Discrete Time Signal Processing
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Lecture – 22
Decimation and DFT of Decimated Sequences

Okay just recall what we did in the last class towards the end.

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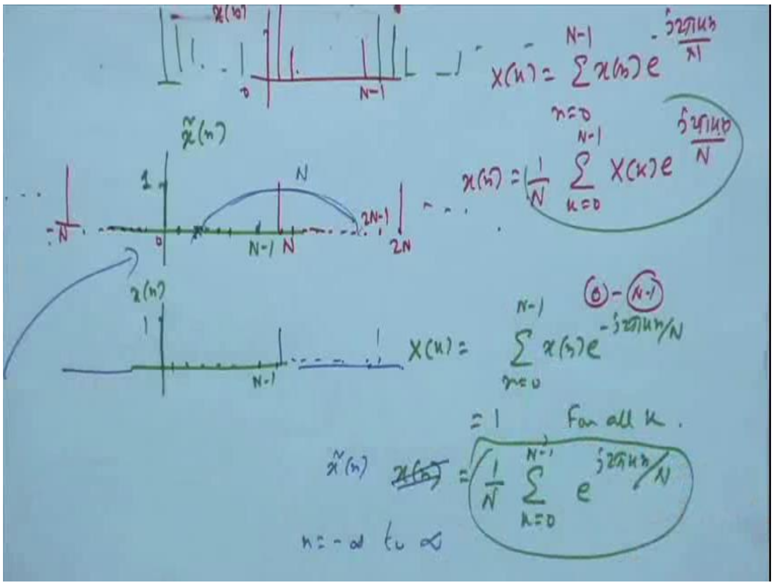

We considered what periodic sequence, for when one period you have value 1 at n equal to 0 and otherwise all 0s and that got we did it. Then I took out just one period separately called it x n, so 1 here and all zeros up to N minus 1, we carried out this d f t, that d f t is 1 because if we take the summation x n e to the power minus j 2 pi k n by capital N. This x n is 1, only for n equal to 0, when e to the power 0 is 1 and all other x values are 0s, so 1 into 1 is the result, which is true for all k. Now in the inverse d f t, if you put that back; x k is 1, so 1 into e to the power j 2; pi k n by capital noise, this is x n; small n you choose from here, fix here and that n goes summation is over k.

And now in this formula, if you allow small n to go from minus infinity to infinity; this formula then what you get as we have seen is nothing, but periodic reputation of that one block which is nothing, but this.

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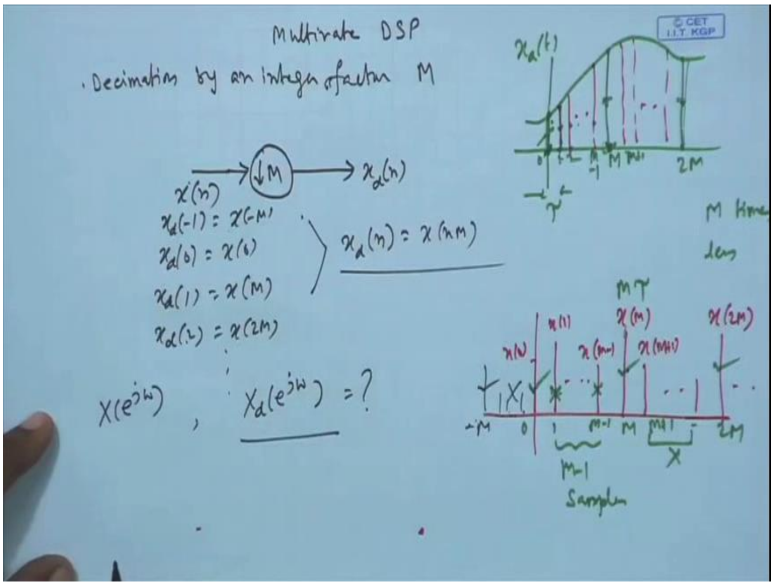

So, that x tilde n which I called sampling function, sampling sequence that is actually can be written by this formula. Now I go to, I start another topic which is just, which is a big topic; I will just give some introduction to that only, very basic things I will discuss because we have very common place to; this topic is called Multirate D S P, I will just consider very basic things; Decimation by an integer factor. So, capital N will be an integer, either at least 2 or more than 2 and there is a decimator, this is the first block. One first important component in multirate d s p, as you going to it will understand why you need multirate D S P.

