Discrete Time Signal Processing
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Lecture - 23
Expansion and interpolation of Sequences

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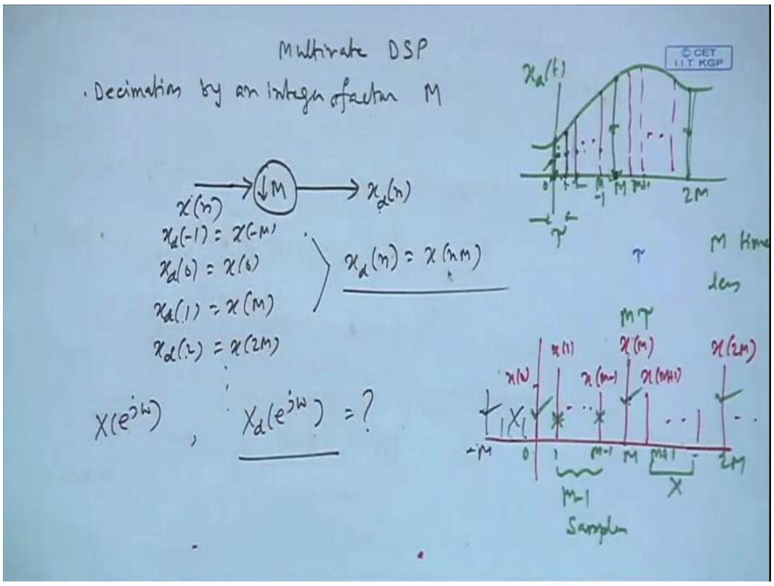

So just a quick recap of what we did yesterday. That is we are considering decimation. Decimation is suppose a sequence is obtained; just for a example, obtained by sampling an analog signal x a t. And, we had this sample; zero th sample. Then tau is this much. This is the sampling period; tau, then 2 tau, 3 tau. I did a sampling here. This is sample number one, this is sample number 2 dot, dot, dot, sample number M minus 1, then again sample, M plus 1, this is M plus 2 th dot, dot, dot, 2 M, etcetera.

Now, suppose I say that I want to bring down the sampling rate, I do not have to use sample at that high rate, that is, sampling period should not be so small; tau. It should go. It should become wider by an integer factor capital M; so that means, instead of tau, my sampling period will be M times tau period; because period going up means, rate going down. Reciprocal of period is rate, speed.

So, that is M tau. So after 0, I have tau, 2 tau, 3 tau, M tau means here; which means if we subtract this zero th sample, I will pick up the M th sample. I will discard the intermediate samples. Then, again M plus 1 th, M plus 2 th, all the samples I will discard. After Mth, I will go to 2 M th; because M tau, 2 M tau, these are the sampling instances. So that means, I will get; from the previous sequence, I will get x 0, then x M. But, x M will be sample number 1; x 2 M that will be sample number 2, like that.

Alternatively, the sequence could have been obtained just by, you know, I mean offline, you just united this data through a computer whatever, not necessary obtained by sampling an analog signal. Even then, you take zeroth sample, then discard the next M minus one sample, then take the next sample, that is, M th. Call it first, sample number 1. Then, again discard, then again discard this next M minus one samples, then see take only 2 M th guy x 2 M; call it sample number 2, dot, dot, dot. Then, this sequence to be called the sequence decimated by a factor M and, this is called the decimator; M is an integer; a minimum value is 2.

So, x n; minimum value 2 because there is no point in decimating by one that you will get back same x n, tau remain tau. So, x n decimated, you will get x d n. One thing, since I am bringing down sampling rate, there is a possibility of myself getting into the danger of aliasing; because maybe original sampling rate was above microstate. But, now I am bringing down the sampling rate making the sampling period wider. So, it is quite possible. I may go down eventually below microstate; which means it might lead us to aliasing. So, that is one thing we should be careful about.

Now that means, x 0 will pass as it is; will be the zeroth sample of the decimated sequence. Then, first sample of the decimated sequence will be M th sample of the original sequence say x M. Second sample of the decimated sequence here will be 2 M th sample of x 2 M. Minus one th sample of this decimated sequence will be the minus M th guy because we are discarding this block. Ok. Minus M th guy of x and dot, dot, dot, which means from here, you can write x d n, for any n it is x n into M, there is, x d one is x M, x d 2 is x 2 M, x d 3 is x M, dot, dot, dot. Then, we wanted to find out the discrete time Fourier transform of this x d n that is decimated sequence, in terms of the original discrete time Fourier transform.

