Discrete Time Signal Processing

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Lecture - 25
Noble Indentifies

Today I will discuss two important identities in mutilates signal processing.

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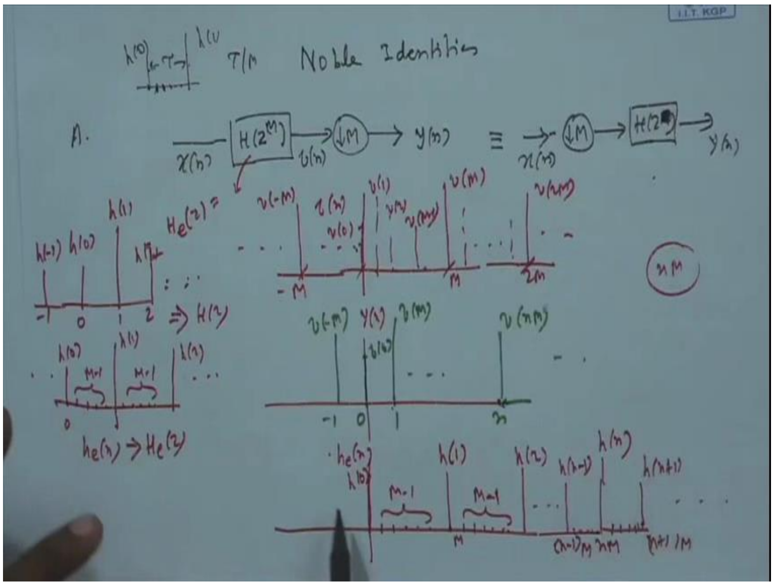

These are called Noble Identities, there are two noble identities – a, suppose there is a sequence x n it is passing through a filter which is not H Z, but H Z to the power M which means there is an original H Z in time domains small h n that is expand it by m. So, transfer function becomes H Z to the power M expanded means between every pair of samples M minus 1 0 were placed M is greater than equal to two that kind of filter. After this filter, if I suppose have the decimeter by the same factor, M let us call it y n and let us call it v n. Then the claim is, this is equivalent to first to decimate then pass it through a filter naught H Z to the power M, but H these two we will get the same y n this is a claim - these the noble identity type one.

But before I give a proof see the advantage of this structure here H Z to the power M. So, between every pair of samples, so H 0, H 1 there are n minus 1 0s – so this is 1 sampling point, this is another sampling point, this is another sampling point. So, clock period is if totally tau it is tau by M. So, clock is high. So, these filter M and therefore the input,

input is what we get the same clock at as that of this filter. So, x of 0 comes, x of 1 comes, x of 2 comes, x of 3 comes, (Refer Time: 02:18) these are, this system is working at a high speed, so it is costly and then you are decimating. So, this is a low speed sequence it is period is tau, but here x n and H Z to the power M they are clock period is tau by M. So, they are operating at a high speed and therefore, this system is costly, but see the equivalent system x n was as before it is period was tau by M, but first I am decimating it.

So, it is becoming slow, its period is becoming just tau and then sorry this is H Z I am very sorry, this is a H Z and then you are passing it through the normal filter there is H z, H z means original filter, so H 0, H 1, H 2, that is tau. So, this will be a filter working at a low clock, lower clock because it is an up a decimation. So, this will be a less costly. So, by this process you can convert a costlier section in to a super section.

Now, I have to prove it. Now you see y n is decimated version of v n; that means, if there is a sequence v n. So, v 0 may be you have got v 1, v 2, dot, dot, dot, dot v M minus 1 and after that you have got v M then again intermediate samples; there are 2 M at 2 M have v 2 M. So, then minus M you got v minus M, dot, dot, dot, dot this is v M sequence afterward decimate we get y n, but what is y n? y 0 will be v 0, y 1 will be arranged here, y 2 will be 2 arranged here, while minus y will be minus arrange by

So, it will be at 0th point it will be v 0 coming at point number 1, v M coming at point number minus 1 v minus M coming; that means nth point will be v M into n n is 0, so v 0. So, at n equal to 0 v 0 n equal to 1 v m. So, at n equal to 1 v M like that. That is what I am saying 0 is sample then m-th is first, two m-th is second third, three m-th is third n m-th is nth all right. So, this is your y n. So, I have to find out y n here. So, y n is nothing but these v n down sample you have decimated. So, y n is after decimation of v n is this sequence. So, general sample of this sequence is what I have at nth point, nth point have got v n M that is a general sample - that is v a M, v 2 M, v 3 M, v n M. So, what is v n M let us find out, before that what is v n right.