But first let us understand some of the basic blocks, this is one block; we denoted like this that the suppose there is a sequence x n and we denoted this way this is called decimation and output you can give any name, I call it x d; d for decimated x d n. It means, it is like this suppose x n was obtained, suppose as an example x n was obtained by sampling an analog function, where is an x a t your sampled here you are here, here here dot dot dot dot here and then again at this point. So, this is suppose 0, 1, 2 dot dot

dot and they this is m, this is m minus 1, then again dot dot dot dot dot dot dot dot dot then again at 2 m, solve this 0 is sample, first sample, second sample, m minus 1 sample mth, m plus oneth dot dot dot dot this also how you sampled.

But suppose I had sampled, I use a sampling rate which is m times less; that is sampling period m times larger, that is after taking one sample here, next sample would have would be here all right. Suppose this much is your one sampling period tau, so earlier sampling period was tau now it will be m tau. So, 1 tau, 2 tau, 3 tau, m tau will be here 1 tau brings to 1, 2 tau brings to 2 m tau brings 2. So, now, sampling period has become m tau because sampling speed has gone down, so instead of sampling at all these intermediate points, you sampling here and the next sample you will take here next will take. So, now, it will be zero th sample, it will be first sample, it will be second sample like that which means; even if it is not obtained by sampling and analog signal you have got some x 0 then x 1 dot dot dot may be x m minus 1 then x m, then again x m plus 1 dot dot dot, then again x 2 m dot dot dot dot. What we do? This is 1, this m minus 1 this is m; m plus 1. What we do in this process, that is here I have drawn a general sequence, it has got nothing to with time and all that may be you generated the data yourself on a piece of paper or may be it is where computed.

So, just zeroth data, first data, second data but in decimation means if you are decimating by a factor name where after taking one sample you discard this samples m minus 1 samples, then again take the next, then again these (Refer Time: 05:30) then again take the next like that, which is what you are doing here as though if you are doing sampling in the real time case, earlier you had zeroth sample, first sample, second sample, m minus one th sample, (Refer Time: 05:43) sample dot dot dot dot. Now you are saying no my sampling rate I am bringing down, so I have speed, I am bringing down which means sampling period is very wider by a factor capital m. So, if tau was the sampling period earlier now it is m into tau, so after this zero m into tau will bring you to this point because tau brought you to number 1 then 2 tau brought you to 2; m tau will give you 2 m that will be mth sample, to be as mth sample in is now be the first sample because m tau is the new period, so after one period I will get sample first sample, after two period you get second sample.

So, this zeroth sample, first sample, second sample like that, so intermediate for us go, you rename this sample zeroth, mth. Earlier mth now becomes first, earlier 2 mth now become second, earlier 3 mth now become start same here, x 0 you keep, m minus 1 sample just throw away, then this mth you retain. So, this will be a first guy now, then again next m minus 1 sample you throw away, x 2 m you retain, but will be the second sample now dot dot dot dot and this will be called a decimated sequence. Now there is a danger of course, you see here you are bringing down sampling rate by a factor m, so there may be a possibility of (Refer Time: 06:55) because if the function is band limited and you must sample it at more than the (Refer Time: 07:00) rate that is twice the band limiting frequency.

