Discrete Time Signal Processing
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Lecture – 21
Zero Padding and Linear Convolution via DFT

So the last class, we considered the graphical approach to circular composition, a very quick recap.

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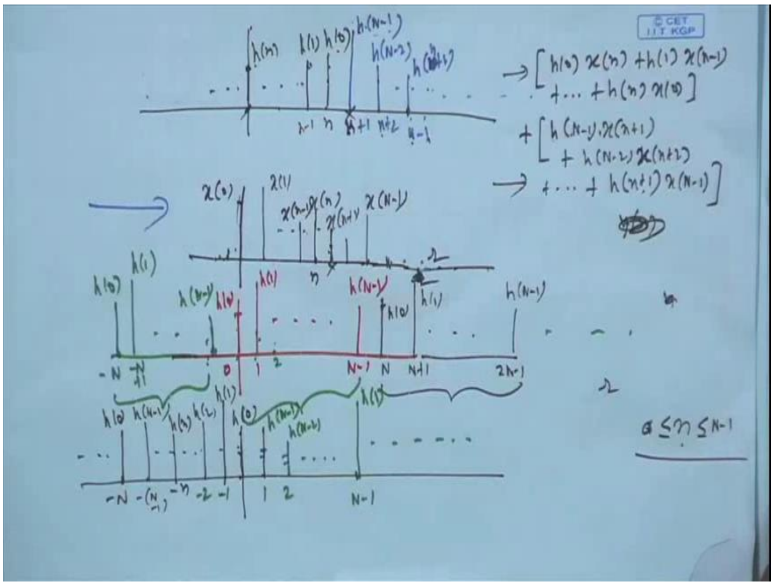

There are 2 sequences to be call x n, and h n, both where length capital N, and this was the expression for circular convolution, how to carry out it graphically, I said keep 1 signal as it is, length n, there is x, write it as a function of r, keep another signal also, as a function of r, but then make it periodic, that is, whatever you have from 0 to n minus 1, just keep repeating it. So, x h of 0, at 0 th point, at again at capital Nth point it comes at, there are 2 nth point also h 0 will come so on and so forth.

So, this whole block keeps repeating. So, it becomes periodic version of that. Next step is flipping it. So right side goes to left hand side, left hand side goes to right hand side. rth point goes to minus rth, minus rth goes to rth. After flipping it, now, if you want to carry out this convolution, circular convolution at an index n, you want to find out the y n at n in this range, then you shift this flipped periodic sequence, to the right, by small n. After that, you hold both, hold this as it is, multiplied term by term, with this original sequence, and the original sequence is 0 outside this window, that is, outside 0 to capital N minus 1 zone, it is 0. So, no point in considering, anything outside that window, from this flip sequence, it will be sample wise multiplication, we see x 0 into h 0, you already present, x 0, h 0.

Then here x 0, h 0, x 1, x 1, I mean, to start from this side. If you shift it to n, shift, then h 0 comes here, h 1 here, dot, dot, dot, So, at nth position h 0, n minus 1 is 1. So, 0th there is n minus nth, nth, and now if you multiply h 0 into x n, it is present, h 1 into x n minus 1, present dot, dot, dot, h n into x 0 present, there is a linear convolution part, we have seen, at that additional term also you get, h capital N minus 1, because this periodic, if h 0 then, the next period, is the trail of the next period. So, h, capital N minus 1, that comes at small n plus 1 is index, that multiplied x n plus 1, you get here. And no if verified, but as I told you, this convolution expression, is not linear convolution between x and h, is a summation of 2, this part linear convolution plot, plus an extra term, that is why, from this y n, which is equal to this, I cannot extract, the linear convolution part as it is. Then the question is, if I want to still carry out linear convolution, by this means, what I do.

