Discrete Time Signal Processing
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Lecture - 27
Linear Phase Filters

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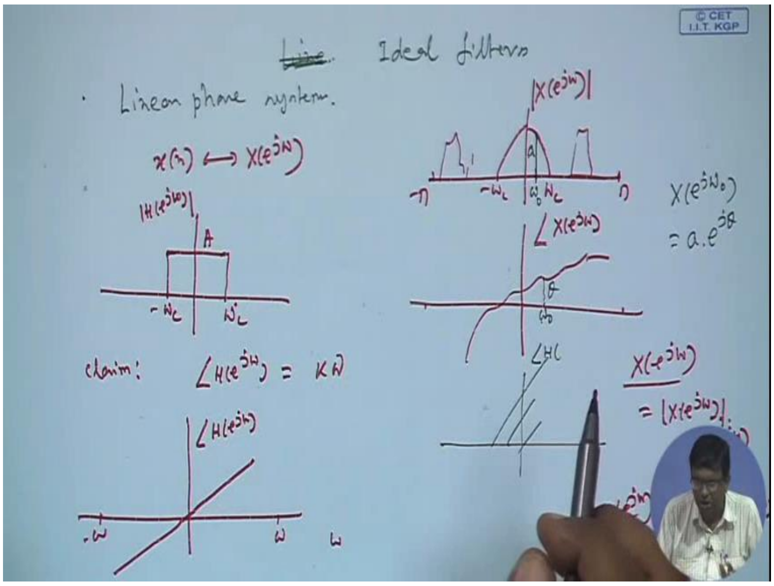

So, today's topic is another very important topic Ideal filter. Under this actually, we will come across a topic which is the main focus is called Linear phase systems, but before I tell you what is Linear phase system, suppose you have got one input sequence x N and it has got a discrete time Fourier transform this. So, this may be within a band may be something like this, within some band, outside the band up to pi N minus pi there may be unwanted signals here you know unwanted say noise and all that. So, you want to filter it out. So, if you want to filter it out, and this is I am writing x e to the power j omega, but here it is very important.

To recall that you cannot plot a DFFT just like that; you can plot a magnitude, which is actually this case, and you have to have a separate phase plot. Like this should actually the magnitude and there will be another phase plot earlier I used to mention this, but I used to carry out with one plot because that was ok for explaining my point. But in this case, I have to make use of this fact that actual plot of the DTFT means magnitude versus frequency plot and phase that is angle. Angle could be anything you know suppose angle is like this from minus pi to pi. That means, at any frequency suppose at this frequency, height magnitude is say a and angle is theta at the same frequency, then suppose the frequency is some omega naught say omega naught that means, at omega naught the DTFT that is x e to the power j omega naught will be magnitude a times e to the power this j theta this is the meaning of this. This is the how to connect these two to get the actual DTFT.

So, suppose this is given to you, but nevertheless this is band limited within some band outside the band there may be unknown there may be you know I mean unwanted components noise and all, so you have to filter it. Filter it means you want to pass this signal as it is without any distortion and remove all components, which are outside this band. So, then filter magnitude response then what should be the filter. Firstly, from here to here, say omega c to minus omega c, its magnitude should be constant; normally instead of showing magnitude, we write full thing DTFT and that is constant, but what will happen to the phase then that time we normally assume phase to be 0. So, e to the power e to the power j 0 is 1 so, but why should the phase be 0, because when you design a filter you cannot guarantee that phase will be 0, there will be phase as a function of omega.

So, what should be the phase, magnitude should be constant, so that mod x e to the power j omega into mod H e to the power j omega will be just capital A times this shape this shape will remain intact capital A can be represent to be 1 also. But how would the phase because this is what this is you know x e to the power j omega is mod into e to the power angle. Same for H, H e to the power j omega is magnitude at any point times the corresponding angle sorry this is x, very sorry this is x. So, when you multiply when you pass a signal to the filter or transfer function H e to the power j omega output will be the product of 2 DTFT. So, output magnitude will be mod of x e to the power j omega into mod of H to the power j omega; and phase e to the power summation these two phases will add. So, summation phase will be added and magnitudes will be multiplied. Now magnitude response is constant. So, mod x e to the power j omega into A within this band, so it will be a times this shape will be constant shape will not change.