Suppose you had been doing it, but now you are beginning down the sampling frequencies, sampling rate here, so there is a possibility that if you by this bringing down the sampling rate you go below (Refer Time: 07:15) rate there will be might in danger there will be (Refer Time: 07:18), so this you have to be careful. May be your IQ state is less, but your actual sampling rate here you know that at this rate, the tau; tau period at that rate was very high. So, you could effort to bring it down by some factor m, still remaining higher than the (Refer Time: 07:33) state that is okay, but by bringing down do not bring it below (Refer Time: 07:36), then you are inviting danger in form of earlier scene, it is just a war of (Refer Time: 07:41). So, now x d n means x d zeroth sample will be original zeroth, x d first guy will be original mth, x d second guy will be original 2 mth, x d minus oneth guy will be original x minus mth because in that this second sample I will throw away, throw away at the I will take from minus mth right. So, which means dot dot dot dot, which means in general x d if you take nth, it will be if it is 2, it is 2 m which is 1; 1 m, which 0, 0 m; minus 1 minus m. So, x d n means x n mth, n times m a in mth will be mth sample mth first sample, two mth second sample, three mth third sample, a in mth will be mth sample. So, this is a mathematical description all right x d n and this is the notation, then question is given this d t f t; what is the d t f t of this into what.

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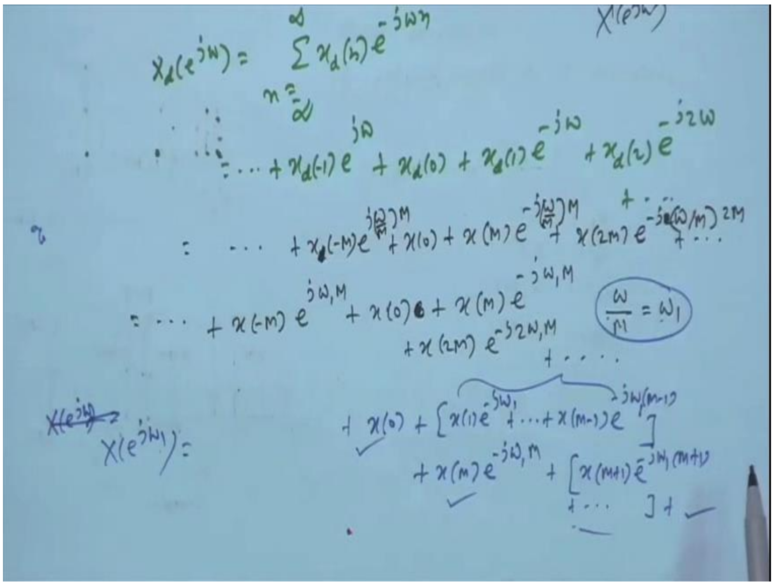

This will work out today these not an easy thing; is a tricky thing. So, let us work out; x d e to the power j omega means by this formula x d n, d t f t formula and I am doing for d t f t same thing you can extend to the more generalize case of z transform same same approach. Now you see x d 0, x d 0 e to the power 0; then x d 1, e to the power minus j omega, x d 2, e to the power minus j 2 omega dot dot dot dot and on the left-hand side x d minus 1, e to the power j omega; it is plus j omega and dot dot dot dot. But earlier we have seen x d 0 is same as x 0, but x d 1 is a mth original sample; first new sample, decimated sample possible of the decimated sequence of the mth of the original. So, I just write x n, these I keep as it is, in x d 2; second sample of the decimated sequence if the two mth sample of the original sequence, again minus oneth means originally this x alright.

Now, this omega I can write as omega by m into m, this I can write again omega by m into 2 m, this you can write into m like that. So, omega is radiant; omega by m also radiant you can call it omega 1. So, this is x minus m; e to the power j; omega 1 m, then x 0, e to the power 0; x m, e to the power minus j; omega 1 m, x 2 m, e to the power minus j twice omega m and dot dot dot dot. See, I have to manipulate so that this term becomes closer to the original x, e to the power j omega. Now if you see x, e to the power j omega, now if you see x e to the power j omega; it will have x 0, but after that there is this terms you know x 1, e to the power minus j omega dot dot dot dot, x m minus 1; e to the power minus j omega, m minus 1; this block then again this term. So, suppose I am carrying out each at omega 1, x this is d t f t at omega 1.