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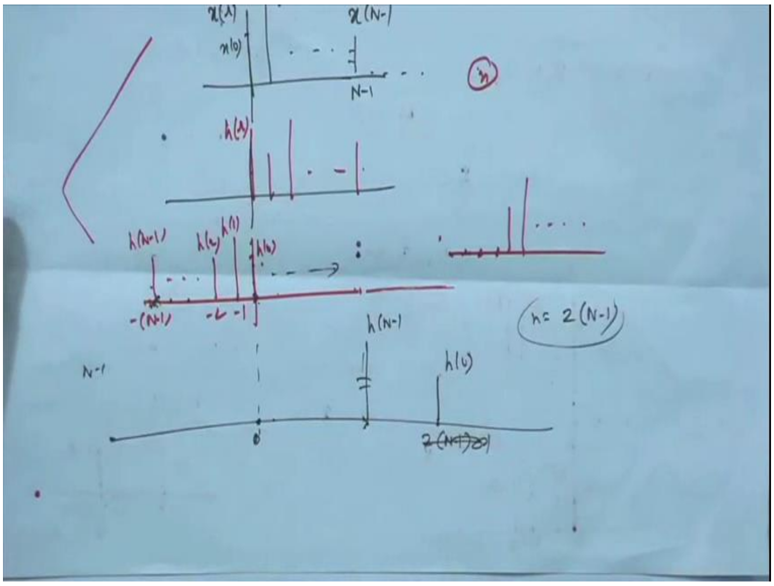

Now, that can be done but before that, you understand when I convolve 2 sequences, suppose I am convolving 2 sequences. One is x, say x r, there x 0, x 1 dot, dot, dot, x n minus 1, and then 0s, and there is another sequence, say h, h r, h 0, h 1, h 2, dot, dot, dot, this, when you convolve, you want to flip this one so, suppose you flip it. So, you have got h 0, h 1, h 2, dot, dot, and dot. So, minus 1, these minus 2, these are minus, n minus 1, you have got h n minus 1. Now without 0 any shift, 0 shift; we have got only these 2 terms, multiply you get some value. So, some value you get there, if you shift this to the left, then there is no overlap you do not get anything; that means, output for shift 1, shift 2, you know is 0, but that is I am finding out convolution at various values of n. This has for n equal to 0, no shift, remembered. So, what now n equal to minus 1, this would be shifted to the right, left by 1 bit, if I do by that, and then multiply term wise, sampled wise, between the 2, there is no overlap between these, and these no overlap, so 0.

If I shift it further to the left, and there again multiply sample wise, between these 2, 0. So, for any minus 1, minus 2, minus 3, this is 0. If you, on the other hand, if I shift it to the right, h 0 come below x 1, h 1 comes below x 0. So, 2 samples multiply, x 0 with h 1, x 1 with h 0, you get some value, and you continue doing it. This h 0, after sometime will come here, and keeping shifting, how long will you get non 0 value, that is what we are seeing. So, this x 0 is moving, as you keep shifting it to the right by 1, n equal to 1, that 2, there is an equal to 2, dot, dot, dot, finally, a stage will come, when h 0, there is the linear in this train, has come here. So, that the last guy, h n minus 1, is here, and so, these 2 multiply, after that if I shift it further again, there is no overlap, samples from here, multiply 0 from here, samples from here, multiply 0 from there, and you continue get 0.

So, after how will, so h 0 was here, after how many shifts, h 0 will come here. So, that h capital N minus 1, moved to this, that is the question; obviously, n minus 1, after one shift comes here, 2 shift comes here, after n minus 1 shift, it comes to 0, and then another n minus 1 shift, it comes here, so 2 into n minus 1. So, if it is, if this sequence is shifted to the right, by 2 into n minus 1, this will come first here, and then here. So, n equal to up to this, that is, this index from 0, if you considered now h 0, this is also shifted by the same amount. So, this will be, 2 n minus 1, minus 1, because it is 1, 2, like that, all right? Anyway, this calculation I have to recheck. So, you get like this, in general let us consider, actually I cut it because I want to go to a more general case, earlier I took both the sequences to be of the same length. So, that is why I want to change, that is why cut.