I remember out of v n I am not interesting all these all these points I am only interesting in either here or here or here or here that is n m-th points.

Because n m-th sampled becomes nth sample, n m-th, n m-th sample becomes nth 1 into m-th becomes first, two into m-th becomes second like that. So, that is why n into M that is importantly for which because only they come in the y n sequence right, but then I have considered this filter this is a convolution between x n and H Z to the power M, but what is H if you call this filter as H expanded version right H e z which is H Z to the power M. So, what is impulse response of this filter? I am call it expanded form, so H e z. So, originally there was an h n then expand it originally there was an h n - H 0, H 1, H 2, dot, dot, it has a z transform capital h z that is H 0, H 1, H 2, dot, dot, dot, dot, these as an z transform h z now you expanded it here you have got. So, H minus 1, we expanded. So, this is 0 this is h H0, H 1, but you have got M minus 1 0s then again M minus 1 0s then I have got H two, dot, dot, dot, dot, dot.

This sequence is the expanded sequence and I call it H e n expanded version of these it has a (Refer Time: 07:17) great transform H e z what is H e z? We have yesterday seen it is H z, but not z n z to the power m. So, this H Z to the power M is call H e z it is simple response is H e n which is the expanded version of this original H n. So, you can draw it like this, H e n means first we will have H 0 then you have got 0s then m-th sample is H 1 then again you have got M minus 1 H 2, in general n m-th, n m-th will be h n, n minus 1 M will be h n minus 1 n plus 1 m, so there are 0s in between, there are 0s in between all M minus 1 0s this will be h n plus 1, dot, dot, dot, dot this is a expanded version with this I have to convolve x n.

So, how to convolve x n graphically is there linear convolution I hold H sequence as it is, flip the x n sequence I have flip the x n sequence.

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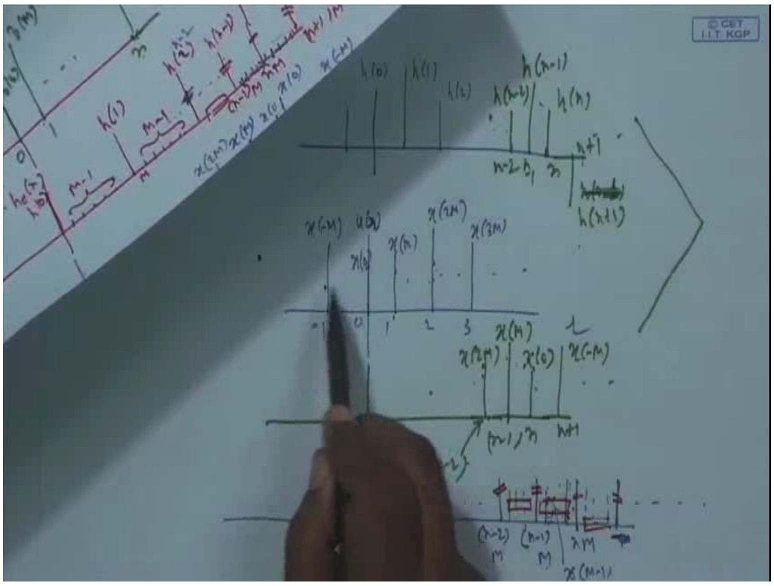

And I am interested in finding out output not at any arbitrary point, but n time's m. So, either capital N or 2 n or 3 n like that because n M n m-th here will be nth here.

This is y n. So, y n at nth point is v at n m th point. So, n m-th points are of interest that is n m-th, 2 nm th, 3 nm th like that. So, I am interesting generally n m. So, I want to find out the output at n M that is here. So, what I have to do how to do how to calculate convolution? I hold this H sequence as it is flip the other sequence. So, H 0, H 1, H 2, H 3, dot, dot, dot and x minus 1 minus 2 minus 3, dot, dot, dot these I shift by n M. If I shift since you know graphical convolution at n m th point will come x 0 then I have got x 1, dot, dot, dot x then again at n minus 1 into M at this point will be 2, x M because I have got capital M minus 1 0. So, capital M minus 1 point share x 1, x 2 up to x M minus 1 this is x so, x 0, x 1, x 2, up to x n minus 1 they meet zeros from here because M minus 1 0s there are n minus 1 into M at n minus 1 into M who comes n minus 1 into m-th guy because at n M x 0 has come it is flipped. So, x 1 was the earlier to the right it has come to the left then x 2 then x 3 how many points? First M minus 1 points correspond to M minus 1 0s.