So, next term will be see; x 1, e to the power minus j omega 1, x 2, e to the power minus j 2 omega 2, this is the original d t f t and these then x n, e to the power minus j omega 1, m which is present here. Then again there will terms, you know x 2 m, sorry x m plus 1, e to the power minus j omega 1, m plus 1; again dot dot dot dot some term and then again this term will be present. These are not present here, only this is present, this is present, this will be present like that, so these summation, these d t f is part of original d t f t at a frequency omega 1, where omega 1 is these, omega one is these. So, original d t f t at omega 1 has got, you know say generalized expression, this has got the terms presenting these d t f t, there plus additional terms. Now, if I can make this additional terms 0 in the original d t f t, then I will get this that will become equal to the desired d t f t, this x d e to the power j omega. For this what I do is this, suppose this is your original x n, you may sample, zeroth sample, intermediate samples; I am not so bothered about again these, these intermediate samples; then x 2 mth on this side also you have got then x minus mth dot dot dot dot these are original sequence x n.

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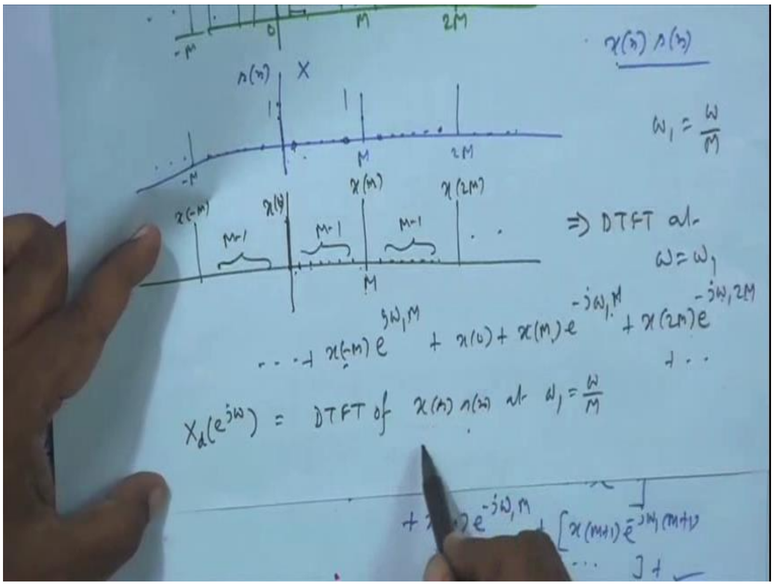

Suppose I multiply it by that sampling sequence I told you; 1, 0, 0, 0 then again at mth point 1, 0, 0, 0; then again 2 mth point 1, 0, 0, 0, then 0, 0, 0, 0, 0 at minus m 1 dot dot dot. If I multiply by the 2, let me call it s n sampling sequence. So, x n into s n suppose I do then what will I get, I will get x 0 into 1 and then these fellows will be raise to 0s, this into 0 is 0, this into 0 is 0, this into 0 is 0 then x m into 1, so like that. So, this will be let the decimated sequence x 0, but in the decimated sequence this fellow x m was called first sample; it was index one, but now these 0s are brought back, so it will be mth again; x m into 1, so mth sample will be x m, this into 0; 0, this into 0; 0. So, m minus 1 0s and x m again 0, 0, 0; m minus 1; 0s, x 2 m because dot dot again m minus 1, zeros x minus m right.

If I take d t f t of this sequence at omega equal to omega 1, what will I get? x 0 as it is plus 0 into something, 0 into something they do not contribute anything. So, next is x m x, so x 0 then these 0s you forget, then next is x capital M, at capital Mth point, so it will be minus j, frequency to omega 1 m, then again 0s you forget, next is x 2 m; e to the power minus j omega 1 into 2 m and dot dot dot on these side these zeros you forget; x, x minus m; e to the power; minus, minus plus will j omega 1 m and dot dot dot.