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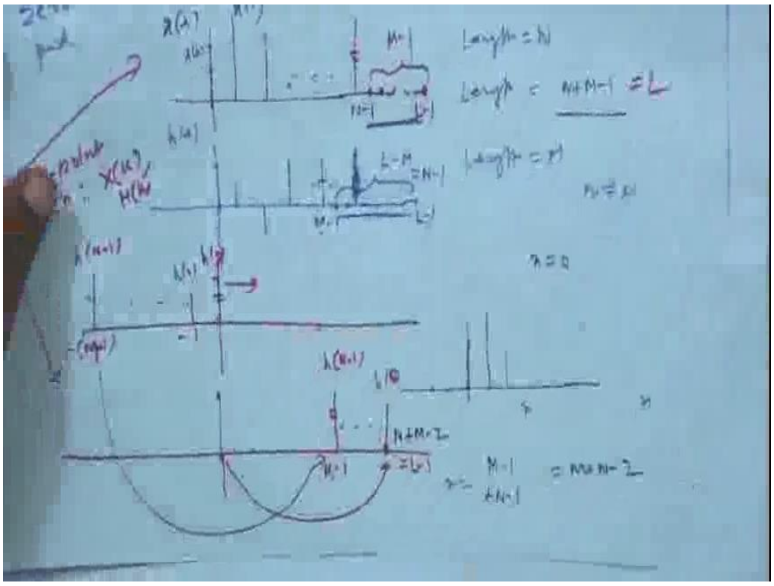

X r, suppose it is as it is, up to n minus 1. So, length n, and h r, not up to this. So, up to up to maybe this and then zeros, this is m minus 1. So, length, n length, m, and n not equal to m. I am taking m to be less, than in it could be the other way. Then if I convolve, what will be the duration of the output sequence. So, first flip it, as before h 0, h 1, dot, dot, dot, dot, h n minus 1. Very much like what I did last time, minus 1, all that and this side is 0. Then for n equal to 0, 0 shift, you have got these simple multiplying, these sample, you get some value. So, at n equal to 0, there is an output, some output, n equal to minus 1, means shifting to the left and then if you multiply sample wise, you get nothing, so 0. If you shift it further to the left, there is n minus 2, minus 3, as I told you, last time, there is no overlaps. So, you continue get 0 this n axis now.

If you shift it to the right by 1, that is n equal to small n equal to 1, then h 0 comes below, this x 1, h 1 comes below x 0, they multiply at, you get some value, and you shift it further to the right, 3 samples will come below 3 samples here, sample as multiply here, you get some value, dot, dot, dot, dot, finally, a stage will come, when h 0, that is h 0, is the leader, followed by this, followed by this, these guy is moving to this side, as I keep shifting to the right, by n equal to n, then n equal to dot, dot, dot,. So, the stage will come when finally, h 0, has come to some index so that, the last guy, in this trail, I have done linear convolution, mind you, in just coincides to, coincides with this guy, because I am multiplying these 2. So, this 2 are coinciding, this is n minus 1, if this is not n minus 1, sorry, this was m minus 1, so m minus 1, this is minus, I am sorry this is m minus 1. So, this is minus, m minus 1.

Earlier I took both of same length, and it was 0 to capital m minus 1. So, it was going to minus of bracket m minus 1, but now since it is capital m minus 1, is going to minus m minus 1. So, there is a mistake, I correct it all right? So, now, my question is I am giving shift, 1 n equal to 1, small n equal to 1, 1 bit to the right small, n equal to 2, another bit to the right, and dot, dot, dot, dot, and I keep getting some samples from this sequence, some sample from here, they are overlapping, multiply them add, you get output, and these happens. So, h 0 is moving, h 1 is moving, they are all moving to this side, finally, stage will come, when h 0 has move to, such, from 0 it has gone to this index.

Such that the last guy is here, at n minus one th point. So, it these 2 are coinciding, till that time, I will get nonzero output. After that, if I further shift it to the right, all output will be 0, and it wills continuity is 0. So, what will be that case, how, by how much shift? Now if 0, h 0 comes here; that means, this guy has come here, isn’t it? h 0 has moved there, h 0 from here, is has moved here. Same amount of shift has been executed on this. So, this has come here. So, it took m minus 1 points, and then another n minus 1 points, m minus 1, another n minus 1 all right? m minus 1 shift brought here and then another, so this must shift, is equal to m plus n minus 2, all right? So, if are n was 1, h 0 would have been here, n is 2, h 0 is here, at 2 n point and, and if n is 3, h 0 would have been here. So, now, n is so much.