Phase, question is what is a phase. Our claim is this angle for perfect reconstruction that is for output to be same as input may be with the delay, but otherwise same as input. Our claim is that it is not enough for the magnitude only to be constant within this band you have to tell about the phase. So, the phase claims that this should be just a constant multiple k times omega, a straight line that is maybe I can draw on this side because this is magnitude, phase. If I plot versus omega it should be some k omega k, if k is positive slope will be positive; k is negative slope will be negative like this. This is the claim if phase is also like this then our claim is that output of the filter will be same as the input of the filter that is x N may be x N will be delayed somewhat. But that delay is ok, delay is known delay does not change or shape of the waveform, these are claim will easily verified.

But one thing remember if k is 0, then phase is 0 that is we often do that is we just saw H e to the power j omega we do not take mod and by that we assume we are considering the special case of 0 phase, but actually why should the phase be 0. That is why we in this case we are considering phase in a more generic case claim and we have claimed we are saying that your digital filter should have not only magnitude response flat within the band, but it is phase also should be linear in omega. It is a linear function k times omega straight line going through origin.

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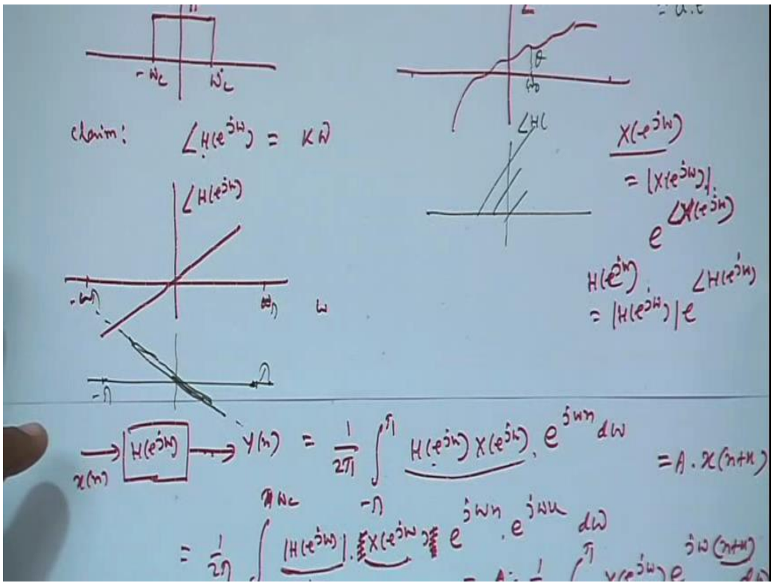

Why, now we can easily see here your x N here is your transform function H this and here is your output y n. So, output Yn will be what inverse transform of output DTFT what is output DTFT? Product of the two DTFTs output DTFT and e to the power j omega n, d omega N is coming from here your choice. This is the formula we have seen it earlier also the two DTFTs are multiplied, there is output DTFT output DTFT that time e to the power j omega d omega minus pi to pi. And now if you carry out this product, it will be mod e to the power this you keep as it is, this I keep as it is. So, e to the power j omega N and e to the power j and this as e to the power j k omega that is omega k that is for the system transforms function capital H e to the power j omega is mod of omega into e to the power j e to the power the angle. And the angle is k omega or omega k so that I can make, and d omega and this is flat at limit actually is from this band minus omega c to omega c, but you can still keep minus pi to pi because outside that band this is also 0, this is also 0.

Similarly, is band limited within this outside 0, filter also designed like this that outside this band is 0. So, integral still you can keep from minus pi to pi because it will not change anything. It is constant to ax, but this DTFT this is 0 outside the band. Even if you do not consider from minus omega c to omega c you from minus pi to pi, this fellow will be zero in the range. So, integral will not you know will not change. So, at this is constant A, what this band. So, it will be your A times 1 by 2 pi minus pi to pi that is you can even keep omega c to plus omega c here and when you take a out and then this integral you can again come back from minus pi to pi because you have inside you have got x e to the power j omega and that is 0 outside this range minus omega c to omega c.