And this is what you have here you see, x minus m, e to the power j omega 1 m, x 0, x 0, x m; e to the power minus j omega given m, x 2 m; e to the power minus j 2 omega m, 2 m these 2; omega 1 or omega 1, 2 same thing. So, this d t f t, the decimated sequence is same as d t f t of the product of these two sequences, original sequence multiplied by the sampling sequence, at a frequency omega 1. Remember this d t f t was at a frequency omega, so decimated sequence it is d t f t at a frequency omega is same as d t f t of this product sequence x n, s n; x n was the original sequence, s n is the sampling sequence. So, x n; s n at omega 1, where omega 1 is omega by m that is decimated sequence is d t f t at any these is d t f t of x n; s n at omega 1 equal to omega by m alright because two expressions are same alright.

So, this we work out we tried to carry out d t f t of these for these two as a omega 1, alright; that means, x d, e to the power j omega.

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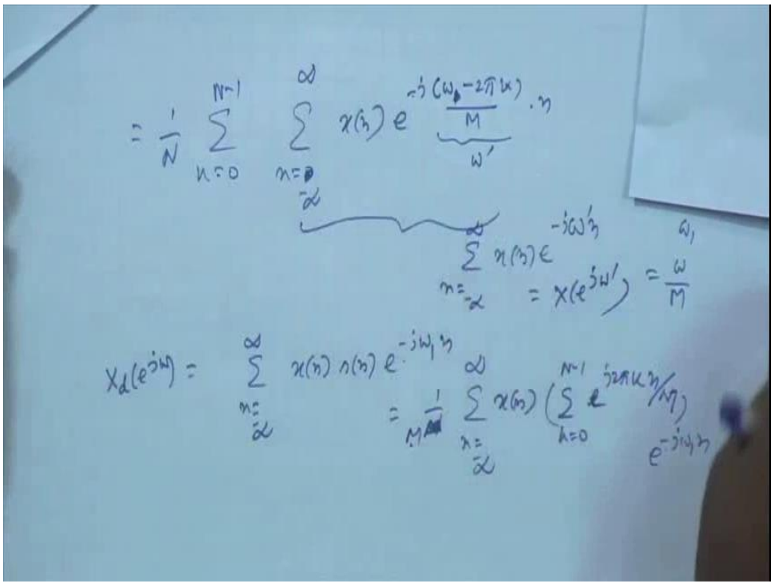

By this expression, it means a product x n; s n, e to the power minus j omega 1 because at omega 1, omega is a frequency where you have to find out this and omega d t f t. Now s n, I told you the sampling sequence can be represented, if these when are just a minute that time it was called x, now this is called s because call x tilde that time the same thing I am calling s n now; the s n can be represented like this. So, that I substitute here this s n are replaced by that; 1 by capital n will go out, outer summation as before x n; s n.

So, summation a equal to 0 to m minus 1; e to the power j 2 pi that came from, here n from outside and then multiplied by e to the power minus j omega 1 n; alright and now these we can simplify whenever I know that whenever there is a double summation, next step is to interchange the two summations. So, this summation over k will go out, n from sorry minus infinity to infinity; x n, e to the power minus j. Now omega 1, I replace by omega by m, now this is not n, this will be m sorry I forgot to mention because it was; here the period was taken to be capital N; 0 to n minus 1, but for our business, for our business it was if this is m right.

Just give me a minute, mth yeah here this is the sampling sequence. Earlier I took 0 to 0 and then capital N; 2 n, three (Refer Time: 20:44) n m capital M; 2 m. So, that is why this formula; I should not write n it should be m, it should be m; everywhere should be m and omega 1 I know is omega 1 is omega by m. So, now here I write x n, e to the power minus j, I take this common; not omega 1 omega by m, so omega minus; n I will put outside, 2 pi k by m into n; omega by m, omega by m is omega 1, omega 1 into n with minus, e to the power minus j omega 1 in there is present and another term is minus, minus plus j 2 pi k, j 2 pi k small n by m small n by m alright. Now what is this summation, omega is radiant; digital frequency minus 2 pi into k another radiant, so omega minus 2 pi k is radiant another digital frequency, divided by m still radiant; another digital frequency, if you call omega prime.