So, this is this index, n plus m minus 2, which means the length, is from 0 to n plus m minus 2. So, length now, new length, is total is this 0. Because, this n plus m minus 2, 1, 2, 3, like that, and then 0, this have to count also. So, if you count 0, and then 1 to up to this, total is n plus m minus 1. So, you see, but by convolution, actually output sequence, becomes y dot, if you convert to finite sequences, which is length m, length m and length n, they are convolution you know, why there is a sequence. So, sequence spreads in time domain.

Now, I want to carry out this convolution, by using DFT, that is, circular convolution, by, because DFT you know advantage is, they are very first algorithm to calculate DFT, called FFT, you can use them, and them calculate inverse DFT also, by these first algorithm therefore, because if the first algorithms, I have to go by the DFT route, but by whether DFT route, I only have circular convolution, there is only a multiply the DFT, is and taking inverse DFT, I have the circular convolution, but circular convolution is not my goal, my goal is linear convolution, there is this one. How to get the linear convolution from that will do some manipulation here, but you understand one thing, final output after linear convolution is not of length n, or not of length m, but is length n plus m minus 1, if you call it L. So, it is length L, 0 to L minus 1. So, call it L minus 1, this is L all right.

So, my output, I should have a provision for so many points in my output sequences, which means, what I should do actually, I will make both the sequences, same length of L. So, I will go and append 0s, up to L minus 1 here, L minus 1, n minus 1. So, how many zeros? L minus n, L minus n is m minus 1. So, m minus 1 0s, and here I will append 0s. So, there I again go to same point L minus 1. So, L minus 1 m minus 1, so many zeros; you subtract this from that, L minus m, which is equal to, if you, this is L, you subtract m, this is actually n minus 1. So, here m minus 1 zeros, here m minus zeros, this is called 0 pad, 0 pad, this technique is called 0 padding. So, I made both the sequences of the same length, L. So, 0 to L minus 1, or L is, in short I am calling for n plus m minus 1, because there is the length of the final output that have seen, after these linear convolution, final output is wider is total length is n plus m minus 1, there is L. So, it goes from 0 to L minus 1, that many points.

So, I am to make input and h, both of them or the same, that, that length only. So, additional points I have plug, you know, set as 0. So, here it was length n, but I want to make it there is from 0 to n minus 1, but I want to go up to 0 to L minus 1. So, additional 0s will be L minus 1, minus these. So, L minus n, that many and L minus n is, if you after m minus 1, so m minus 0s here, by the same way, n minus 1 0s here. So, now, what I will do, next step, I will take DFT of this, DFT of this, but not n point, and m point, I will take L point DFT, L point DFT, multiply the 2, and carry out inverse DFT, that is, circular convolution, between this sequence, with the 0 pad, and this sequence with the 0 pad, and that also will be length L, because I will take the L point inverse DFT, and will see, that will turn out to be this. Even though I am carrying out circular convolution, the presents of 0s, in these 2, will make sure, that the additional term which was coming earlier, you know, this additional term, this will term to be 0, you will be left with linear convolution, because of additional inclusion of this 0s.

So, I repeat again, I have added this 0 pad, this 0 pad of length m that is total, length m minus 1. So, many 0s, here also 0 pad, n minus 1. So, both the sequences are not same length, like L, that is 0 to L, by L minus 1. I take L point, L point DFT, DFT. So, what the DFTs? x k, h k, L point, and multiplied, and they are taking L point inverse DFT. So, you get a L point sequence, that will be circular convolution by your definition, between this extended sequence with 0 pad, and this extended sequence 0 pad, but that circular convolution, if you carry out by the graphical bits, you will see that, this additional term, is becoming 0 because of that presence of this 0 pad, and will be left with linear convolution. So, by that process, you will get the linear convolution, this is what we will verify now. So, will carry go, go by the graphical method.