So, no point even if I exchange the integral from minus pi to pi outside this interval minus omega c to omega c this is 0. So, it will not cause any change integral whereas, e to the power j omega N plus k, so this is the inverse transform of x N at time index N plus k. So, what will you get is this capital A times original x N plus k. So, you get back x only after a delay here k is positive, but I want the filter to be causal in that case k must be negative. So, actually the slope should be for all causal systems this is the phase value goes down like this, from minus omega to omega or from minus pi to pi you can take, k should be negative. So, actually k is a negative number.

So, in that case x n plus k n minus something, so filter is now output is nothing but delayed version of original input. So, it is causal also that is why for all causal systems, phase is actually having downwards slope not upward slope, but what we are getting is output is nothing but input, thus amplified or attenuated by a constant factor A, A could be chosen in the filter to be 1. In that case not change in amplitude and it delayed by some constant amount. The k is known to be, because I designed the filter. So, I know what k is. So, if I know x n plus k. So, k is (Refer Time: 11:40) minus 2. So, I know what the value of k is minus 2. So, it will A into x N minus 2, I also know A. So, from A into x N minus 2, I can easily find out what is you know x n, N minus 2 and all that because I will get the entire sequence.

So, it will be just shifted version of the original by two original sequences and the input output will be shifted by two in case minus 2, and amplified by A, attenuated by A, A is known. So, I can get back my original that means, if phase is linear, for causality the constant k in the phase, it was linear phase the constant factor cosine factor should be negative. So, under that case, the filter will be able to reconstruct the input x n perfectly just, but for a delay and amplitude factor amplitude factor is coming from the magnitude response it is in my hand. So, I know what is z and delay is also in my hand because it is coming from the phase, phase is linear, k is the constant k is linear constant.

Now, how to construct such filters, which will guarantee that phase is linear, magnitude may not be fully flat may be little wavy type and all it is ok may be like this you know something like this. May not be fully constant, but which is ok, but phase is linear. We will one technique and that is only applicable for FIR filter, if you have only for FIR filters can be made to be linear phase no IIR filter till date, you know have been able to be made linear phase. That is we say that when is IIR filter as of now as of today there is no technique by which you can design IIR filters that are linear phase, but for FIR filters you can impose some condition on the FIR filters coefficients and by that you can make sure that the filter phase response is linear in omega. So, our consideration now, now next topic will be linear phase filters which are actually FIR filters.

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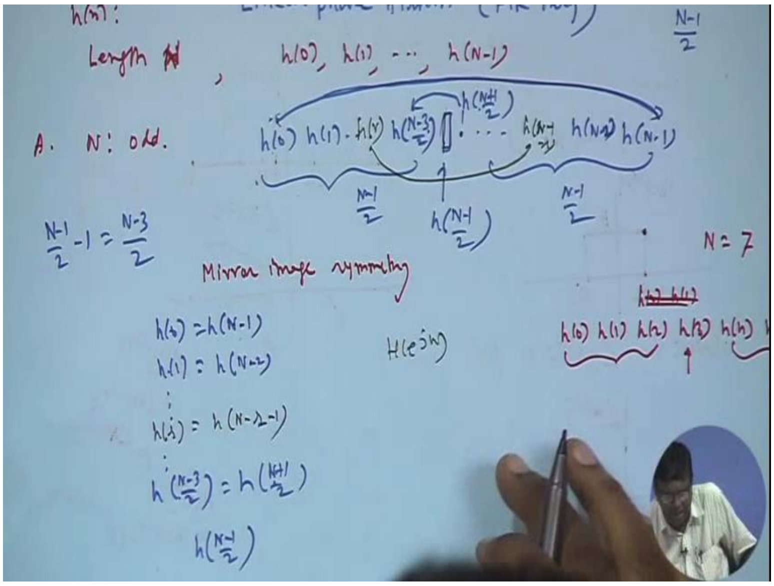

So, Linear phase filters FIR only, IIR filters cannot be gain made to be you know I mean cannot give any guarantee that IIR filters will be you know giving linear phase response. But FIR filters yes we can make sure in our design process, we can impose some condition on the filter coefficients, which will guarantee that phase is linear. So, we will consider a filter length n, h N filter impulse response let it N that is where we have got h 0, h 1 dot, dot, dot up to h N minus 1. So, you are considering an FIR filter whose impulse response is h 0, h 1 up to h N minus 1. So, total number of proficiency is capital N length capital N sorry length capital N. Two cases are possible length capital N positive sorry length capital N odd or capital N even.