Then one more point. So, that point is n minus 1 into M that is nm minus M. So, they are x M comes there again some samples, I do not care then yet again this point next point suppose n minus 2 into M who comes x 2 M, dot, dot, dot, dot. Around this side some intermediate samples then here who comes x minus M from the left hand side because M minus 1 sample x 0 x minus 1 minus 2, so total n minus 1. So, after that here we will have x minus m, dot, dot, dot, dot and there you carry out sample wise multiplications.

Now, you see at nth point n m-th point I have got data multiply, but next these block is meeting these 0s here n minus 1 n minus 1 0s. So, then would contribute anything then again these with these then again these block and these block they multiply, but these are 0. So, they do not want this fellows they do not contribute anything. So, then again these, these, again here this block this data, but here you have a 0. So, they do not contribute anything forget them then this with this, with this and this get added and if you add what you get is the following x 0 h n, the left hand side x 1 h n minus 1, these with these x sorry x M these x M not x 1 I am very sorry x M h n minus 1, x 2 M x 2 M, next will be h n minus 2. So, this is n minus 2 actually, dot, dot, dot, dot n minus 2. Earlier I wrote from this sides 1 2, but I now coming from this sides, so these be n minus 2 adjacent to this. So, x 2 M h n minus 2, dot, dot, dot, dot and on the other side x minus M h n plus 1, dot, dot, dot this is a summation.

What is the meaning of this summation you see? It is a convolution between whom, it is a convolution between a sequence between these and these because what is these sequence M times decimated version. So, these sequence here if you call it u n, in fact, you can write as r, u r because I am interesting finding out y at for specific n. So, u r is decimated version. So, 0 x sample will be x 0 first sample will be not x 1, x 2, but x M. Next sample we will be x 2 M, next sample will be x 3 M, dot, dot, dot, dot and here we have got x minus M at minus 1 0 1 2 3, dot, dot, dot, dot because decimated x 0 then through as the next n minus 1 sample then take x n that is the first sample then again through away the next n minus 1 sample. Then next taken x that will be second sample and, dot, dot, dot that you we will do that is u n with that if you convolve h z what you have? This was your H original H, original H is not expanded version this original H if you convolve these two what will you get you flip this to the left flip this.

So, what you get is x m on this side x minus m on this side x 2 M on this side, x minus 2 M on this side and you have to find out the convolution now at nth index and you have to. So, this is same as this side. So, nth index means the flip side sequence will be shifted to the right by n. So, if it is nth point where you have got h n then I have got h n minus 1, h n minus 2, dot, dot, dot, and dot. So, after you flip x zero will come and sit bellow x h n 2 x 0 you have flipped. So, x capital M has gone to this side. So, now, you shift means at n minus 1 x M will come because after flipping x M has gone to this side, this side has gone to this side, then you are shifting and before x 0 here it will be x minus M then x 2 M this is at index n minus 2, dot, dot, dot, dot. Now if you convert if you multiply these with these what you will get x 0 into h n nth point x 0 into H n.

Then, x M into h n minus 1 x m into h n minus 1 x 2 m into x 2 M minus 2, so at n minus 2 is the h n minus 2. So, x 2 m into h n minus 2 x 2 m into h minus 2 on this side these n plus 1 th index x minus M H this is 1, sorry yeah this is n minus 1 this sequence is this point is n plus 1, so I will have a h n plus 1. So, h n plus 1 x minus M these and so and. So, this is a convolution right this convolution is this term which was getting at here this was historic equivalence what I did I found out y n, what is y n? If it is nth it is n into m-th here because this is decimated and then I found out in m th sample by convolving this with this, this is your original x n, these are the original x n hm.

But this is expanded version. So, expanded version is you start with original H and then H 0 and H 1 between then M minus 1 0 H 1 and H 2 between they get minus values expanded version. With that if you convolve x n and find out the output at n M, how to do the convolution hold this 1 sequence, x sequence expanded H version H e n as it is take the other sequence there is x n reverse it then flip it and shifted to the right by n into M because I am interested finding output n into m. So, n into M x 0 comes after that x 1 x two, dot, dot, dot x n minus 1 there hope this next n minus 1 points then x M and so and so forth.