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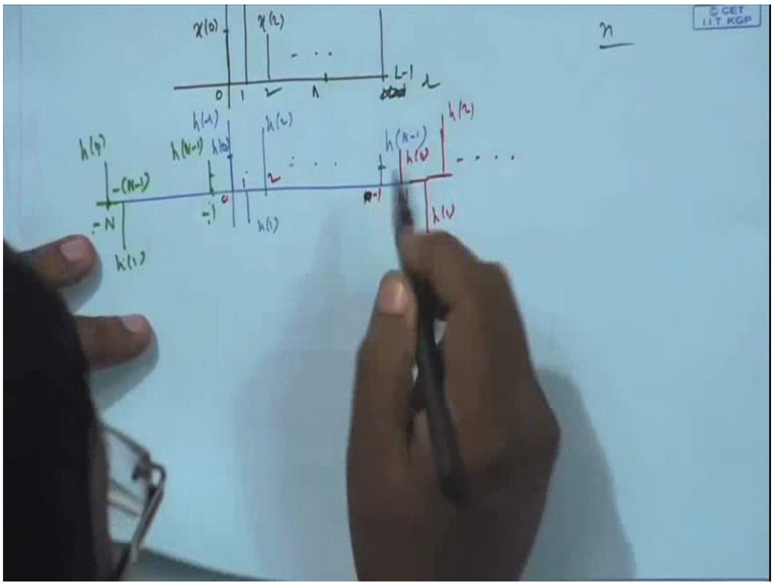

So, I draw what we have here, graphical method means, one sequence I will hold as it is, and quality function of r, our graphical methods. So, this is r, and this is suppose your x, 0, on that sequence first I write as function of r, h 0, h 1, h 2, dot, dot, dot, dot, h, n minus 1, then next step was to make it periodic. So, same block will keep repeating. So, now, this will be repeating. So, h 0 will come here, this was point number 1, 2, 0, this was n minus 1, this is n, then again these, h 1, then again these, h 2, dot, dot, dot, dot, and this side also the periodic repetition right? So, at minus n, you will have again another h 0, then at minus h 1, dot, dot, dot, dot, at minus 1, this is periodic, this thing will repeat. I mean, whatever you have at any point, if you jump to the right by capital N, you will get the same point. So, at minus n, minus capital is of h 0. So, jump to the right by capital N, you have 0, is index you get h 0.

Then again at n h index, you get h 0, like that. So, h 1, h 1, and last 1, is minus 1, it will be same as what I had here, h, n minus 1, this guy. If you jump by capital N, you will get this index, you will say, like this. This was step number 2, next if you have to find convolution at any index n, what will you do? You will shift it to a right by n, and then you see only within this block, and do sample wise multiplication at, because any data from this side and that side, you will get 0s only from top. So, they do not contribute anything, so you do not consider them. Now, this was the (Refer time: 19:35) convolution, now what we have here, this is just to remember, but what we have here is, we know that there was a 0 pad. So, we went up to, this is not n sorry, this is L minus 1, let me re draw it.

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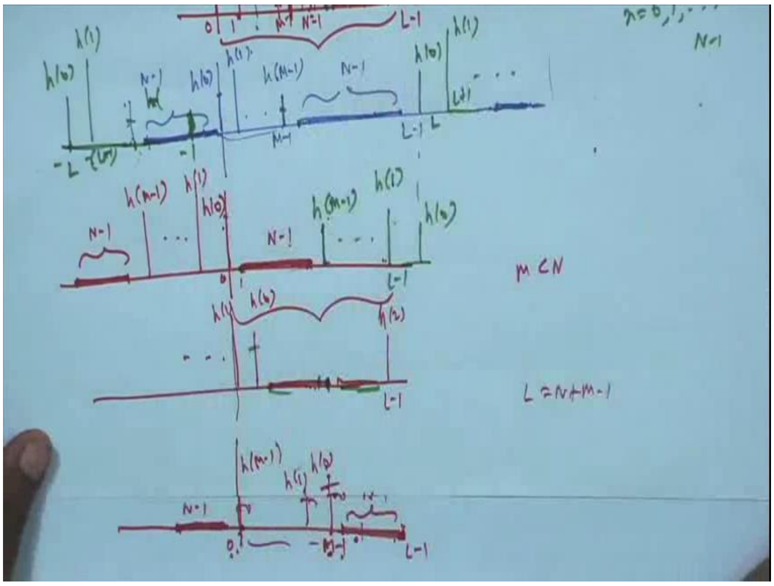