Suppose we consider case a N odd, then what happens is we have got h 0, and h N minus 1; h 1 h N minus 2 or may be, let me see how to explain this, total we have got N coefficients right, N is odd. Suppose, so I take one out then N minus 1 that is even and N minus by 2 then that means it is integer. So, what happens there is a central guy; and to the left, I have got N minus 1 by 2 terms; to the right, I have got N minus 1 by 2 terms and this guy, so N minus by 1 by 2 N minus 1 minus 2. So, the total is N minus 1 plus one total N. So, when N is odd we have got term at the center and to the left I have got N minus 1 by 2 term to the right we have got N minus 1 by 2 terms.

Like suppose N is say 7, then I have got h 0, h 1, h 2, h 3, h 4, h 5, h 6, 0 to 6 that is 7 and you see seven means if you take one I out 7 minus 1 – 6, 6 by 2 - 3. So, you take 3 on this side, 3 on this side and this is the center guy. So, I arranged the coefficients like this h 0 h 1 question is what will be the this guy 0, 1 total N minus 1 by 2, so it will be h dot, dot, dot h N minus 1 by 2 minus 1 is not it; 0, 1, 2 dot, dot, dot total will be how many terms N minus 1 by 2 terms. So, if you start from 0, and counting 0 1 2, so it will be N minus 1 by 2 minus 1. So, if you have N minus 1 by 2 minus 1, this is N minus 3 by 2. So, it is N minus 3 by 2 then what is this guy this guy is a h, next guy, so N minus 1 by 2.

If you want to add one to it you get back N minus 1 by 2. And then here again you have got next term, so you add one to this N plus 1 by 2 all are increasing by 1 0 1 2 3, So, N minus 3 by 2 plus 1. So, N minus 1 by 2 that was this guy this index plus 1. So, N plus 1 by 2 is this guy, dot, dot, dot, dot up to h N minus N minus 2 h N minus 1. So, I am just doing nothing, I am just writing all the terms in this manner, but I am identifying the central guy separately and this one set of points, one set of coefficient total N minus 1 by 2 number N number one side another N minus 1 by 2 coefficient on the right hand side.

Now, for linear phase, we will impose something called Mirror image symmetry that is we will assume that as though there is mirror here. So, mirror here means this guy the equidistance, so they will be same. That is h 0 is same as then h 1 is same as h N minus 2 dot, dot, dot, it will take a general case h r, it will be see 0 N minus 1, 0 N minus 1. If you go 1 N minus 1 2; if you go up to h r, it will be N minus r 0 N minus 1, 1 N minus 2 or N minus r minus 1, so N minus 1 0 N minus 1 minus 0, 1 N minus 1 minus 1 that is N minus 2, 2 it will be h N minus 3 that is a minus 1 minus 2 or N minus 1 minus r.

So these two, they will be same; this is the condition we are imposing. And the central guy will be as it is. So, dot, dot, dot, dot h these two will be same h N minus 3 by 2 is same as h this guy, N plus 1 by 2 this is the thing I am imposing and the central fellow will be independent guy N minus by 2. So, this is same as this, this is same as this, this is same as this, this is same as this and this. This is a mirror image of symmetric condition, I am imposing on the filter coefficients that are filter coefficient are designed from frequencies point of view under this condition that this coefficient I am taking to be same. If so then what happens to the filter transfer function h e to the power j omega, we will see it is phase will turn not to be linear in omega, how we will see this now.