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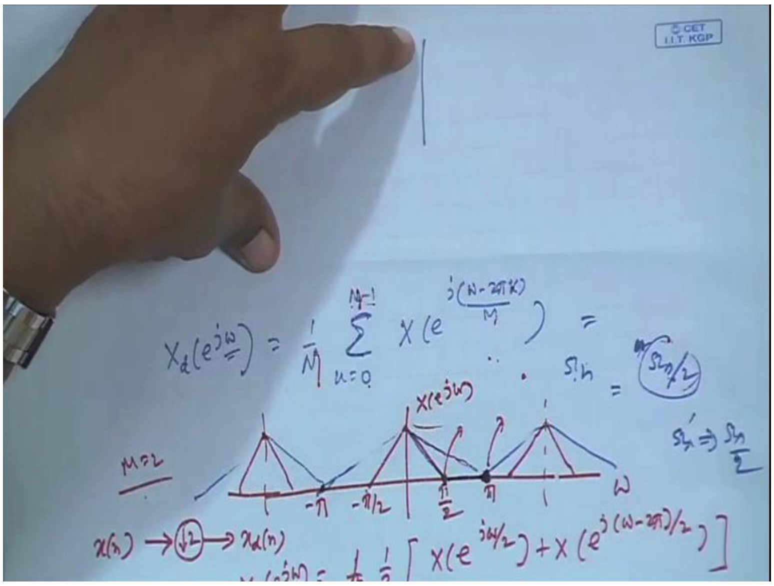

Whenever, we derived it; the derivation I will not, you know, I mean discuss about today again. I will not repeat. But that formula was this that if you are decimating by a factor n, then the d t f t of the decimated sequence at any frequency omega will be one by capital M summation k equal to 0 to M minus one. So, total M components. Capital X, the d t f t. D t f t is what? At frequency, omega minus 2 pi k by M, k equal to 0 means omega by M, k equal to 1 means omega minus 2 pi by M, k equal to 2 means omega minus 4 pi by M, dot, dot, dot. How to evaluate these? For that I yesterday gave you one problem.

And, I will work it out today. That suppose original d t f t is like this; goes up to pi by 2 minus pi by 2 and then 0 up to pi minus pi and repetition. And, I want to decimate by factor 2. So, if you have to decimate by factor 2, like here, this summation will have upper one by 2 outside. Summation will have 2 components; 1 with k equal to 0, which is e to the power j omega by 2. Another k equal to one because M minus 1 is with M 2 is 1. So, e to the power j omega minus 2 pi into 1 2 pi by 2.

Now, remember one thing. If, before I am proceed I told you yesterday, pi always corresponds to half sampling frequency. Omega s by 2 analog will map to this digital always, irrespective of what is omega s by 2. And, this corresponds to the band limiting frequency omega h. Now omega h, since pi by 2 is half of pi, omega h is obviously half of omega s by 2 in this 1. But to avoid aliasing, it is enough if omega, it is equal to omega s by 2 that this is pi by 2 goes up to pi, which means omega s by 2 is unnecessarily twice of omega h. It could have; the sampling rate could be brought down by 2, so that omega s by 2 is actually omega h. In that case, pi, if I bring down the sampling rate, that is this to the omega s, I make it omega s by four, omega s by two, is a new omega s, omega s prime.

So, new omega s prime by 2 will be pi again. But that will coincide with the band limiting frequency because originally these half sampling frequency is twice omega h; because pi is transferred into half sampling frequency, pi by 2 corresponding to the band limiting frequency because after that this is 0 and this is developed. This means, if the analog domain, half sampling frequency is twice omega h. But, this is unnecessarily twice. I can bring it down. Instead of omega s, I make it omega s prime; is omega s by 2. So, half sampling frequency is half of these; that means, that is equal to omega h. Now because omega h was the half of original half sampling frequency, now half sampling frequency has been brought down by 2. So, new half sampling frequency omega s prime by 2; that will be same as omega h. But, omega s prime by 2 always maps to pi. So, therefore omega h will map to pi. So, this would be like this. Then, no aliasing, though. This is what we should have.