Because making correction there, you know, might be clumsy. So, this was your, anyway this was just recap that, graphical procedure again. So, this was x r, it was up to n minus 1 and then I have got 0, 0, 0, 0, this 0 I left out that time up to this where L was, n plus m minus 1.

So, total m minus 1, zeros, zero situation, I hold it as it is, next sequence, the other sequence, has h 0, say h 1, dot, dot, dot, may be here, I mean 0, 0, 0, 0, 0, n minus 1. So, this actually length, L minus 1, both, and I will make it periodic. So, you will have periodic (Refer Time: 21:06) of this. So, to the right again, will have h 0 in Lth point, then h 1 at L plus Lth point, dot, dot, dot, dot, and this side, at minus L, we have got, h 0, then h 1, at minus, L minus 1, dot, dot, dot, at minus 1 you have got, h, this guy. Now, it will be moved, if I, minus L, then you have got this 0 pad, this, this part is here, periodic, this part, this part, then upper that followed by 0 pad. So, here also we have got 0 pad, these are 0 pad, these are 0 pad I am thickening it, all right? These are 0 pad, how many 0s? m minus 1, I am not writing the indices, you can easily calculate from minus 1, to whatever, till that time you have got some data, but after that, 0 point. Same n minus 1, 0s, this, this, this, and then this 0 pad, 0 pad, here also, will have after, after sometime, will have a 0 pad, this is a block, like that, this is the situation.

Now, you have to find out convolution at a index n, n will be from 0 1, dot, dot, dot, up to n minus 1, this is the business. So, I will be flipping it and then shift it. So, if I flip it, h 0 remains here, h 1 goes here, dot, dot, dot, up to, then this 0 pad, this one, how many? N minus 1, and from this side, this 0 pad first comes, what is the length? N minus 1, see, 1 to n minus 1, and these total N minus 1. So, this directly fits below this data, data block right? Because these zero th index, and total N minus 1, 0s, that will from 1 to N minus 1, and then again, you have the data, data means, after the 0 pad, you had h, N minus 1. So, this will be h, this N minus 1, all right? The same, same guy is here, because of periodicity. So, that will come to the right hand side, these 0 pad has moved, and then this will move. So, this will go here, next; h, m minus 1, dot, dot, dot, dot, and from minus L. So, this will, this will be coming at, L minus 1, and then, Lth, here h 0 is coming, but I will be multiplying, term wise over these block, 0 to L minus 1, to these 0 to L minus 1, anybody outside, its a no consequence, because they do not contribute.

So, what we have here you see, here how many data, 1 to m minus 1, here what is the length of this 0 pad, m minus 1. So, this is directly below this, when you multiplying this sample, with these samples you may 0. Similarly here this data are over 1 to n minus 1, but here, it the, this is 0 pad. 0 index, 0 index, then you have got a 0 pad of length capital N minus 1, I mean these data from 1 to N minus 1, so this length also N minus 1. So, this data gets multiplied by this 0 pad, these data which is from 1 to m minus 1.

So, it total length is m minus 1; they get multiplied by these 0 pad, all right? So, they do not contribute, we have got h 0, into x 0. Like you know, linear convolution, in linear convolution, you just flip the original sequence, do not shift. So, h 0 and x 0, that only comes, then n equal to 1, next is you shifted to the right by 1 and again see within this block, and multiply. That time h 0 will come below these, h 0 will come below these, h 1 will come here, like linear convolution, h 0 below these, h 1 below these, and other data, but 0 pad is moving, 0 pad from point 1, which has shifted to point 2. So, it is coming here.