So, if you now multiply and add these n minus 1 sample contributes nothing because they finds 0s here, so this into this plus this into this, and then again these n minus 1 sample contribute nothing because if find another 1 minus 0s here, so again this into this, this way you get this term x 0 h n then x m comes h n minus 1 from top. Then h n minus 2 if it is x 2 m if it is h n plus 1 it is x minus m these 0, do not contribute anything this samples do not contribute anything. So, next is x minus M then only that only finds h n plus 1 here that zeros. So, that is what I want and these I showed is equivalent to these because if you decimate x n by m the sequence you get is x 0 then first sample is x m not x 1 then second sample is x 2 m because your throwing a intermediate samples n minus 1 sample then third sample is three m, dot, dot, dot and this is your h z means original H sequence.

Now you convolve the two, reverse this x and shifted to the right by n x 0 finds h n because there are look at n M s. So, x 0 into h n then x 1 there is no x 1 x 0 x M, x M was on the after flipping it went to the left side it has now come before the left of x 0. So, this is x M x M into h n minus 1 because these are at n minus 1 th point.

So, x M into h n minus 1 then before these who comes this was going to this left this was into minus 1 this was into minus 2. So, there it is upper shifting x two M this guy has come that will find sampled at x n minus 2 because that is what they are located. So, you have got x 2 M into h n minus 2, dot, dot, dot around this side before x minus n that came to the right of x 0. So, if it is x 0. So, here is x minus n by that is at the next point in plus 1 th point an H h n plus 1, so h n plus 1, h n plus 1 x minus 1 and dot, dot, dot. So, that is you get these expression right. So, this proves identity 1, and then we go to normal identity two.

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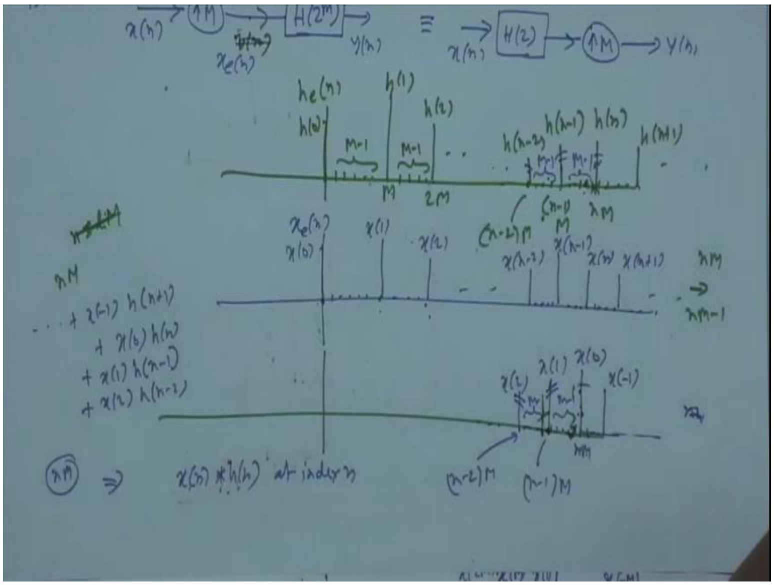

Suppose there is sequence x n I first explained it by M that is be in (Refer Time: 21:12) samples I bringing M minus 1 0s then pass it through a filter of this kind. So, again, but this is low speed, but here after this introducing this M minus 1 0s in between a go to sample this is high speed. If the original time period was tau between two samples the time period was tau, now it is tau by M here and this filter is working at that rate. So, this is costlier side these by these identities equivalent to these that first you convolve with x n with just h z only not H Z M. So, it before expands and whatever it was. So, it is at low rate filtering is done at low rate which is less costly and then again we explained by M. We will get the same thing, if you have y n you will have y n here this is another identity this also been easily proved. Suppose I consider this these two is a v n, v n is expanded version and what is this filter we have already expand it. So, this filter is H you can call it H e z equal to this. So, x bar is impulse response is expanded version of this.