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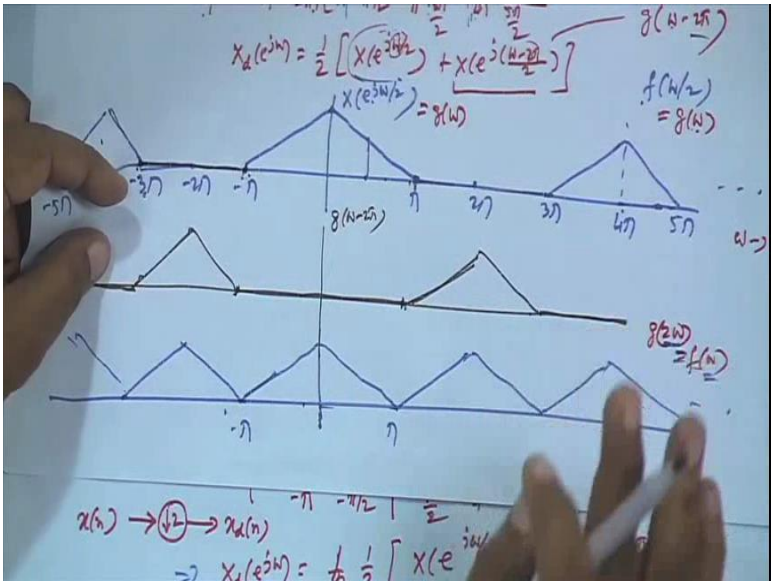

And, we will see that by this formula also we will get the same figure. That is what we will see. So, what you are given? I am rewriting. This is what is given. Ok. You have to evaluate X d e to the power j omega (Refer Time: 08:12). By that formula, we have already at this X e to the power j omega by 2; another component X e to the power j omega minus 2 pi by 2. So let us find out these, then find out these, add and divide it by 2. This divided by 2 is something I am not much bothered about.

So, let us start with this guy. Now, x g e to the power j omega, that is, some function of omega, you can make it simplified function of omega is given. So, this is a function of omega by two. It was a function of omega; e is constant, j is constant. It is; just (Refer Time: 08:54). And, actually it is a function of omega. It is a function omega by two. So, if it is called f omega, it is f omega by two; f omega is plotted, how will f omega by 2 works?

Now, if f omega by 2, to make it crystal clears to you, so that you do not make any mistake and do not get confused. If omega by 2 is not f omega, I can call it; some give it a new name; g omega. And, I have to plot g omega, that is, f omega by 2, that is, g omega versus omega; this is a task. But, g omega is f of omega by 2. f of omega is known to me. So at any twice omega, what is the value of g? g twice omega is f of omega. So, whatever happens at any omega, say f omega, suppose this is a omega and this much is f omega, these value will go to twice omega in this new plot; which means, pi by 2, this will go to pi; minus pi by 2 will go to minus pi; because whatever value I have in the original plot at any omega, at any omega, for g, that is this, same thing will happen at twice omega because g 2 omega is f omega.

So at any omega, whatever value we have f omega. For g, it will come the same thing because there is a equality; will come at twice that omega. So, pi by t 2 as 0. So, at pi it will be 0; minus pi by t 2 at minus pi, it will be 0. At every value, these, if it, this omega, this will come at, if we call these as omega, any omega. This will be come at twice omega, the same value, likewise. So, it will be like this. It will wide, get widen. Then pi by 2 has come to pi. Pi will this part; pi will go to 2 pi, this 2 pi; three pi by 2 will go to three pi; pi goes to 2 pi; three pi goes to three pi.

Then, so this pi by 2 to 3 pies by 2, there are zeros. So, these 0 values are coming here; because at any frequency omega, whatever is the value that will come with twice that, at twice that frequency. So, pi by 2 becomes pi. Pi goes to 2 pi, 3 pi by 2 goes to 3 pi. All these values are 0. So zeros here. Then, from 0 pi by 2, which is three pi here, then 2 pi means 4 pi and then 5 pi by 2 is 5 pi. So, in these zones again I have some repetition; 3 pi

2, then setup point is 4 pi twice that. Then, here they are twice that; here twice that is 5 pi, dot, dot, and dot. Around this side, minus pi; minus pi by 2 become minus pi twice that and then this minus 3 pi by 2, so that will be become minus 3 pi.