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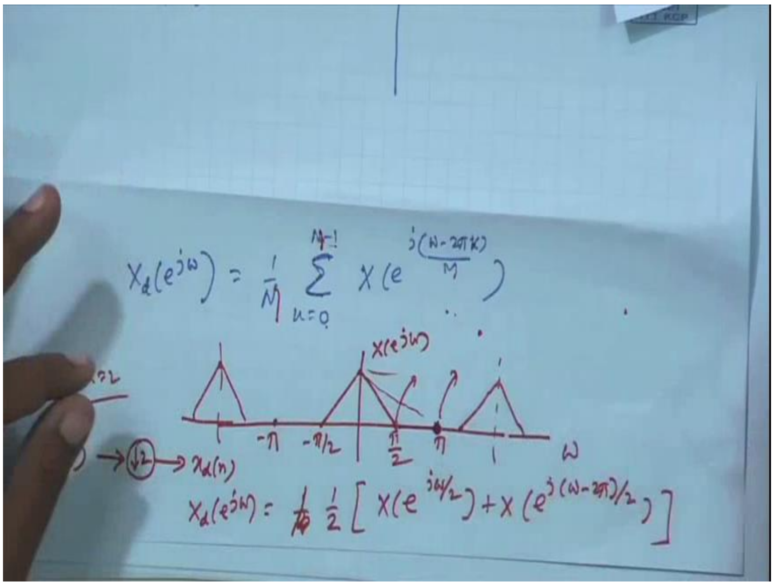

So, this is nothing, but this inner summation is what; summation x n, e to the power minus j omega prime n alright which is nothing but d t f t at omega prime. So, x e to the power j omega prime, so this entire thing then turns out to be this; x d, e to the power j omega is 1 by capital N, this much frequency omega prime, so I am writing omega prime; I am writing this full, so this is the d t f t. Now what is the interpretation of this, for that we will take an example; I do not know whether I have that much time today, but nevertheless we will take an example of just m equal to 2 and suppose I take a first page, suppose this formula I need; suppose what is given is x e to the power j omega, again I tell you cannot plot a complex function by just one plot, I should have mod of these versus omega and angle of these versus omega, but just to explain my point I am here doing that I am plotting capital X, e to the power j omega versus omega.

So suppose this is given to you like this okay, now this x n; that has did these d t f t and these x n, I am decimating by a factor 2, to get x d n, that is M equal to 2; question is what will be these, x d; e to the power j omega that by this formula will be 1 by n, so it is not n it is m; everywhere I am making the mistake, there is no capital N here, it is M you be correct, this would have been M. So, now these summation means 1 by M, M is 2, so 1 by 2 and there were only 2 cases; k equal to 0 and this is M. Look at here, this would be M, wherever you had N; that becomes M, so M minus 1, sorry for this mistake there is no capital N for our notation here, these all capital M. So, instead of capital N everywhere, it will become capital M, there is where is capital M; capital M, half k equal to 0 and M 2 means there is 1, so 0 case and 1 case; if you put k equal to 0 here, you have got x, e to the power j omega by 2 because n is 2 and then x, e to the power j omega minus 2 pi into 1 because k is 1 now, M is 2 alright if you plot this then you will get that.

This will be in the next class, you can see one thing if you look at this d t f t; pi corresponds as I told you half some free frequency. Now this is the analog, this is the digital version of the analog band limiting frequency, so here half sampling frequency itself is twice with band limiting frequency; whereas to avoid earlier see it was enough if band limiting frequency equal to (Refer Time: 27:27) sampling frequency so that means, I can still bring down the sampling; rate half sampling frequency is pi is giving you analog half sampling frequency going to pi, analog band limiting frequency going to pi by 2. So, I am finding half sampling frequency is already twice the band limiting frequency; which is not required, half sampling frequency could be equal to band limiting frequency, and band limiting frequency is fixed. So, I can bring down the half sampling frequency and therefore, sampling produced by factor 2, so in that case half sampling frequency is always, analog half sampling frequency will always map to pi.

So, band limiting frequency will go like this then they will coincide with that and it will be like that, this will see in the next class. Thank you very much, you try this as an exercise by yourself; we will meet in the next class.

Thank you very much.