So, H 0, H 0 0 0s, M minus 1 0s when H 1 these are m-th point 2 m-th here you have got 0s H 2, dot, dot, dot, dot, on this side in n m-th these h n 0 0 0 n minus 1 times M to begin 0 0 0 because n minus 2, dot, dot, dot, dot this is a filter and after we expand; after we expand the x n x n also like this instead of v n let me call it x e n because we already know how to represent what is the notation we are adapted for expand that is x e n. So, do not call it v n directly x e n. So, x e n also I have got x e 0, dot, dot, dot, dot then x 1, dot, dot, dot, dot x n minus 2 x n minus 1 x n there are 0s dot, dot, dot.

So, to convolve them now I have to convolve them I hold this as it is flip this, x sequence and then shift it to a right by any n not basically n into n any n. So, after those shift see either small n could be 0 capital N there is 2 N there is an multiple of n or not multiple see either you are (Refer Time: 24:38) shifting you want to find out y n for all n. So, n can be as a 0 capital N, 2 N, 3 N, like that. So, this is the multiples of M or not multiple immediate points say both the cases you have to consider. Suppose I consider first these case where n is a multiple of M say l into m. So, I have to shift it, so n M I consider suppose instead of n, n is n M all right, n is n m.

So, have to if I shift it; if I flip it and then shift it by n M say x 0 will come here, dot, dot, dot, dot, dot, 0s will come followed by because of flip x 1 then 0s will come then x 2, dot, dot, dot, dot here will have x minus 1, dot, dot, dot, dot and if you now multiply samples of these are these what will you get x 0 in to h n then 0s; n minus 1 0s M minus 1 0s right, because both are expanded by M source M minus 1 0s y M minus 1 because I am at n m th point and expand it n into M followed by M minus 1 0s on this side and this side. So, these will be n minus 1 into M, this will be n minus 2 into M because M minus that is why this is considering with these this is considering with these, this is considering with these, like that and this 0 block, 0 block, 0 block. So, 0 in to 0 contribute nothing, so you have got x 0 in to h n then x 1, h n minus 1 x 2 h n minus 2 and on this side x minus 1 h n plus 1 plus, dot, dot, dot, dot all right.

Set n m-th position what you have is these, at n m-th position output is this, but what is your output? You see x it is simply convolution between x n and h n at index n, is not it x 0 you know original convolution for without this 0s, these 0s are not there say originally h. So, H 0, H 1, H 2 and x - x 0 is what this first sample - first sample, second sample - second sample, not m th not m th you flip it flip this and take to the right. So, x 0 into H 0 x 1 into H 1 like that. So, at n m th point not n m-th, n m-th convolution will be x and H is this right; that means, upper expansion or if I convolve the two at n into M that convolution is original convolution before expansion of the two sequences at index n.

On the other hand, it I will shift not by n m, but say 1 less than that or 2 less than that then x 0 will come under 0 this will come under 0. See here I have shifted by n M to x 0 has here nicely come at n m-th point here it fight another nonzero fellow from here. What is about? Instead of n M, I have n M minus 1 see say x 0 would have come here, this here if it is x 0 this points 0 here, at this point is 0 here, then these 0s this would be have been 0 this would coming here. So, these 0 multiplies these, this 0 multiplies this sample get multiplied by 0 like that. So, there will be, total will be 0.

So, similarly if n M minus 2, n M minus 3 like if you are not understanding, if I am delaying by say as an example only say n M minus r, r can be 1 2 up to say M minus 1 (Refer Time: 29:10) n minus 1 n minus 2 n minus 3 like that.

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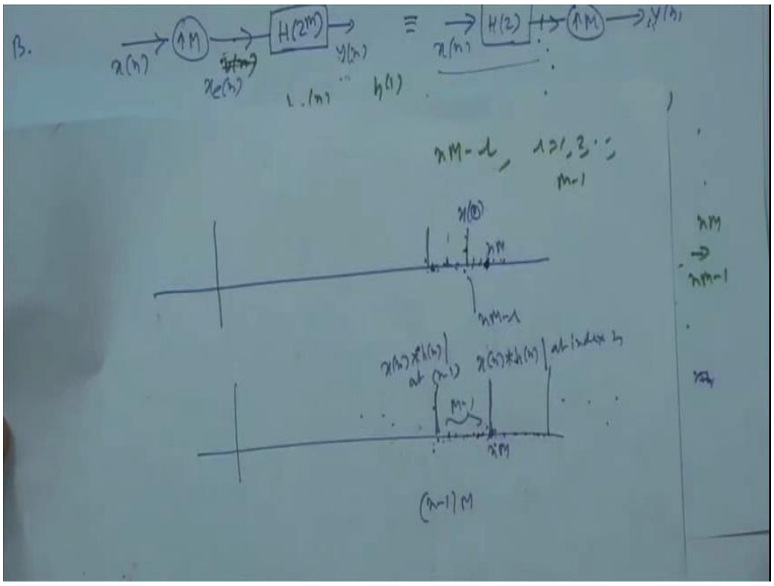