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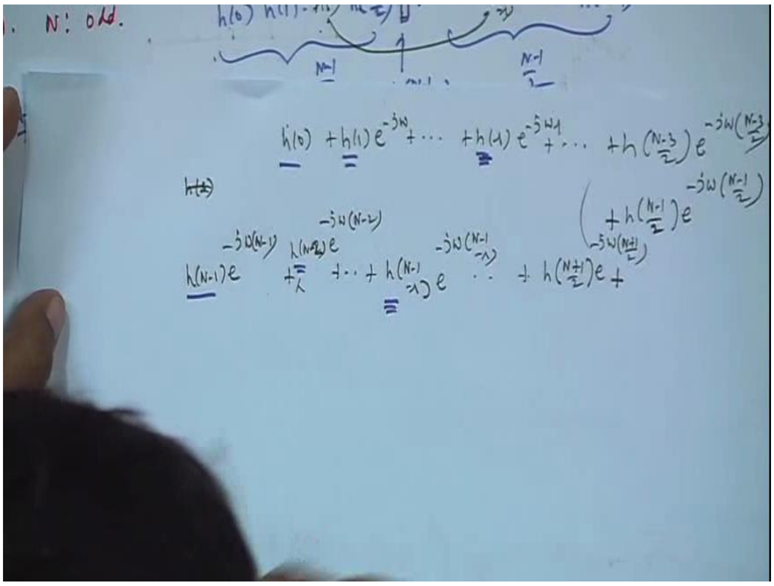

So, actually there we have got h 0, when you take DTFT, you will have h 0 plus h 1 e to the power minus j omega, dot, dot, dot h r e to power minus j omega, dot, dot, dot up to to this right. So, in this total DTFT summation, you have got one see h 0 plus h 1 e to the power j omega, dot, dot, dot h r e to the power minus j omega r plus, dot, dot, dot plus h this term e to the power minus j omega this N minus 3 by 2 up to this. Then you have got this middle fellow; and then you have got counterpart of these counterpart of these which are same that is N plus 1 b y 2 e to the power minus j omega plus, dot, dot, dot, dot counterpart of this it was h r it was h N minus 1 minus r, dot, dot, dot. Finally, the last term, I am writing in a fully expanded form the DTFT.

And now you see these two terms are same by my imposition h 0 is h N minus 1 then this term I am not drawn written it here I could written h N minus 2 . So, these two terms are same h r N minus r 1 minus r this guy this guy. These coefficients are same because of this mirror symmetric condition I have imposed h 0 is same as h N minus 1. So, h 0 h N minus 1 they are same I can write it as h 0 only h 1 is h N minus two. So, h 1 h N minus two this I can write as h 1 only; h r is same as h N minus r minus 1. So, h r h N minus r minus 1, so I can write this also h r, dot, dot, dot, dot.

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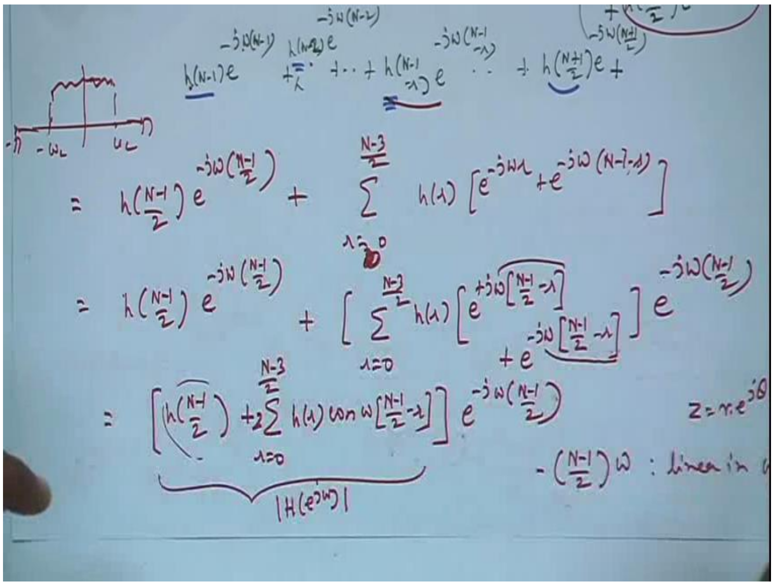

