus z inverse.

So a filter was 1 and minus 1. So, they are not same length 2, but they are not same. But in this case you will have 1 plus z inverse because a is minus 1 to this amongst to us lengths two filter with impulse response this 1 and 1 from here 1 1. So, 1 1 and this is symmetric length 2 there is a mirror in between there. In this case you have you can have only one factor like this. That is all for the linear phase system.

Thank you very much.