So, from here to here I had zeros. So, minus pi by 2 will go to minus pi, minus pi will go to minus 2 pi, minus 3 pi by 2 will go to minus 3 pi. And then from here to here, minus 2 pi becomes minus 4 pi, minus 5 pi by 2 becomes minus 5 pi, and we have one more such block and dot, dot, dot.

Now, you see this is not periodic over a period 2 pi because minus pi to pi, then zeros, then this. So, it is not coming here. If it is you are periodic over a period 2 pi s, whatever you have from minus 2 pi to pi that would go here. But, it is not. That is because this is not the d t f t. X e to the power j omega was d t f t, that is, periodic over pi omega over a period 2 pi from minus pi to pi, whatever you have that gets with it. But, this is just X e to the power j omega by 2, which have plotting not versus omega by 2, but versus omega. That is why this is not periodic over a period 2 pi. Fair enough. This is what I have here.

How about this function? If this was, if I call this g omega because X e to the power j omega was f omega, what is g omega? f omega by 2; that is, X e to the power j omega by 2; that is, g omega. Then, how about these X e to the power j? Instead of omega, omega minus 2 pi by 2, this will be; this will be g omega minus 2 pi because, when it was omega, g omega as plotting it was f of omega by 2. f of omega by 2 means capital X e to the power, earlier it was j omega, now instead of omega, omega by 2. So, now what is g omega minus 2 pi? Omega minus 2 pi means X e to the power j, instead of frequency omega, it is omega minus 2 pi and then by 2, as it is.

So, this is g omega. What is this? Instead of omega, you see, it is a function of omega by 2. Instead of omega, this total thing if you call g omega, then instead of omega frequency has become omega minus 2 pi. So, it is not g omega, g omega minus 2 pi. As simple as that, if this was g omega, then g omega minus 2 pi is, you know, right shifted version of these by 2 pi, this omega minus 2 pi. So, if you plot it, that is g, if you have to plot it, this entire thing will get shifted to the right. So, minus pi will come to the right by 2 pi because shifted by 2 pi. So, this will come here; pi will go to 3 pi. So, we will have here things like this. This minus pi is here and minus 3 pi, it will come to here because if you add 2 pi to it, it becomes minus pi. So, these 0 block will come here. Then, again minus 3 pi here, minus 5 pi will come to minus 3 pi because if you add 2 pi to these, you are shifting to the right, it will come to minus 3 pi. So, these blocks will come here, then 0 from here will come here, likewise, then, zeros.

If you add that 2, you will get the final d t f t; minus pi to pi and then pi to 3 pi, dot, dot, dot. This is the d t f t. And, you see this is the figure I said, you will get. It will go from 0 to pi. Again, the same thing will be repeated because half sampling frequency and band limiting frequency there are not coinciding; because I brought down the sampling frequency. And therefore, half is sampling frequency by a factor two. So, earlier half sampling frequency was twice the band limiting frequency. That is why it was pi by 2 is pi. Now, half sampling frequency is same as my limiting frequency, but half sampling frequency always maps to pi. Therefore, band limiting frequency also maps to pi. That is why band limiting ends here and then you block, again stuff like that. This is for decimation. We can work out other examples.

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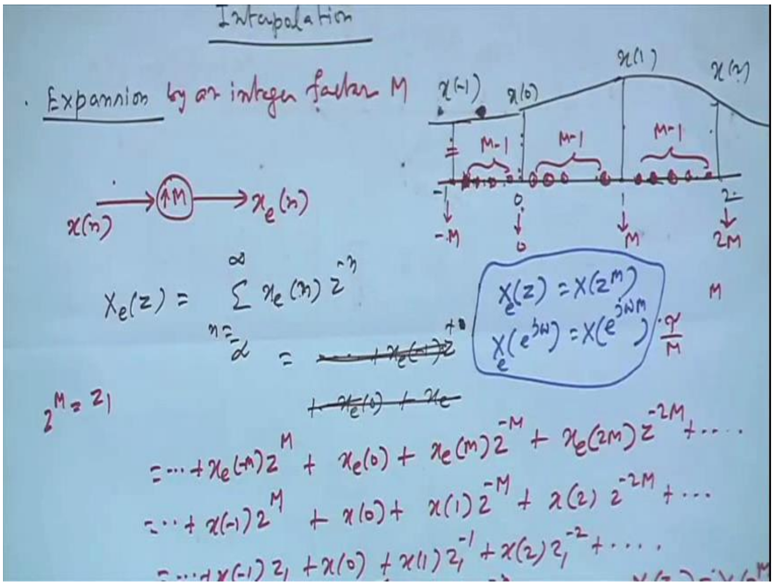

There is another very important thing that is called expansion and interpolation. Before interpolation, before that I will consider something called expansion. Expansion is denoted by say by a factor, by an integer M. It is denoted by this. You give X e for expanded X e n; it means it is a very simple thing. Suppose, originally I had x 0, zeroth sample; x 1, first sample; x 2, second sample; 2, 1, 0, now I create some intermediate points; M minus one points. And, I put zeros here.