H m minus 1, next guy, instead of h 1 at L minus oneth point, I have got now h 2, because this is shifted to the right, it has gone, but this data, now you see if I multiply this, and this, like linear convolution, h 0 into x 1, h 1 into x 0, this h 0 has shifted, h 1 has come in, these 2 have multiplied, you have give the result. How about contribution from here? Here you see, you have got data, like this part of the data, h, these already in, this part, these data, this is from 2 to n minus 1. So, this is multiplied by these 0 pad, this extra 0, these multiplying the first 0 here. Because the 0 pad has also shifted, earlier 0 pad was studying from 1, now it has gone to 2. So, instead of coming from between 1 to n minus 1, this is now got 1 point to the right. So, it will take care of these n minus 1, take out this n minus 2 samples now, then this will be multiplied by 0, and 1 more, this will multiplied 0 here, and remaining samples will be multiplied by 0s here.

So, I will get the same result. And this way, if you continue, after sometime what will happen? Let me take out a first page, after sometime what will happen, that suppose I am shifting now, so n equal to 0 cases, I have taken n equal to 1 case, after sometime. Now the whole chunks has come here, say h 0, h 1 and followed by this h n minus 1. So, how many m points, if m is less than m, in they are meeting data from here, but this is followed by, this 0 pad. 0 pad is of length, we know n minus 1. So, this is direct, this 0 pad is directly going up to, you know, full 0 pad is going up to L minus 1, because I have got total length, h 0, h 1 up to m minus 1. So, m, capital M samples are already here. 0 pad was length n minus 1. So, total length is m plus n minus 1, that is L, which means from 0 to L minus 1, this is my entire 0 pad. Just count, h 0 (Refer Time: 28:56) case, where h 0, h 1, up to h minus 1 has come, and then, this is, before these I have got another 0 pad, of length n minus 1, but if this be the case, total length is L, out of which capital M, gone here, 0, 1, up to, so m gone.

So, L minus m, L minus m, is a remaining length, and L minus m is, what is L? L was n plus m, minus 1. So, L minus m, is n minus 1, and the 0 pad is length n minus 1; that means, the full 0 pad is up to this, there is 0 to L minus 1, if you see the length, if you see the duration, out of which, so total length is L, out of which, m length gone here, and n minus 1 here, all right? and now if you do the sample has multiplication, as you being suppose m less then n, there is, h 0 will multiply some data here, because it is m minus 1, it is 0, at m minus oneth point, h 0 at zero th point m minus 1, this will multiply some data, this will multiply some data, like that, you will get some value, and then these 0.

Additional data, this m minus 1, this m minus 1, m minus 1 might be here, m minus 1. So, you will have, this data getting multiplied by these data, this data getting multiplied by these data, like that, but how about this data? They will get multiplied by the 0, they will not contribute anything. After that, this will shift further, but as you shift further, this block moves, but from behind which comes is zeros. So, again, they do not contribute anything. These samples from here, they will multiply those zeros. So, only like linear contribution, these blocks will be shifting, and it will multiply from whatever data you have on top, and you will get the result.

So, this is shifting. This is preceded by zeros, and succeeded by zeros. This, I mean, a situation where full block has come in, m minus 1. So, it is taking the m minus oneth point, m is less than n, multiplying these data with these data, then next data with next, and all that. Here whatever data I had, from m minus 1 to n minus 1, they are multiplied by this part of these 0 blocks, and remaining 0s here, multiply by whatever 0s I have here, they do not contribute anything. So, it is still you know linear convolution way, now next if I shifted further to the right, this is getting further to the right, but will this. So, under linear convolution, suppose I shifted to the right by 1. So, this guy you would come to the next, this guy would come to the next, this guy would come to the next h 0, earlier was m minus one th point, because multiplying the m minus 1 sample, now it will be multiplying the next sample, and so on and so forth.

But this guy, will it be multiplying something? It means multiply some non 0 value from here, then there is an error, because some the linear convolution, this block is moving to the right, earlier it came fully under this, now it is moving to the right. So, there is should not be any overlap between this and this. Now if I carry out the product who is entering, if I shift it to the right by 1, 0 from here, that will be multiplying x 0, so no contribution. If I said further to the write by 1 more, another 0 comes in. So, they do not contribute anything. So, what contributes is this part, and the way it is moving, it is like linear convolution. You can verify this by some example. So, this way, this is called 0 point in technique. This is the way to carry out linear convolution, by a circular convolution. This is one important result, find it, it is called 0 point technique.