So that means, this summation this can be taken as common h 0, h 0 here; h 1, h 1 here; h r again h r here because they are same. So, I take that common and therefore the summation will become consider the general case h r and this and this. And the single term fellow I write first separately out this independent guy this I write separately out and then I take add them I take h 0 h 0 common, h 1 h 1 common, h r h N minus r common. So, general term I consider h r, I take common. So, it is e to the power minus j omega r here and it is e to the power minus j omega N minus 1 minus r not. So, r is 0, 1, r up to this. So, r is 0 up to N minus 3 by 2 r 0 this case these two are common. So, you have got e to the power minus j is 1. So, one coming from here and r 0 e to the power minus j omega N minus 1 coming from here like that general case, this is a situation.

Now, I will do some just simply manipulations. If I take this thing common, so e to the power minus j omega like this, this factor if I take common out h r. So, e to the power minus this taking out, so I have to add e to the power plus j omega this here. So, e to the power plus j omega minus r, because it was minus j omega r, so minus j omega r, and since I am taking these out e to the power minus j omega N minus 1 by tau by 2 out to cancel it here we have e to the power plus j omega into N minus 1 by 2 there. And here already I had N minus 1 e to the power minus j omega into N minus 1 out of which e to the power minus j omega N minus 1 by 2 has gone.

So, another N minus 1 by 2 is still left. So, only this will be there minus r because e to the power minus j omega N minus 1 was present from that I have taken out e to the power minus j omega N minus 1 by 2. So, another N minus 1 by 2 is here. And what is this e to the power j, if you call it theta e to the power j theta plus e to the power minus j same theta you see this and this are same. If you call this omega into this bracketed quantity equal to theta then omega into same thing here theta, so e to the power j theta plus e to the power j theta which is twice cos theta. So, then this factor can be taken as common, this will be twice h r cosine theta, theta is omega into. So, you see this is real, this is real, this is real, and so there is no phase as such.

So, this is like the magnitude like any complex number j it is r e to the power j theta. So, this magnitude part e to the power j minus omega this, this is the angle part. So, what is the angle, what is the phase, phase is minus times omega N is odd. So, odd minus 1 that is even, even by 2 this is integer minus integer times integer omega minus is coming because of causality I have told you it has to be minus negative slope. So, phase is linear and this part is the magnitude, but there is lot of quotient, this is cosine, cosine omega into you know this part changes from r to r, r equal to 0, one value r equal to 2, another part like that what are the value like that.

So, this whole thing may become negative sometimes for certain value of parts suppose. If this whole thing becomes negative then negative means these is a pi phase shift. So, definitely that time e to the power j pi will get added and the magnitude part of that will remain here. So, then phase will no longer be linear there because a constant thing will get added this was you know it was coming down like this, suddenly it will get added it will go up like by some pi and again it will start falling. So, it will not be linear, so that is why we then try to design this filter coefficients such that within the pass band minus omega c to omega c, this does not changes its sign number one and then it is value which is a positive.

Now, it is not changing it is sign from negative it is not turning out to be negative it is value also does not fluctuate much it is kind of constant like you know if this is your minus omega c to omega c, and this pi here N minus pi there it designs the filter. So, what this band design the filter. So, what this band you know it does not change, does not alter much then it is linear phase. Then it is linear phase and it does not distort the input signal because magnitude responses certain kind of constant it is not undergoing any sign change within the pass band.

So, if the signal of that band limited signal within this band confined to this band is passed through the filter. There will be no phase distortion it will delayed by so much N minus 1 by 2 that much delay and magnitude response also will not undergo much change. This kind of constant I will design coefficients like this that first this whole thing does not change it is sign does not become negative within the pass band of the signal. And also its value remains kind of constant does not change much that is part of the design of the magnitude part. So, this possible if the FIR filter only you will get a linear phase. Next will be the case of N equal to even, even case also can be direct dealt to this similar manner will take it up in the next class.

Thank you.