So, M minus one point, this will then, if we, this is zeroth sample, this will not become M th because already M minus 1 gone, 0, and then 1, 2, 3 M minus 1; so, this is M. Again, here I create intermediate points and bring zeros, again 1, 2, 3 up to M minus 1. So, this will become 2 M. This is M plus 1, M plus 2, M plus 2, M minus 1, M plus M minus 1. So, next is M plus M, that is, 2 M, dot, dot, dot. On this side also; so earlier if I had a sample here at minus 1, I bring in zeros, M minus 1 0. So, this will become minus M th point, then this sequence will be called expanded sequence, expanded version of x M. It is like this. If suppose this was obtained by sampling an analog signal of this kind, you are sampling, so you use so much sampling period. This was sampling period. Your 0 is first sample, second sample, third sample, like that.

Now, what is happening? I do not have; after I sampled, analog wave form has disappeared. So, I do not have the wave form again. But, what I am doing? I am creating new sampling point as though my sampling period has gone down, sampling frequency or sampling split has gone up by a factor M. So, if it was tau, if it was tau sampling period, it will be tau by M. If it is tau by M, then M times that will be tau and, M th sample will be here. So that means tau by M, another tau by M, another tau by m; so, I am creating sampling points at tau by M. What tau was the original sampling period? So, I divide the sampling period by M, very small sampling period brings faster clock sampling period has gone up by M. So, I have new sampling periods tau by M to tau by M three, tau by M, dot, dot, dot and then M into tau by M will be M th; So, this will be M th sample now.

At these sampling points, since I do not know the analog wave form, now this has disappeared. This has gone already. I am putting just 0 values. So, that is the minimum expanded sequence. For this expanded sequence, what is say X e z? If I do X e z, you can find out d t f t also. What is X e z? X e z will be by the gate transform formula, X e n assuming (Refer Time: 20:40) all that, but this is what dot, dot, dot, x e if you take minus one z to the power plus one, then X e 0, then x e.

I mean, I do not have to explain here. From here only you can see, if you carry out this summation, we will have X e 0 term. X e 0 term will be this guy; then, x e one that will be 0; x e 2 that will be 0; x e 3 that will be 0. So, those terms I ignored because they do not contribute anything. Straight I go to X E M. So, n equal to M, X e M, and z to the power minus n. Then, again next term; M minus 1 terms I ignore, I go to x e 2 M, z to the power minus 2 M. On the left hand side, I ignored this minus 1, minus 2, up to this. And, we straight go to x e minus M, this guy, is minus M th point now, z to the power plus n and dot, dot, dot. So, this summation is X e 0 as it is. Then, X e M z to the power minus M, x e 2 M is on this side. What is X e 0? This is same as x 0. What is x e m? That is original sample number one; original sample number 1, x 1; in terms of x, x 1. What is x e 2 m? Original x 2. What is x e minus m? Original x minus 1.

Now, z is a complex number. So, z to the power M is another complex number; you can call it z 1. So, this is z one to the power one, then the x 0 with x one z one to the power minus one. Then, x 2 z 1 to the power minus 2, dot, dot, dot, which is nothing but summation x n z 1 to the power minus M, you see. Which is capital X? Z 1 which is equivalent to? And, what is z 1? Z to the power M, what I get is just the z transform of the expanded sequence; capital X e z is same as capital X z to the power M. So, this relation, X e z is original X z to the power M. And therefore, d t f t, it should z e to the power j omega will be capital X e to the power j omega M. Remember this. This, I will not come back to again.