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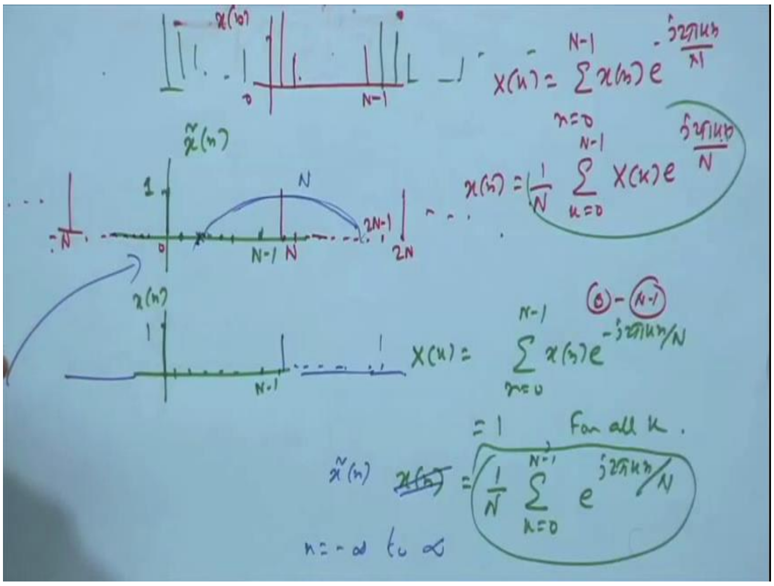

Then one more result from this DFT, which I would like to cover. You have seen earlier, that suppose you have got a sequence x n, 0 to N minus 1, and then if you have DFT, you find out DFT, all right?

Then using this DFT you can write x n, whether inverse relation. Here some over k, and then I said, if I allows in this formula, if I allows small n now, to go beyond this range, go to the right, go to the left, from minus infinity to infinity. Then that will be a sequence, which is nothing but periodic repetition of this. That is, then it will be, if you allow, then it will be like this. Contribute. If you in this formula, if you allows small n to vary from minus infinity to infinity, what you will get you? You will get infinite low sequence, which is nothing, but, periodic repetition, or repetition of this periodic.

Now, I will just take an example, because this is useful for me. Suppose I have, a sequence which is having 1 in 0, 0, 0, up to n minus 1, first, and then again repeated, 0, 0, 0, 0 from nth point, then I get 2 nth, point another 1. So, this 2 n minus 1, zero th and this side is 0, 0, 0 at minus n, you have got 1, dot, dot, dot, dot, all right? This is given, suppose I take only this part, finite length, call it x n and call this as periodic x tilde n. So, I take this one chunk, this, this block only 1, and 0, 0, 0, 0 up to this n minus 1. What is capital X k? Capital X k, is x n, e to the power minus j, 2 pi, k n, by these, was sum to over n 0 to N minus 1, but you see this sequence, has non 0 value, only at n equal to 0. So, only x 0 is 1, all other x value are zeros. At when, x 0, n is 0, and e to the power 0, that is 1, 1 into 1, so X k is 1, for all k, therefore, if I calculate the x n by the inverse DFT, it will be 1 by n, capital X k, whether it is 1, put 1, 1 into something that something only e to the power j, 2 pi, k n, by n.

Now, if I allow, small n to be, if I, in this formula, if I allow small n to go from minus infinity to infinity, what will I get? I will get periodic reputation of this, which is nothing, but this x tilde n. So, these there will become x tilde n, where n is from minus infinity to infinity, this formula if you allow, right? So, original x tilde n, sequence like this, can be written like this. You put any value of n, on this side also, you will get the same value, if you walk back here say, whatever you have the same thing, you will have, if you go capital N to the right, from the capital N to the right, like that. This is call sampling function, and I will use it, in my subsequent discussions.

Thank you very much.