Then what will happen earlier at n M x was coming it will not both there to it come here at n M minus r. So, I will have 0s either 0s at n M these sample will find this is nm or 0 from here, this will find a 0 from here, this will find a 0 from here and this 0 block after then will be a sample that will find a 0 from there. So, like that if you see it will be have 0 because either nonzero in to 0 or again nonzero from here into 0 otherwise 0 in to 0. That means this is what? If you see this, if you convolve the two like thi, convolve the two and then if you convolve the two at nth point you have got x n convolved with h n at n m-th point, at n m-th point output is; but if I do not (Refer Time: 30:23) told you if I do not consider n M n M; see if I consider less than that like this, these points are 0s. This point output as I told you by this shift and all output is 0. So, you get 0 then again if you go to n minus 1 into M here we will get again the same convolution, this is convolution at index n this is again the same convolution, but (Refer Time: 30:55) instead of n M it is n minus 1 m. So, at this is at index at M minus 1 like that all right.

So; that means, x n h n convolved x n H an convolved this was giving M minus 1 output this was giving a nth output because n minus 1 it was n th, but now it is n minus 1 into M it is n m. So, in between 0s have come up that is why you are convolving. So, between nth and n minus 1 if your plugging in this M minus 1 0s and so on and so forth. So, basically this will be nothing but this sequence, this sequence, why this sequence? As I told you that if your shift is n M then what at index n you get if n is a multiply of M that is n M then what you get you are shifting, you are flipping these and shifting by n m. So, n M is here it will multiply h n to x sorry n M is here x 0 is multiplying this is flipped and from shifted. So, x 0 h n before that x 1 x 1 also earlier on the right side now left side. So, x 0 x n then 0s take in and 0s n minus 1 n minus 1 then x 1 h n minus 1 then again n minus 1 0 x 2 h n minus 2.

So, you get this product H x 0, h n x 1 h n minus 1 x 2 h n minus 2 plus dot, dot, and here x minus 1 H M plus 1 0s into 0 no contribution this is what you get. But M a n into m-th point this is the output is what this summation, but this summation you look at it is nothing but convolution between original x and original h n evaluated at n. So, 0 x 0 is multiplying h n. So, after flipping it is x 0 is shifted by n times to the right. So, x 0 into h n and other sample also this is linear convolution, original linear convolution between the two original sequences x n and h n evaluated at index n that will come at n m-th point.

Similarly, if instead of n it is n minus 1 m. So, at n minus 1 into M by the same logic output will be original convolution at n minus 1 because at n m-th output is original convolution at n. So, at n minus 1 n m-th it will be output will be original convolution linear convolution at n minus 1 and so on and so forth. So, that is what n minus 1 into M output is original convolution at M minus 1 and then if I do not shift by n M or n M minus 1 a for n plus n a then plus 2 M shift by something else like say n minus r - r can be 1 2 like that then what happens? As I told you that n M, x 0 does not come x 0 comes here it does not come at n m. So, x 0 comes here at n M minus r there it finds are 0 values. So, this into 0 h n on the other hand finds some 0 value then 0s. So, 1 0 multiplies h n minus 1 then the x dot are multiplies at 0 here, like that, so 0 into, dot, dot, dot into 0 and 0 into 0, so output gets 0s.

So, you will have the situation for all intermediate point this you can verify all intermediate points between n M and M minus 1 M. At n M output is convolution original convolution evaluated at index n at n minus 1 into M output is original convolution evaluated at index M minus 1, dot, dot, dot, and dot. That means, original convolution at n, n minus 1, n minus 2 after that between n and n minus 1 you fill 0s n minus 1 0s between n minus 1 and n minus two you will have n minus 1 0s like that. So, that is the expansion and you get this. So, they are expressed then these two are equivalent.

In the next class I will explain its advantage it is one some example I will give what is this is used.

Thank you very much.