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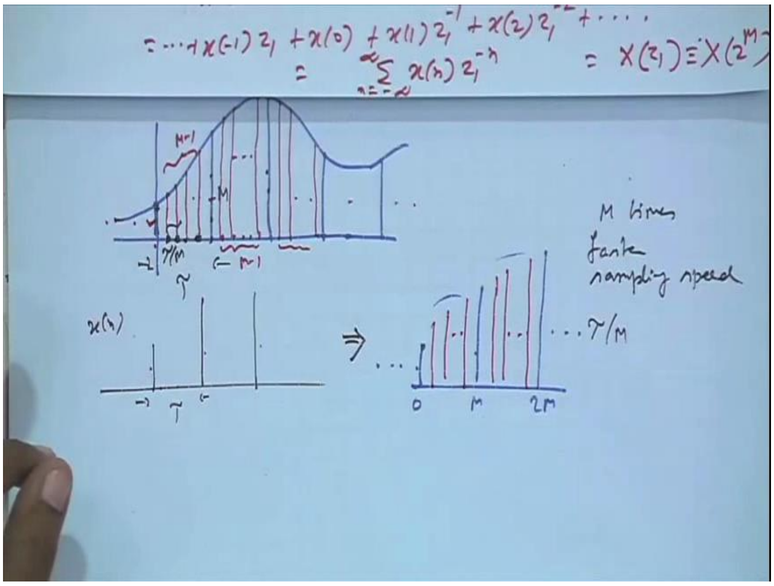

Now, I come to the topic of interpolation; where we will be using this expansion. Suppose, as before I had an analog signal, and suppose it was sampled at a lower rate, so much was my sampling period tau; so much was my sampling period tau. That this is done taking care, so that microstate is not, I mean, we are above microstate. If its sampling rate is not high, this is good enough. We are above microstate; there is no aliasing and all that. Now, my question is if I employ a higher sampling rate, there is, of course I am doing better. Instead of calling below microstate, I am going further ahead, further above, so no question of aliasing. But, if I had employed, suppose M times faster, sampling speed that the sampling rate, sampling period would have been tau by M. Speed goes up by M time's means, period goes down by M.

So instead of tau, I would have tau by M. So, this is tau by M. And then, here, here, like this. You know, and this would have been 1 tau by M, 2 tau by M, 3 tau by M, M minus 1 and this is M th; because M in 2 tau M is M tau. Since I know this point is tau, it should be the M th sample because tau by M, 2 tau by M, 3 tau by M. So, how many tau by M will be (Refer Time: 26:25) tau that is M. So, this will be the M th sample. And so how many intermediate? M minus 1.

Now, suppose I had originally sampled these at that high rate, then along with this and this, I also would have got this red color samples. These M minus 1 sample along with these and these, I would have got them. Then, again here I would have got these samples, another M minus one, again here, dot, dot, dot, dot, dot, and dot. But, since I did not do that I sampled only at the lower sampling rate, I got only the blue colored samples, that is, this guy. Then, at tau sampling period time tau, a sample, which I call the first sample; where that 2 tau, which I call the second sample, this is all that.

But if I had sampled at the higher rate, I would have so many intermediate samples M minus 1. So, these would have been M th; another M minus 1, so this would have been 2 M th, like that. My question is if I am giving this lower, this original, you know, sample (Refer Time: 27:40) this sequence, that is this guy, this guy, this blue, blue, blue colored samples. Sampling period is a tau, which I call x n. Can I get back this? Can I get back these high speed one? That is these as it is these as it is. What we have got? These red ones, then again, then again this 1, dot, dot, dot, dot; this is, see it would have been, this would have been M th, then this is zeroth. This would have been 2 m th and this intermediate samples would have been here.

So, from these can I get back this intermediate samples shown by red color here? In this sequence, they are not (Refer Time: 28:41). So, you can ask me in this a very peculiar question. Out of (Refer Time: 28:44), is it a magic? It is a magic that out of (Refer Time: 28:48), this samples, which I cannot take any more because analog sample, analog function has gone. Our sample (Refer Time: 28:54) is gone. So, where from will I get this red ink? Red colored intermediate samples? I am only giving this black color samples, these, these, these. Or, here only the blue value sample, blue colored samples. The intermediate samples, I could have got. If the analog function was still available, I could have resembled it. But that is gone.

So, from this how to recover those intermediate samples, which I would have got if I had sampled it originally at higher rate. And, it is sampling period tau by M. But, yes, still by processing this low sampling rate sequence x n, there is a way by which I can reconstruct this intermediate samples exactly without any error, without any approximation. And, this business is called interpolation. That is in the next class.

Thank you.